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Construction of higher groupoids via matched pairs actions

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Abstract: In this work, we construct a relationship between matched pairs and triples of groupoids. Given two 3-groupoids with a common edge, we construct a triple groupoid by using the matched pairs actions.

Key words: Triple groupoid, matched pairs, matched triple

1. Introduction

Matched pairs of groups were introduced by Takeuchi [17] as a group version of Singer's work [16] for Hopf algebras. Majid introduced the Lie algebra analogue of matched pairs and applied this to quantum groups [15]. The theory of matched pairs was also used as a tool for set theoretic solutions of the Yang–Baxter equation in [10].

Groupoids were introduced by Brandt [1] in 1926 as algebraic structures also known as virtual groups. A group-like approach to the groupoid is a category $\mathcal{C}$ with objects set C_0 and morphisms set C_1 in which each morphism is invertible. These structures are useful in a variety of mathematics from geometry to homotopy theory, algebra, and topology. For more information on groupoids see [2–5, 11]. Double groupoids were introduced by Ehresmann in [9]. A double groupoid can be seen as a set of boxes with horizontal and vertical compositions together with interchange law. For more information see [7, 8, 12].

In his brief note [6], Brown introduced a geometric approach to double groupoids. The existence of a triple groupoid by matched triples of groups, mentioned by Brown [6], is a useful way to approach geometric considerations. Later, Majard [13] generalized this concept for n-tuple groups. In this work, following Brown, we investigate this situation for triple groupoids, diagrammatically.

2. Matched pairs of group(oid)s

In this section, we recall some basic information about matched pairs of groups and groupoids.

Definition 2.1 *A matched pair of groups means a triple* (G_1, G_2, σ) *where* G_1 *and* G_2 *are groups and the map*

$$\begin{array}{rcl}\sigma & : & G_1 \times G_2 \to G_2 \times G_1 \\ (g_1, g_2) & \mapsto & (g_1 \rightharpoonup g_2, g_1 \leftharpoonup g_2)\end{array}$$

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satisfies the following conditions:

$$\begin{array}{rcl} g_2 & \rightharpoonup & (h_2 \rightharpoonup g_1) = g_2 h_2 \rightharpoonup g_1 \\ g_2 h_2 & \leftharpoonup & = (g_2 \leftharpoonup (h_2 \rightharpoonup g_1))\,(h_2 \leftharpoonup g_1) \\ (g_2 \leftharpoonup g_1) & \leftharpoonup & h_2 = g_2 \leftharpoonup g_1 h_2 \\ g_2 & \rightharpoonup & g_1 h_1 = (g_2 \rightharpoonup g_1)\,((g_2 \leftharpoonup g_1) \rightharpoonup h_1) \end{array}$$

for $g_1, h_1 \in G_1$ *and* $g_2, h_2 \in G_2$.

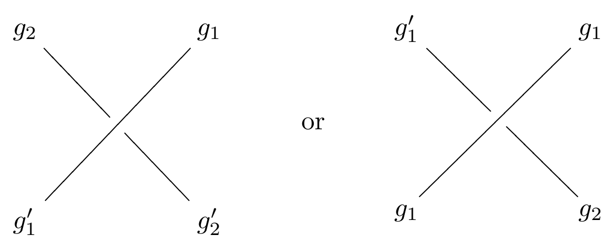
$G_1 \times G_2$ forms a group with the product, denoted by $G_1 \bowtie G_2$. Conversely, if G_1 and G_2 are subgroups of a group G such that the product map $G_1 \times G_2 \to G$ is bijective, then (G_1, G_2) forms a matched pair with structure $\sigma(g_1, g_2) = (g_1 \rightharpoonup g_2, g_1 \leftharpoonup g_2)$ defined by $g_1 g_2 = (g_1 \rightharpoonup g_2)(g_1 \leftharpoonup g_2)$.

The structure map σ of a matched pair (G_1, G_2) is bijective. The triple $\left(G_1, G_2, \sigma^{-1}\right)$ forms a matched pair called the opposite of (G_1, G_2). The group $G_2 \bowtie G_1$ is isomorphic to $G_1 \bowtie G_2$ by $(g_2, g_1) \longmapsto (1, g_2)(g_1, 1)$.

Let $g_2, g_2' \in G_2$ and $g_1, g_1' \in G_1$. We denote the relation

$$(g_1', g_2') = \sigma(g_2, g_1)$$

by the diagram



Since the structure map is nondegenerate in the sense of [13] and [14], upon determining one element (g_1, g_2), the rest of the elements are determined by the diagram above.

A groupoid is a small category in which all arrows are invertible. It consists of a set of arrows G_1, a set of objects G_0 (called the base), source and target maps $s, t : G \to P$, composition $\circ : G_1 \times G_1 \to G_1$, and identities $id : G_0 \to G_1$.

Alternatively, a groupoid may be defined as a set G with a partially defined associative product and partial units, whose elements are all invertible.

Definition 2.2 *Let*

$$\begin{tikzcd} (\,G \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & G_0, \circ) \arrow[l, bend left=60, "e"] \end{tikzcd}$$

be a groupoid. For a map $\wp : \varepsilon \to G_0$ *a left action of* G *on* $\wp$ *is a map*

$$\triangleright : G \times \varepsilon \to \varepsilon$$

satisfying the following rules:

$$\begin{array}{lrcl} 1. & \wp(\alpha \triangleright_e) & = & s(\alpha) \\ 2. & \alpha \triangleright (\beta \triangleright e) & = & (\alpha\beta) \triangleright e \\ 3. & id(\wp(e)) \triangleright e & = & e \end{array}$$

for all $\alpha, \beta \in G$ *and* $e \in \wp$. *A right action of* G *on* $\hbar : \varepsilon \to G_0$ *is a map*

$$\lhd : \varepsilon \times G \to \varepsilon$$

satisfying the rules

$$\begin{array}{lrcl} 1. & \hbar(e \lhd \alpha) & = & t(\alpha) \\ 2. & (e \lhd \alpha) \lhd \beta & = & e \lhd (\alpha\beta) \\ 3. & e \lhd id(\hbar(e)) & = & e \end{array}$$

for all $\alpha, \beta \in G$ *and* $e \in \wp$.

Definition 2.3 *A matched pair of groupoids consists of two groupoids* (G_1, G_2) *with the same base* G_0 *together with the following data:*

Let $s_1, t_1 : G_1 \rightrightarrows P$ *and* $s_2, t_2 : G_2 \rightrightarrows G_0$ *be the source and target maps of* G_1 *and* G_2, *respectively. Then we have a left action,*

$$\rhd : G_2 \times G_1 \to G_1$$

of G_2 *on* $s_1 : G_1 \to G_0$ *and a right action,*

$$\lhd : G_2 \times G_1 \to G_2$$

of G_1 *on* $t_2 : G_2 \to G_0$. *All the data given above satisfy the following:*

i. $s_1(\beta \rhd \gamma) = s_2(\gamma \lhd \beta)$,

ii. $\beta \rhd (\sigma\alpha) = (\beta \rhd \sigma)\left[(\beta \lhd \sigma) \lhd \alpha\right]$,

iii. $(\beta_1\beta_2) \lhd \alpha = \left[\beta_1 \lhd (\beta_2 \rhd \alpha)\right](\beta_2 \lhd \alpha)$,

for all $\alpha, \gamma \in G_1$, $\beta, \beta_1, \beta_2 \in G_2$, *for which the operations are defined.*

Lemma 2.4 *For all* $\alpha, \gamma \in G_1$, $\beta, \beta_1, \beta_2 \in G_2$ *for which the operations are defined, we have*

i. $t_1(\beta \rhd \gamma) = s_2(\beta \lhd \gamma)$,

ii. $(\beta \rhd \sigma)^{-1} = (\beta \lhd \sigma) \rhd \sigma^{-1}$,

iii. $(\beta_2 \lhd \alpha)^{-1} = \left[\beta_2^{-1} \lhd (\beta_2 \rhd \alpha)\right]$.

Proof *i.* For $A_1, A_2, A_3 \in G_0$ and $\beta : A_1 \to A_2 \in G_2, \gamma : A_2 \to A_3 \in G_1$ consider the following diagram:

$$\begin{tikzcd} A_1 \arrow[r, "\beta \rhd \gamma"] \arrow[d, "\beta"'] & X \arrow[d, "\beta \lhd \gamma"] \\ A_2 \arrow[r, "\gamma"'] & A_3 \end{tikzcd}$$

where

$$\begin{array}{rcccl} s_1(\beta \rhd \gamma) & = & s_2(\beta) & = & A_1, \\ t_2(\beta \lhd \gamma) & = & t_1(\gamma) & = & A_3. \end{array}$$

The possibility of $X \in G_0$ gives us

$$t_1(\beta \rhd \gamma) = s_2(\beta \lhd \gamma).$$

ii. Let $A_1, A_2, A_3, A_4 \in G_0$, and $\beta : A_1 \to A_2 \in G_2, \sigma : A_2 \to A_3, \alpha : A_3 \to A_4 \in G_1$. Considering the following diagram,

$$\begin{tikzcd}
A_1 \arrow[r, "\beta \rhd \sigma"] \arrow[d, "\beta"'] & * \arrow[r, "(\beta \lhd \sigma) \rhd \alpha"] \arrow[d, "\beta \lhd \sigma"] & * \arrow[d, "(\beta \lhd \sigma) \lhd \alpha"] \\
A_2 \arrow[r, "\sigma"'] & A_3 \arrow[r, "\alpha"'] & A_4
\end{tikzcd}$$

we get

$$\begin{array}{rcl} \beta \lhd (\sigma\alpha) & = & (\beta \lhd \sigma) \lhd \alpha \\ \beta \rhd (\sigma\alpha) & = & (\beta \rhd \sigma)\left[(\beta \lhd \sigma) \lhd \alpha\right] \end{array}$$

and taking $\sigma = \alpha^{-1}$ in the last equality we have

$$(\beta \rhd \sigma)^{-1} = (\beta \lhd \sigma) \rhd \sigma^{-1}.$$

iii. For $A_1, A_2, A_3, A_4 \in G_0$, and $\beta_1 : A_1 \to A_2, \beta_2 : A_2 \to A_3 \in G_2, g : A_3 \to A_4 \in G_1$. Considering the following diagram,

$$\begin{tikzcd}
A \arrow[r, "\beta_1 \rhd (\beta_2 \rhd \alpha)"] \arrow[d, "\beta_1"'] & * \arrow[d, "\beta_1 \lhd (\beta_2 \rhd \alpha)"] \\
B \arrow[r, "\beta_2 \rhd g"] \arrow[d, "\beta_2"'] & * \arrow[d, "\beta_2 \lhd \alpha"] \\
C \arrow[r, "\alpha"'] & D
\end{tikzcd}$$

we get

$$\begin{array}{rcl} (\beta_1\beta_2) \rhd \alpha & = & \beta_1 \rhd (\beta_2 \rhd \alpha) \\ (\beta_1\beta_2) \lhd \alpha & = & \left[\beta_1 \lhd (\beta_2 \rhd \alpha)\right](\beta_2 \lhd \alpha) \end{array}$$

and taking $\beta_1^{-1} = \beta_2$ in the last equality we note that

$$(\beta_2 \lhd \alpha)^{-1} = \left[\beta_2^{-1} \lhd (\beta_2 \rhd \alpha)\right].$$

$\square$

3. Matched pairs and matched triple of groups and a geometric approach to 3-groupoids

In this section, we investigate matched pairs and matched triples as in Brown [6] to understand the geometry of triple groupoids. For more information on matched pairs see [14].

Let G_1, G_2 be subgroups of G such that $G_1 \cap G_2 = \{e_G\}$. For $x \in G_1$ and $y \in G_2$ we will consider the group operation xy as a composite of arrows such that $t(x) = s(y) = *$. Then we have

$$\begin{tikzcd} * \arrow[r, "{}^{x}y"] \arrow[d, "x"'] & * \arrow[d, "y^{x}"] \\ * \arrow[r, "y"'] & * \end{tikzcd}$$

where the horizontal and vertical arrows via actions are denoted by $\varepsilon_h(y,x) = ^x y$ and
$\varepsilon_v(y,x) = y^x$, respectively.

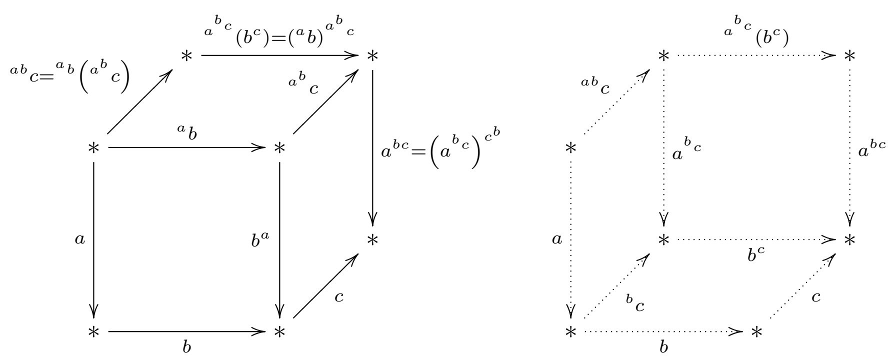
Example 3.1 *Let V be any groupoid with base P. There is a matched pair (V, P) with actions*

$$t(f) \triangleleft f = f \text{ and } t(f) \triangleright f = b(f).$$

Similarly, for any groupoid H with base P, there is a matched pair (P, H) with actions

$$x \triangleleft r(x) = l(x) \text{ and } x \triangleright r(x) = x.$$

Example 3.2 *Let M, N, and P be the matched triple of subgroups of a group G. Take $a \in M, b \in N$ and $c \in P$ such that $t(a) = s(b)$ and $t(b) = s(c)$. Then the cubical model is of the following form:*



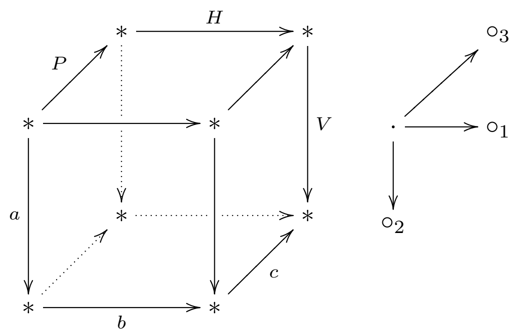
An n-fold groupoid is an internal groupoid in $(n-1)$-fold groupoids. That is, a 0-fold groupoid is a set, a 1-fold groupoid is a groupoid, a 2-fold groupoid is a double groupoid, and so on, where the structure of a double groupoid consists of a set G and two groupoid structures in which the compositions satisfy the usual interchange law; that is, for $x_1, x_2, y_1, y_2 \in G$ we have

$$(x_1 \circ_i y_1) \circ_j (x_2 \circ_i y_2) = (x_1 \circ_j x_2) \circ_i (y_1 \circ_j y_2). \qquad (*)$$

For $n = 3$ a triple groupoid is a set G with three groupoid structures satisfying the interchange law in pairs when defined: for example, $\circ_i$ with $\circ_j$, $\circ_j$ with $\circ_k$ and $\circ_i$ with $\circ_k$ satisfy $(*)$.

From now on our interest will be in triple groupoids, or the triple categories in which each underlying set category is a groupoid. By a triple category we mean a 3-fold category that is an internal category in double categories. Now we give a description of a 3-groupoid by using the matched triples of groups diagrammatically.

We will examine the matched triple of subgroups M, N, and P of a group G in which each pair is a matched pair. With such data, we can consider a triple groupoid as

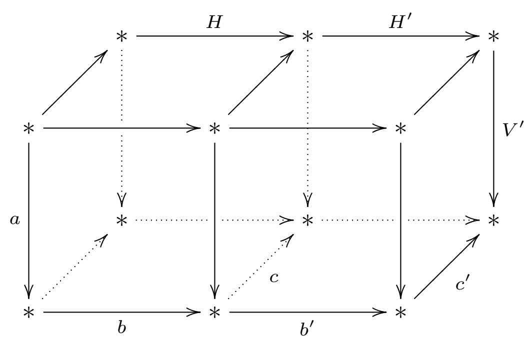


where $V = \varepsilon_v(a, bc)$, $H = \varepsilon_h[\varepsilon_v(b,c), \varepsilon_v(a, \varepsilon_h(c,b))] = \varepsilon_v[\varepsilon_h(b,a), \varepsilon_h(c, \varepsilon_v(a,b))]$, and $P = \varepsilon_h(c, ab)$.

The triple groupoid should have the algebraic analogue of the horizontal, vertical, and parallel compositions of cubes and also should permit cancellations.

Proposition 3.3 *Horizontal composition of matched triples of groups defines the inverse elements $\varepsilon_h(b,a)^{-1}$ and $\varepsilon_h[\varepsilon_v(b,c), \varepsilon_v(a, \varepsilon_h(b,c)))]^{-1}$.*

Proof For $i = 1$, we obtain



$$
\begin{aligned}
\varepsilon_v(bb', a) &= \varepsilon_v(b', \varepsilon_v(b', a),) \\
\varepsilon_h(bb', a) &= \varepsilon_h(b,a).\varepsilon_h(b', \varepsilon_v(b,a)).
\end{aligned}
$$

In the last equality, taking $b' = b^{-1}$, the left side becomes $\varepsilon_h(e_G, a) = e_G$, so we can find the inverse of $\varepsilon_h(b,a)$ as

$$\varepsilon_h(b,a)^{-1} = \varepsilon_h(b', \varepsilon_v(b,a)) = \varepsilon_h\left(b^{-1}, \varepsilon_v(b,a)\right)$$

for $a \in M$, $b, b' \in N$, and $c \in P$. For the back side of the cubes, the actions can be given by the following diagram:

$$\begin{tikzcd}
* \arrow[r] \arrow[d, "\varepsilon_v(b,\varepsilon_h(c,b))"'] & * \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r, "\varepsilon_v(b,c)"'] & * \arrow[r, "\varepsilon_v(b',c')"'] & *
\end{tikzcd}$$

We obtain the following result:

$$\begin{aligned}
\varepsilon_v\left[\varepsilon_v(b,c)\,\varepsilon_v(b',c'),\varepsilon_v(a,\varepsilon_h(c,b)),\right] &= V' \\
&= \varepsilon_v(\varepsilon_v(b',c'),V) \\
&= \varepsilon_v\left[\varepsilon_v(b',c'),\varepsilon_v(a',\varepsilon_h(c',b'))\right] \\
&= \varepsilon_v\left[\varepsilon_v(b',c'),\varepsilon_v(a,bc)\right]
\end{aligned}$$

and we obtain

$$\begin{aligned}
\varepsilon_h\left[\varepsilon_v(b,c)\,\varepsilon_v(b',c'),\varepsilon_v(a,\varepsilon_h(c,b))\right] &= H.H' \\
&= \varepsilon_h\left[\varepsilon_v(b,c),\varepsilon_v(a,\varepsilon_h(b,c)))\right].\varepsilon_h\left[\varepsilon_v(b',c'),\varepsilon_v(a',\varepsilon_h(b',c')))\right].
\end{aligned}$$

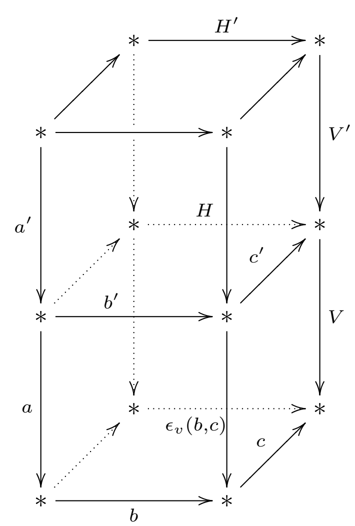
If we take $\varepsilon_v(b',c') = \varepsilon_v(b,c)^{-1}$, we get

$$\begin{aligned}
\varepsilon_h\left[\varepsilon_v(b,c),\varepsilon_v(a,\varepsilon_h(b,c)))\right]^{-1} &= \varepsilon_h\left[\varepsilon_v(b',c'),\varepsilon_v(a',\varepsilon_h(b',c')))\right] \\
&= \varepsilon_h\left[\varepsilon_v\left(b,c\right)^{-1},\varepsilon_v(a,bc)\right)\right].
\end{aligned}$$

□

Proposition 3.4 *Vertical composition of matched triples of groups defines the inverse elements* $\varepsilon_v(a,b)^{-1}$ *and* $\varepsilon_v(a,bc)^{-1}$.

Proof For the operation $\circ_2$, we have the following diagram:



For the actions on the front side of the cubes, we have

$$\varepsilon_h(b, a'a) = \varepsilon_h(b', a') = \varepsilon_h(\varepsilon_h(b, a), a'),$$

$$\varepsilon_v(a'a, b) = \varepsilon_v(a', b')\,\varepsilon_v(a, b),$$

and if we take $a' = a^{-1}$, we get

$$\varepsilon_v\left(a^{-1}, b'\right) = \varepsilon_v(a, b)^{-1}.$$

For the actions on the back side of the cubes, we have the following diagram:

$$\begin{tikzcd}
* \arrow[r, "H'"] \arrow[d] & * \arrow[d, "V'"] \\
* \arrow[r, "H"] \arrow[d] & * \arrow[d, "V"] \\
* \arrow[r, "\epsilon_v(b,c)"'] & *
\end{tikzcd}$$

where $V = \varepsilon_v(a, bc)$, $V' = \varepsilon_v(a', b'c')$, $H = \varepsilon_h[\varepsilon_v(b, c), \varepsilon_v(a, \varepsilon_h(c, b))] = \varepsilon_v[\varepsilon_h(b, a), \varepsilon_h(c, \varepsilon_v(a, b))]$, and $H' = \varepsilon_h[\varepsilon_v(b', c'), \varepsilon_v(a', \varepsilon_h(c', b'))] = \varepsilon_v[\varepsilon_h(b', a'), \varepsilon_h(c', \varepsilon_v(a', b'))]$, and then we get

$$\begin{aligned}
\varepsilon_h[\varepsilon_v(b,c), \varepsilon_v(a', \varepsilon_h(c', b'))).\varepsilon_v(a, \varepsilon_h(c, b))] &= \varepsilon_h[\varepsilon_v(b', c'), \varepsilon_v(a', \varepsilon_h(c', b')))] \\
&= \varepsilon_h[\varepsilon_h(\varepsilon_v(b, c), \varepsilon_v(a, \varepsilon_h(c, b))), \varepsilon_v(a', \varepsilon_h(c', b')))]
\end{aligned}$$

and

$$\begin{aligned}
\varepsilon_v[\varepsilon_v(a', \varepsilon_h(c', b')).\varepsilon_v(a, \varepsilon_h(c, b))), \varepsilon_v(b, c)] &= V'.V \\
&= \varepsilon_v(a', b'c')\,\varepsilon_v(a, bc) \\
&= \varepsilon_v[\varepsilon_v(a', \varepsilon_h(c', b')), \varepsilon_v(b, c)].\varepsilon_v(a, bc).
\end{aligned}$$

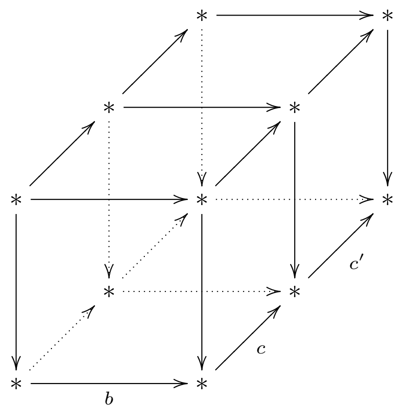
If we take $\varepsilon_v(a, \varepsilon_h(c, b))^{-1} = \varepsilon_v(a', \varepsilon_h(c', b'))$, we get

$$V^{-1} = \varepsilon_v(a, bc)^{-1} = \varepsilon_v\left[\varepsilon_v(a, \varepsilon_h(c, b))^{-1}, H\right].$$

$\square$

Proposition 3.5 *Parallel composition of matched triples of groups defines the inverse elements* $\varepsilon_h(c, b)^{-1}$ *and* $\varepsilon_h(c, ab)^{-1}$.

Proof For the operation $\circ_3$, consider the following diagram:



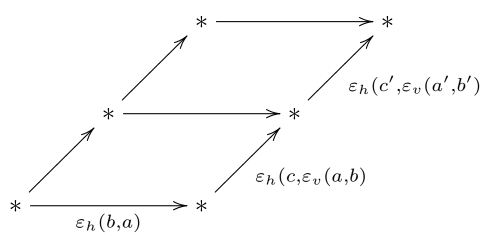
$$\varepsilon_v(b, cc') = \varepsilon_v[\varepsilon_v(b,c), c'] = \varepsilon_v(b', c'),$$

$$\varepsilon_h(cc', b) = \varepsilon_h(c,b)\,\varepsilon_h(c', b'),$$

and taking $c' = c^{-1}$ we get

$$\varepsilon_h(c,b)^{-1} = \varepsilon_h\left(c^{-1}, b'\right) = \varepsilon_h\left(c^{-1}, \varepsilon_v(b,c)\right).$$

For the other side, using the following diagram,



we get

$$\begin{aligned} \varepsilon_v[\varepsilon_h(b,a), \varepsilon_h(c, \varepsilon_v(a,b)).\varepsilon_h(c', \varepsilon_v(a,b))] &= \varepsilon_v[\varepsilon_h(b',a'), \varepsilon_h(c', \varepsilon_v(a',b'))] \\ &= \varepsilon_v[H, \varepsilon_h(c', \varepsilon_v(a',b'))], \end{aligned}$$

and

$$\begin{aligned} \varepsilon_h[\varepsilon_h(c, \varepsilon_v(a,b)).\varepsilon_h(c', \varepsilon_v(a',b'))), \varepsilon_h(b,a)] &= P.P' \\ &= \varepsilon_h(c, ab)\,\varepsilon_h(c', a'b'). \end{aligned}$$

Taking $\varepsilon_h(c', \varepsilon_v(a',b'))^{-1} = \varepsilon_h(c, \varepsilon_v(a,b))$, we get

$$\varepsilon_h(c, ab)^{-1} = \varepsilon_h[\varepsilon_h(c, \varepsilon_v(a,b)), H]^{-1}.$$

□

We also obtain that

$$\begin{aligned} c^{-1}b^{-1}a^{-1} &= V^{-1}H^{-1}P^{-1} \\ &= \varepsilon_v\left(a,bc\right)^{-1}.\varepsilon_h\left[\varepsilon_v\left(b,c\right),\varepsilon_v\left(a,\varepsilon_h\left(c,b\right)\right)\right]^{-1}\varepsilon_h\left(c,ab\right)^{-1}. \end{aligned}$$

Replacing by

$$\begin{aligned} a^{-1} &\mapsto a \\ b^{-1} &\mapsto b \\ c^{-1} &\mapsto c \end{aligned}$$

and writing $\dot{a}=a^{-1}$, $\dot{b}=b^{-1}$, and $\dot{c}=c^{-1}$, we deduce that

$$\begin{aligned} cba &= \varepsilon_v\left(\dot{a},\dot{b}\dot{c}\right)^{-1}.\varepsilon_h\left[\varepsilon_v\left(\dot{b},\dot{c}\right),\varepsilon_v\left(\dot{a},\varepsilon_h\left(\dot{c},\dot{b}\right)\right)\right]^{-1}.\varepsilon_h\left(\dot{c},\dot{a}\dot{b}\right)^{-1} \\ &= \varepsilon_v\left[\varepsilon_v\left(\dot{a},\varepsilon_h\left(\dot{c},\dot{b}\right)\right)^{-1},\dot{H}\right].\varepsilon_h\left[\varepsilon_v\left(\dot{b},\dot{c}\right)^{-1},\varepsilon_v\left(\dot{a},\dot{b}\dot{c}\right)\right].\varepsilon_h\left[\varepsilon_h\left(\dot{c},\varepsilon_v\left(\dot{a},\dot{b}\right)\right)^{-1},\dot{H}\right]. \end{aligned}$$

In an analogous way, we have

$$c^{-1}b^{-1}a^{-1}=V^{-1}\varepsilon_h\left(c,\varepsilon_v\left(a,b\right)\right)^{-1}\varepsilon_h(b,a)^{-1},$$

where

$$\begin{aligned} cba &= \varepsilon_v\left(\dot{a},\dot{b}\dot{c}\right)^{-1}.\varepsilon_h\left[\dot{c},\varepsilon_v\left(\dot{a},\dot{b}\right)\right]^{-1}.\varepsilon_h\left(\dot{b},\dot{a}\right)^{-1} \\ &= \varepsilon_v\left[\varepsilon_v\left(\dot{a},\varepsilon_h\left(\dot{c},\dot{b}\right)\right)^{-1},\dot{H}\right].\varepsilon_v\left[\dot{P}^{-1},\varepsilon_h\left(\dot{b},\dot{a}\right)\right].\varepsilon_h\left[\dot{b}^{-1},\varepsilon_v\left(\dot{a},\dot{b}\right)\right] \end{aligned}$$

and

$$c^{-1}b^{-1}a^{-1}=\varepsilon_v\left(b,c\right)^{-1}\varepsilon_v\left(a,\varepsilon_h\left(c,b\right)\right)^{-1}P^{-1},$$

and we get

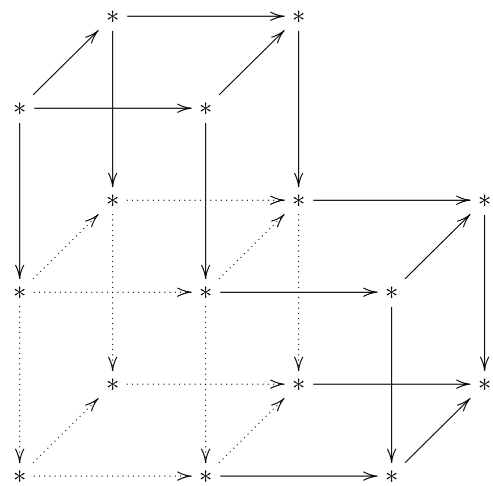
$$\begin{aligned} cba &= \varepsilon_v\left(\dot{b},\dot{c}\right)^{-1}.\varepsilon_v\left[\dot{a},\varepsilon_h\left(\dot{c},\dot{b}\right)\right].\varepsilon_h\left(\dot{c},\dot{a}\dot{b}\right)^{-1} \\ &= \varepsilon_h\left[\dot{c}^{-1},\varepsilon_v\left(\dot{b},\dot{c}\right)\right].\varepsilon_v\left[\dot{a}^{-1},\varepsilon_h\left(\dot{c},\dot{b}\right)\right].\varepsilon_h\left[\varepsilon_h\left(\dot{c},\varepsilon_v\left(\dot{a},\dot{b}\right)\right)\dot{b}^{-1},\dot{H}^{-1}\right]. \end{aligned}$$

These calculations of triple groupoids can be expressed as the sets $(M\times N)\times P$ and $M\times(N\times P)$, which can be given by the eight groupoid actions $(M\ltimes N)\times P$, $(M\rtimes N)\times P$, $M\times(N\ltimes P)$, $M\times(N\rtimes P)$, $(M\times N)\rtimes P$, $(M\times N)\ltimes P$, $M\ltimes(N\times P)$, and $M\rtimes(N\times P)$. We give the operation of some of them as an example.

$$\begin{gathered} (a,b,c)\circ_1\left(\varepsilon_v\left(a,b\right),b',c'\right)=\left(a,bb',c\right)\in(M\rtimes N)\times P \\ \left(a',\varepsilon_h\left(b,a\right),c'\right)\circ_2\left(a,b,c\right)=\left(aa',b,c\right)\in(M\ltimes N)\times P \\ (a,b,c)\circ_3\left(a',\varepsilon_v\left(b,c\right),c'\right)=\left(a',b,cc'\right)\in M\times(N\rtimes P) \end{gathered}$$

Remaining group operation structures can be given by a similiar way.

Conclusion 3.6 *Given two triple groupoids with a common edge with the properties above, one can construct a new triple groupoid via matched triple actions of groups.*



We give the following result from [6].

Conclusion 3.7 *The groupoid composition*

$$(a,b,c)\,(a',b',c') = \left(a.\delta_i\left[\delta_i\left(a',c\right),b\right], \delta_t\left[\delta_i\left(a',c\right),b\right].\delta_i\left[b',\delta_t\left(a',c\right)\right]\right), \delta_i\left(b',\delta_t\left(a',c\right)\right).c'$$

gives a group structure where

$$\delta_i\,(a,b) = \varepsilon_v\left(a, \varepsilon_h\left(\dot{a},\dot{b}\right)\right),$$

$$\delta_t\,(b,a) = \varepsilon_h\left(b, \varepsilon_h\left(\dot{b},\dot{a}\right)\right).$$

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