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On expansions of prime and 2-absorbing hyperideals in multiplicative hyperrings

Gülşen ULUCAK*
Department of Mathematics, Faculty of Science, Gebze Technical University, Gebze, Kocaeli, Turkey

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Abstract: In this paper, we study δ-primary and 2-absorbing δ-primary hyperideals which are the extended classes of prime and 2-absorbing hyperideals, respectively. Assume that R is a commutative multiplicative hyperring with nonzero identity. We call $I \in \mathcal{I}*(\mathcal{R})$ a δ-primary hyperideal if $a, b \in R$ and $a \circ b \subseteq I$ imply either $a \in I$ or $b \in \delta(I)$ and also, I is called 2-absorbing δ-primary hyperideal if $a, b, c \in R$ and $a \circ b \circ c \subseteq I$ imply $a \circ b \subseteq I$ or $b \circ c \subseteq \delta(I)$ or $a \circ c \subseteq \delta(I)$. Moreover, we give the basic properties of these new types of hyperideals and investigate the relations among these structures. Then a number of main results and examples are given to explain the general framework of these structures.

Key words: δ-primary hyperideal, 2-absorbing hyperideal, 2-absorbing δ-primary hyperideal

1. Introduction

In 1934, Marty defined hypergroups as a generalization of groups and so he firstly studied the theory of algebraic hyperstructures. Hyperstructures take an important place in both pure and applied mathematics. Afterwards, many authors have studied the theory of hyperstructures which has a pivotal role on applications to other areas such as geometry, lattices, automata, cryptography, coding theory, artificial intelligence, and probabilities [1–3, 8].

In this paper, we dwell on hyperrings which have an important role in the theory of algebraic hyperstructures. Various types of hyperrings have been introduced and studied by many authors (e.g., Krasner, Davvaz, Ameri and Norouzi) in [2, 9, 12]. Let R be a hyperring. By $P^*(R)$, we mean the set of all non empty subset of R. Let $\circ$ be a hyperoperation from $R \times R$ to $P^*(R)$. Krasner said that the structure $(R, +, \circ)$ is a hyperring if it satisfies the following properties: (i) $(R, +)$ is a canonical hypergroup, (ii) $(R, \circ)$ is a semigroup and (iii) $'\circ'$ has the property of distributive over addition (see [9]). Then, it is known as Krasner hyperring. In [13], Rota presented a different type of hyperring. According to him, $(R, +, \circ)$ is a multiplicative hyperring if it has the following properties: (i) $(R, +)$ is an abelian group whose identity element 0_R has the absorbing property in $(R, +, \circ)$ (i.e. $\{0_R\} = 0_R \circ x = x \circ 0_R$ for every $x \in R$.), (ii) $(R, \circ)$ is a hypersemigroup, (iii) $r \circ (r' + s) \subseteq r \circ r' + r \circ s$ and $(r + r') \circ s \subseteq r \circ s + r' \circ s$, (iv) $r \circ (-r') = (-r) \circ r' = -(r \circ r')$ for all $r, r', s \in R$. By this definition, it can be obtained that $r \circ (r' \circ s) = \bigcup_{t \in (r' \circ s)} (r \circ t) = \bigcup_{k \in (r \circ r')} (k \circ s) = (r \circ r') \circ s$ for all elements r, r', s of R. A multiplicative hyperring R is said to be commutative if $r \circ r' = r' \circ r$ for all $r, r' \in R$. An element $e \in R$ is called identity if $\{r\} = e \circ r = r \circ e$ for all $r \in R$.

*Correspondence: gulsenulucak@gtu.edu.tr
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Let R be a commutative multiplicative hyperring with nonzero identity. A hyperoperation $\circ$ holds $U \circ V = \bigcup_{u\in U, v\in V}(u \circ v)$ and $U \circ x = U \circ \{x\}$ for every two nonempty subsets U, V of R and $x \in R$ [9]. Recall from [8] that a nonempty subset I of R is a hyperideal if it has the following: (i) $I - I \subseteq I$, that is, $x - y \in I$ for each $x, y \in I$ and (ii) $r \circ x \subseteq I$ for each $r \in R, x \in I$. Let the subset $\{r_1 \circ ... \circ r_n | r_i \in R \ for \ some \ n \in \mathbb{N}\}$ of $P^*(R)$ be denoted by C. From [8], I is known as a C-hyperideal of R if for each $A \in C$, $A \cap I \neq \emptyset$ implies $A \subseteq I$. As well, a proper hyperideal I is known as prime (primary) if for any $x, y \in R$, $x \circ y \subseteq I$ implies $x \in I$ or $y \in I$ ($x \in I$ or $y^n \subseteq I$ for some positive integer n where y^n denotes $y \circ y \circ ... \circ y$, n times). $rad(I)$ denotes the radical of a hyperideal I of R, defined by $rad(I) = \bigcap_{I\subseteq P} P$ where P's are prime hyperideals of R. By Proposition 3.2 in [8], we have $D(I) = \{r \in R | \ r^n \subseteq I \ for \ some \ n \in \mathbb{N}\} \subseteq rad(I)$ and also we get $D(I) = rad(I)$ where I is a C-hyperideal in R. For some hyperideals I and J of R, $(I : J)$ is the set $\{s \in R | s \circ J \subseteq I\}$. Let $(R, +, \circ)$ and $(S, +, \circ)$ be two hyperrings and $f : R \to S$ be a map. Then f is called a homomorphism if it satisfies these properties: $f(a + b) = f(a) + f(b)$ and $f(a \circ b) \subseteq f(a) \circ f(b)$ for all $a, b \in R$. In particular, f is called a good homomorphism if $f(a \circ b) = f(a) \circ f(b)$ for all $a, b \in R$. Furthermore, the kernel of a homomorphism is defined by $ker(f) = f^{-1}(< 0 >) = \{r \in R | f(r) \in < 0 >\}$ and note that $f(r)$ may not be a zero element. Let I be a hyperideal of R. The quotient abelian group $R/I = \{a + I | a \in R\}$ is a hyperring with the multiplication $(a + I) \circ (b + I) = \{r + I | r \in a \circ b\}$. Then R/I is called quotient hyperring. It can be easily proved that all hyperideals of R/I is of the form J/I, where J is a hyperideal of R containing I. The natural homomorphism $\pi : R \to R/I$ is defined by $\pi(r) = r + I$. Note that it is a good epimorphism.

Badawi presented the notions of 2-absorbing ideals and 2-absorbing primary ideals in commutative ring theory and then, he extensively give the basic properties of these concepts in [6, 7]. The notions of 2-absorbing and 2-absorbing primary hyperideals in multiplicative hyperrings have been introduced as the generalizations of prime and primary hyperideal in [3]. The author defined a 2-absorbing (primary) hyperideal I as follows: for all $x, y, z \in R$, $x \circ y \circ z \subseteq I$ implies $x \circ y \subseteq I$ or $y \circ z \subseteq I$ or $x \circ z \subseteq I$ ($x \circ y \subseteq I$ or $y \circ z \subseteq rad(I)$ or $x \circ z \subseteq rad(I)$), respectively.

Let R be a multiplicative hyperring. By $\mathcal{I}(\mathcal{R})$ and $\mathcal{I}*(\mathcal{R})$, we mean all hyperideals of R and all proper hyperideals of R, respectively. A function δ from $\mathcal{I}(\mathcal{R})$ to $\mathcal{I}(\mathcal{R})$ is said to be an expansion function of $\mathcal{I}(\mathcal{R})$ if it satisfies the next two conditions: (i) $I \subseteq \delta(I)$, (ii) If $I \subseteq J$, then $\delta(I) \subseteq \delta(J)$ for any hyperideals I, J of R. In [10], Zhao introduced a new concept which is called δ-primary ideals in commutative rings. This concept is considered to unify prime and primary ideals. The author defined I as a δ-primary ideal if whenever $xy \in I$ for each $x, y \in R$ implies $x \in I$ or $y \in \delta(I)$. Then defining 2-absorbing δ-primary ideals in commutative rings, Zhao brought the theory of algebraic in a new concept which unifies 2-absorbing ideals and 2-absorbing primary ideals [11]. According to the author, a proper ideal I is defined as a 2-absorbing δ-primary provided that $xyz \in I$ for each $x, y, z \in R$ implies $xy \in I$ or $yz \in \delta(I)$ or $xz \in \delta(I)$. Later, Yesilot introduced the concept of δ-primary hyperideal on Krasner hyperrings in [5]. He called I a δ-primary hyperideal of R if $xy \in I$ and $x \notin I$ for some $x, y \in R$ imply $y \in \delta(I)$.

In this paper, first we study δ-primary hyperideal of commutative multiplicative hyperring as an expansion of prime hyperideals and primary hyperideals. In Section 2, we obtain that I is δ-primary hyperideal when $\delta(I)$ is prime. Then we determine that $L \circ K \subseteq I$ for some $I, L, K \in \mathcal{I}(\mathcal{R})$ implies $L \subseteq I$ or $K \subseteq \delta(I)$ if and only if I is a δ-primary hyperideal of R. A hyperring homomorphism is shown to preserve the concept of δ-primary hyperideal under the special conditions. In Section 3, we present the new notion of 2-absorbing

δ-primary hyperideals, which is an expansion of 2-absorbing hyperideals and 2-absorbing primary hyperideals. We give explanatory specific examples and results of this concept in a similar manner of Section 2. It is shown that I is a $\delta \circ \gamma$-primary hyperideal of R if $\gamma(I)$ is a δ-primary hyperideal for expansion functions δ and γ of $\mathcal{I}(\mathcal{R})$. It is defined a strongly 2-absorbing δ-primary hyperideal I of R in such a way that: if $I_1, I_2, I_3 \in \mathcal{I}(\mathcal{R})$ and $I_1 \circ I_2 \circ I_3 \subseteq I$ imply $I_1 \circ I_2 \subseteq I$ or $I_2 \circ I_3 \subseteq \delta(I)$ or $I_1 \circ I_3 \subseteq \delta(I)$. We obtain that $I \in \mathcal{I}(\mathcal{R})$ is 2-absorbing δ-primary if and only if it is strongly 2-absorbing δ-primary. Notice that $R = R_1 \times R_2$ is a multiplicative hyperring where R_i is a multiplicative hyperring with nonzero identity for each $i \in \{1, 2\}$ [4]. Let δ_i be an expansion function of hyperideals of R_i for every $i \in \{1, 2\}$. We define δ_R by $\delta_R(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$ for all hyperideals I_i of R_i for every $i \in \{1, 2\}$. It can be easily seen to be an expansion function of hyperideals of R. Finally, we give a characterization δ_R-primary hyperideals and 2-absorbing δ_R-primary hyperideals of $R_1 \times R_2$.

Throughout this paper, we suppose that every hyperring is commutative multiplicative with nonzero identity.

2. On expansion of prime hyperideals

Definition 2.1 *[5] A function $\delta : \mathcal{I}(\mathcal{R}) \to \mathcal{I}(\mathcal{R})$ is said to be an expansion function of $\mathcal{I}(\mathcal{R})$ if it satisfies the following two conditions: (1) $I \subseteq \delta(I)$, (2) If $I \subseteq J$, then $\delta(I) \subseteq \delta(J)$ for all hyperideals I, J of R.*

In the following examples, we explain the definition of expansion function over multiplicative hyperrings.

Example 2.1

1. *The function δ_0 is an expansion function of $\mathcal{I}(\mathcal{R})$ with $\delta_0(I) = I$ for every hyperideal $I \in \mathcal{I}(\mathcal{R})$.*
2. *The function δ_1 is an expansion function of $\mathcal{I}(\mathcal{R})$ with $\delta_1(I) = D(I)$ for every hyperideal $I \in \mathcal{I}(\mathcal{R})$.*
3. *The function δ_2 is an expansion function of $\mathcal{I}(\mathcal{R})$ with $\delta_2(I) = rad(I)$ for every hyperideal $I \in \mathcal{I}(\mathcal{R})$.*
4. *The function δ_r is an expansion function of hyperideals of R with $\delta_r(I) = R$ for every hyperideal $I \in \mathcal{I}(\mathcal{R})$.*
5. *Let δ_i and δ_j be expansion functions of hyperideals of R. δ is defined by $\delta(I) = \delta_i(I) \cap \delta_j(I)$ for each hyperideal I of R. Notice that δ is an expansion function of $\mathcal{I}(\mathcal{R})$.*
6. *Let $\delta_{\mathcal{I}(\mathcal{R})}$ be defined by $\delta_{\mathcal{I}(\mathcal{R})}(J) = \bigcap\{I \in \mathcal{I}(\mathcal{R}) | J \subseteq I\}$. Then, $\delta_{\mathcal{I}(\mathcal{R})}$ is an expansion function of hyperideals of R.*
7. *The compound function $\delta \circ \gamma$ of two expansion functions δ and γ of $\mathcal{I}(\mathcal{R})$ is an expansion of $\mathcal{I}(\mathcal{R})$ with $\delta \circ \gamma(I) = \delta(\gamma(I))$ for each $I \in \mathcal{I}(\mathcal{R})$.*
8. *Let δ_+ be defined by $\delta_+(I) = I + J$ for every hyperideal I of R where J is a hyperideal of R. It can be easily seen that δ_+ is an expansion function of $\mathcal{I}(\mathcal{R})$.*

Definition 2.2 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$. $I \in \mathcal{I}*(\mathcal{R})$ is called a δ-primary hyperideal if $a, b \in R$ and $a \circ b \subseteq I$ imply either $a \in I$ or $b \in \delta(I)$.*

We give the following examples to better explain the structure of δ-primary hyperideal.

Example 2.2 *Assume that* $(\mathbb{Z},+,\cdot)$ *is the ring of integers. Let* $(\mathbb{Z},+,\circ)$ *be a multiplicative hyperring with a hyperoperation* $x\circ y$. *Consider the expansion function* δ_+ *of* $\mathcal{I}(\mathbb{Z})$ *with* $\delta_+(I)=I+q\mathbb{Z}$ *where* q *is a prime integer. Then* $I=p\mathbb{Z}$ *is a* δ_+*-primary hyperideal of* $\mathcal{I}(\mathbb{Z})$ *where* p *is a prime integer with* $p\neq q$ *since* $\delta_+(p\mathbb{Z})=(p\mathbb{Z})+(q\mathbb{Z})=\mathbb{Z}$.

Example 2.3 *Consider the expansion function* δ_r *of* R *(See Example 2.1(4)). Then every proper hyperideal of* R *is a* δ_r*-primary hyperideal.*

Example 2.4

1. *It is clear that a hyperideal is* δ_0*-primary if and only if it is a prime.*
2. *If a hyperideal of* R *is* δ_1*-primary, then it is primary. The converse holds if* $D(I)=rad(I)$.
3. *Let every hyperideal of* R *be a* C*-hyperideal. Then, a hyperideal of* R *is* δ_2*-primary if and only if it is a primary.*

Proposition 2.1 *Let* δ, γ *be expansion functions of* $\mathcal{I}(\mathcal{R})$ *and* $\delta(I)\subseteq\gamma(I)$ *for each hyperideal* I *of* R. *Every* δ*-primary hyperideal of* R *is a* γ*-primary hyperideal. Thus, we conclude that a prime hyperideal is a* δ*-primary hyperideal for each expansion function* δ *of* $\mathcal{I}(\mathcal{R})$.

Let $(R,+,.)$ be a ring and $A\in P^*(R)(|A|\geq 2)$. There exists a multiplicative hyperring $(R_A,+,\circ)$, where $R_A=R$ and $x\circ y=\{x\cdot a\cdot y|a\in A\}$ for each $x,y\in R_A$.

In Proposition 2.1, we know that a prime hyperideal is a δ-primary hyperideal for each expansion function δ of a multiplicative hyperring. However, the next example shows that the inverse of Proposition 2.1 is not true, in general.

Example 2.5 *Let* $\mathbb{Z}$ *be the ring of integers and* A *be the set of all positive even integers of* $\mathbb{Z}$. *Consider the multiplicative hyperring* $\mathbb{Z}_A$. *Then, the set* $\mathcal{E}$ *of all even integers of* $\mathbb{Z}$ *is a* δ_1*-primary hyperideal of* $\mathbb{Z}_A$, *but it is not a prime hyperideal of* $\mathbb{Z}_A$ *(See [8, Example 3.5]).*

In the following theorem, it is stated the relationship between primary hyperideals and δ-primary hyperideals.

Theorem 2.1 *If* I *is a primary hyperideal of* R *and* $rad(\delta(I))=\delta(I)$, *then* I *is a* δ*-primary hyperideal of* R.

Proof We assume $a,b\in R$ with $a\circ b\subseteq I$. By our assumption, $a\in I$ or $b\in D(I)\subseteq rad(I)\subseteq rad(\delta(I))$, and so $a\in I$ or $b\in\delta(I)$ since $rad(\delta(I))=\delta(I)$. Hence, I is a δ-primary hyperideal of R. □

Lemma 2.1 *Let* $I\in\mathcal{I}*(\mathcal{R})$. *Then* I *of* R *is a* δ*-primary if and only if* $L\circ K\subseteq I$ *for each* $L,K\in\mathcal{I}(\mathcal{R})$ *implies* $L\subseteq I$ *or* $K\subseteq\delta(I)$.

Proof $(\Rightarrow)$: Suppose that $L\circ K\subseteq I$, $L\nsubseteq I$ and $K\nsubseteq\delta(I)$ for some $L,K\in\mathcal{I}(\mathcal{R})$. We have $a,b\in R$ satisfied $a\in L-I$ and $b\in K-\delta(I)$. Then $a\circ b\subseteq L\circ K\subseteq I$, yielding a contradiction.
$(\Leftarrow)$: We assume $a,b\in R$ with $a\circ b\subseteq I$. By [8, Proposition 2.15], it is obtained that $<a>\circ<b>\subseteq<a\circ b>\subseteq I$. Consequently, $<a>\subseteq I$ or $<b>\subseteq\delta(I)$ by our assumption. □

Theorem 2.2 *Let I be a δ-primary hyperideal of R.*

1. *$(I:K)=I$ for each hyperideal K of $\mathcal{I}(\mathcal{R})$ with $K\nsubseteq\delta(I)$.*
2. *$(I:H)$ is a δ-primary hyperideal of R for each subset H of R.*

Proof

1. Clearly, $r\circ K\subseteq I$ for every $r\in I$. Thus, $I\subseteq(I:K)$. Conversely, consider $(I:K)\circ K$. Then $(I:K)\circ K$ $=\bigcup_{r\in(I:K),x\in K}(r\circ x)\subseteq I$. We obtain $(I:K)\subseteq I$ as I is a δ-primary hyperideal and $K\nsubseteq\delta(I)$.
2. Let $a\circ b\subseteq(I:H)$ and $a\notin(I:H)$ for some elements $a,b\in R$. It is clear that $a\circ b\circ H\subseteq I$. Take an element $h\in H$ with $a\circ h\nsubseteq I$. Thus, $a\circ b\circ h=a\circ h\circ b\subseteq I$ and $a\circ h\nsubseteq I$, that is, we get $<a\circ h>\circ<b>\subseteq I$ and $<a\circ h>\nsubseteq I$. Hence, we obtain $<b>\subseteq\delta(I)\subseteq\delta(I:H)$. Consequently, $b\in\delta(I:H)$.

□

Theorem 2.3 *If I is a δ-primary C-hyperideal of R with $rad(\delta(I))\subseteq\delta(rad(I))$, then $rad(I)$ is a δ-primary C-hyperideal of R.*

Proof Take $a,b\in R$ with $a\circ b\subseteq rad(I)$ and $a\notin rad(I)$. Then $a^n\circ b^n\subseteq I$ for some positive integer n and $a^k\nsubseteq I$ for each positive integer k. By our assumption and $a^{kn}\circ b^{kn}\subseteq I$, we obtain $b^{kn}\subseteq\delta(I)$. It means $b\in rad(\delta(I))\subseteq\delta(rad(I))$, we are done. □

Definition 2.3 *If δ holds $\delta(I\cap J)=\delta(I)\cap\delta(J)$ for each $I,J\in\mathcal{I}(\mathcal{R})$, it has the property of intersection preserving.*

We denote the property of intersection preserving with *. Notice that the expansion functions δ_1 and δ_2 of hyperideals of a multiplicative hyperring are examples which hold the property of intersection preserving.

Theorem 2.4 *Let δ has the property *. If I_i is a δ-primary hyperideals of R and $\delta(I_i)=P$ for all $i\in\{1,2,...,n\}$. Then $I=\bigcap_{i=1}^n I_i$ is so.*

Proof Let $x\circ y\subseteq I$ and $x\notin I$ for some $x,y\in R$. Then $x\notin I_j$ for some $j\in\{1,2,...,n\}$. Thus, $y\in\delta(I_j)=P$ and $\delta(I)=\delta(\bigcap_{i=1}^n I_i)=\delta(I_1)\cap\cdots\delta(I_n)=P$. By the assumption, we get $y\in\delta(I)$. □

Definition 2.4 *Let $f:R\to S$ be a good hyperring homomorphism, expansion functions δ and γ of $\mathcal{I}(\mathcal{R})$ and $\mathcal{I}(\mathcal{S})$, respectively. Then f is called a $\delta\gamma$-homomorphism if $\delta(f^{-1}(J))=f^{-1}(\gamma(J))$ for each hyperideals J of S.*

Consider the expansion functions γ_1 of $\mathcal{I}(\mathcal{S})$ and δ_1 of $\mathcal{I}(\mathcal{R})$ defined in a similar manner of Example 2.1 (2). It is seen that each homomorphism from R to S is an example of $\delta_1\gamma_1$-homomorphism. If every hyperideal of R is a C-hyperideal, any homomorphism from R to S is a $\delta_2\gamma_2$-homomorphism where the radical operations γ_2 of $\mathcal{I}(\mathcal{S})$ and δ_2 of $\mathcal{I}(\mathcal{R})$ (See Example 2.1 (3)). Also, note that $\gamma(f(I))=f(\delta(I))$ where f is a $\delta\gamma$-epimorphism and $I\in\mathcal{I}(\mathcal{R})$ with $\ker(f)\subseteq I$.

Theorem 2.5 *Let $f : R \to S$ be $\delta\gamma$-homomorphism. Then:*

1. *Let J be a γ-primary hyperideal of S. $f^{-1}(J)$ is a δ-primary hyperideal of R.*
2. *Let f be epimorphism and $I \in \mathcal{I}(\mathcal{R})$ with $\ker(f) \subseteq I$. $f(I)$ is γ-primary if and only if I is a δ-primary hyperideal.*

Proof

1. It is well known that $f^{-1}(J)$ is a proper hyperideal of R. Let $a \circ b \subseteq f^{-1}(J)$ for each $a, b \in R$. We get $f(a \circ b) = f(a) \circ f(b) \subseteq J$. Since J is γ-primary, we obtain that $f(a) \in J$ or $f(b) \in \gamma(J)$. Hence, $a \in f^{-1}(J)$ or $b \in f^{-1}(\gamma(J))$, that is, $a \in f^{-1}(J)$ or $b \in \delta(f^{-1}(J))$.
2. Clearly, $f(I)$ is a proper hyperideal of S. Let $a \circ b \subseteq f(I)$ for each $a, b \in S$. As f is an epimorphism, we take $a', b' \in R$ with $f(a') = a$, $f(b') = b$ and so we obtain $f(a') \circ f(b') = f(a' \circ b') \subseteq f(I)$. Let $k \in a' \circ b'$. There is a $y \in a' \circ b'$ such that $f(y) = x$ for every $x \in f(a' \circ b')$. Then $f(k) = x$ for any $x \in f(a' \circ b')$. Additionally, there is a $y' \in I$ such that $f(y') = x$ for every $x \in f(a' \circ b')$ since $f(a' \circ b') \subseteq f(I)$. Thus, $f(k - y') = f(k) - f(y') = 0 \in < 0 >$ since $f(k) = f(y')$. Since f is an epimorphism, then $k - y' \in f^{-1} < 0 >= ker(f) \subseteq I$. Thus, we conclude that $k \in I$, that is, $a' \circ b' \subseteq I$. In that case, $a' \circ b' \subseteq I$. Therefore, $a' \in I$ or $b' \in \delta(I)$ and we obtain $f(a') \in f(I)$ or $f(b') \in f(\delta(I))$. We obtain $a \in f(I)$ or $b \in f(\delta(I)) = \gamma(f(I))$ by our assumption. Consequently, $f(I)$ is γ-primary. The converse part is quite clear from (1).

□

Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and $I \in \mathcal{I}(\mathcal{R})$. Let the function $\delta_q : R/I \to R/I$ be defined by $\delta_q(K/I) = \delta(K)/I$ for all hyperideals $K(\supseteq I)$ of R. Note that δ_q is an expansion function of $\mathcal{I}(\mathcal{R}/\mathcal{I})$.

Corollary 2.1 *Let I and K be hyperideals of R hold $I \subseteq K$. Then K is a δ-primary hyperideal if and only if K/I is a δ_q-primary hyperideal of the quotient hyperring R/I.*

Proof The claim is verified from Theorem 2.5. □

Definition 2.5 *We let $(R, +, \circ)$ is a multiplicative hyperring.*

1. *An element $r \in R$ is defined as zero divisor if there is an element $0 \neq r' \in R$ such that $\{0\} = r \circ r'$.*
2. *An element $r \in R$ is a δ-nilpotent if $r \in \delta(0)$.*

Theorem 2.6 *A hyperideal I of R is a δ-primary if and only if every zero divisor of R/I is a δ_q-nilpotent.*

Proof $(\Rightarrow)$: Assume that $I \in \mathcal{I}(\mathcal{R})$ is δ-primary. We denote $r + I$ with $\bar{r}$. Take a zero divisor element $\bar{r}$ of R/I. Then there is an element $I \neq \bar{r'} = r' + I$ with $I = (r + I) \circ (r' + I)$. As the result of $I = r \circ r' + I$, we have $r \circ r' \subseteq I$. Thus, $r \in \delta(I)$ as $r \circ r' \subseteq I$ and $r' \notin I$. Consider the expansion function δ_q of $\mathcal{I}(\mathcal{R})$ and the natural homomorphism $\pi : R \to R/I$. We obtain that π is a $\delta\delta_q$-epimorphism. Thus, we have $\delta(I) = \delta(\pi^{-1}(0_{R/I})) = \pi^{-1}(\delta_q(I))$. Note that $\bar{r} = r + I \in \delta(I)/I = \pi(\delta(I)) = \delta_q(0_{R/I})$. Hence, $\bar{r} \in \delta_q(0_{R/I})$.
$(\Leftarrow)$: Let every zero divisor of R/I be a δ_q-nilpotent. Let $r \circ r' \subseteq I$ and $r \notin I$ for each $r, r' \in R$. Then $r' + I$

is a zero divisor element in R/I as $r \circ r' + I = (r + I) \circ (r' + I) = I$ and $r + I \neq I$. By assumption, we get $r' + I \in \delta_q(0_{R/I}) = \delta(I)/I$. Consequently, $r' \in \delta(I)$. □

Theorem 2.7 *Let $I_1, ..., I_n \in \mathcal{I} * (\mathcal{R})$ and I a δ-primary hyperideal of R with $\bigcap_{i=1}^{n} I_i \subseteq I$. Then, $I_i \subseteq \delta(I)$ for some $i \in \{1, ..., n\}$. If $\bigcap_{i=1}^{n} I_i = I$ and $\delta(\delta(J)) = \delta(J)$ for each $J \in \mathcal{I}(\mathcal{R})$, then $\delta(I_i) = \delta(I)$ for some $i \in \{1, ..., n\}$.*

Proof We suppose $I_i \nsubseteq \delta(I)$ for every $i \in \{1, ..., n\}$. Then there exist elements $x_1, ..., x_n$ of R with $x_i \in I_i - \delta(I)$. We get $x_1 \circ ... \circ x_n \subseteq I_i$ for every i and so $x_1 \circ ... \circ x_n \subseteq \bigcap_{i=1}^{n} I_i$. Since I is δ-primary and $x_1, ..., x_n \notin \delta(I)$, then $x_i \in I \subseteq \delta(I)$ for each $i \in \{1, ..., n\}$, contradiction. Let $\bigcap_{i=1}^{n} I_i = I$. Then $\delta(I_i) = \delta(I)$ since $I \subseteq I_i$ and $\delta(I) \subseteq \delta(I_i)$. □

3. On expansion of 2-absorbing hyperideal

Definition 3.1 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and $I \in \mathcal{I} * (\mathcal{R})$. I refers to a 2-absorbing δ-primary hyperideal if $a, b, c \in R$ and $a \circ b \circ c \subseteq I$ imply $a \circ b \subseteq I$ or $b \circ c \subseteq \delta(I)$ or $a \circ c \subseteq \delta(I)$.*

We start with the following examples to explaining this structure.

Example 3.1 *Consider the ring of integers $(\mathbb{Z}, +, \cdot)$. Then we have $(\mathbb{Z}, +, \circ)$ is a multiplicative hyperring with the hyperoperation $x \circ y = \{x \cdot y\}$. Then the hyperideal $6\mathbb{Z} = \{6k | k \in \mathbb{Z}\}$ is clearly a 2-absorbing δ_2-primary hyperideal.*

Example 3.2 *Let $(\mathbb{Z}, +, \circ)$ be defined as in Example 2.2. Consider the expansion function δ_+ of $\mathcal{I}(\mathbb{Z})$ with $\delta_+(I) = I + (q)$ where q is a prime integer. Then $I = (p)$ is a 2-absorbing δ_+-primary hyperideal of $\mathcal{I}(\mathbb{Z})$ where p is a prime integer with $p \neq q$ since $\delta_+(p) = (p) + (q) = \mathbb{Z}$.*

Example 3.3 *Consider the expansion function δ_r of R (See Example 2.1(4)). Then every proper hyperideal of R is a 2-absorbing δ_r-primary hyperideal.*

Remark 3.1

1. *Every δ-primary element of $\mathcal{I}(\mathcal{R})$ is a 2-absorbing δ-primary hyperideal.*
2. *I is a 2-absorbing δ_0-primary hyperideal if and only if I is a 2-absorbing hyperideal.*
3. *I is a 2-absorbing δ_2-primary hyperideal if and only if I is a 2-absorbing primary hyperideal.*

The converse of Remark 3.1 (1) may not be always true as it is shown in the following example.

Example 3.4 *Consider the multiplicative hyperring $(\mathbb{Z}, +, \circ)$ in Example 3.1. Then the hyperideal $6\mathbb{Z}$ is clearly a 2-absorbing δ_2-primary hyperideal but it is not δ_2-primary since $2 \circ 3 \subseteq 6\mathbb{Z}$, $2, 3 \notin 6\mathbb{Z}$, and $2, 3 \notin \delta_2(6\mathbb{Z})$.*

Theorem 3.1 *The following hold:*

1. *Let γ be expansion function of $\mathcal{I}(\mathcal{R})$ satisfied $\delta(I) \subseteq \gamma(I)$ for each $I \in \mathcal{I}(\mathcal{R})$. Then every 2-absorbing δ-primary hyperideal of R is 2-absorbing γ-primary. Additionaly, every 2-absorbing hyperideal is a 2-absorbing δ-primary since $I \subseteq \delta(I)$ for each expansion function δ of $\mathcal{I}(\mathcal{R})$.*

2. *Let $I \in \mathcal{I}(\mathcal{R})$ be 2-absorbing primary and $\delta(I)$ be a radical hyperideal (i.e $rad(\delta(I)) = \delta(I)$). Then, I is 2-absorbing δ-primary.*

Proof

1. The claim is clear by the assumption.

2. We suppose $a, b, c \in R$ and $a \circ b \circ c \subseteq I$. It means $a \circ b \subseteq I$ or $b \circ c \subseteq rad(I)$ or $a \circ c \subseteq rad(I)$ by our assumption. We have $rad(I) \subseteq rad(\delta(I))$ since $I \subseteq \delta(I)$. Thus, we obtain $a \circ b \subseteq I$ or $b \circ c \subseteq rad(\delta(I))$ or $a \circ c \subseteq rad(\delta(I))$. Consequently, $a \circ b \subseteq I$ or $b \circ c \subseteq \delta(I)$ or $a \circ c \subseteq \delta(I)$.

□

The following example shows that the converse part of Theorem 3.1(1) may not be true, in general.

Example 3.5 *Consider the hyperring $\mathbb{Z}_A$ where $A = \{3, 4\}$. It is easily seen that the principal hyperideal $< 8 >$ of $\mathbb{Z}_{\mathbb{A}}$ is 2-absorbing δ_2-primary hyperideal. However, it is not a 2-absorbing hyperideal as the fact that $2 \circ 2 \circ 2 \subseteq < 8 >$ but $2 \circ 2 = \{12, 16\} \nsubseteq < 8 >$.*

Theorem 3.2 *Let $\delta(I)$ be a prime hyperideal of R. Then I is a 2-absorbing δ-primary of R.*

Proof Take $a, b, c \in R$ with $a \circ b \circ c \subseteq I$ and $a \circ b \nsubseteq I$. Let us consider two situations. Firstly, let $a \circ b \nsubseteq \delta(I)$. Then we obtain $c \in \delta(I)$ by our assumption. Thus, we get $a \circ c \subseteq \delta(I)$ and $b \circ c \subseteq \delta(I)$. Secondary, take $a \circ b \subseteq \delta(I)$. By our assumption, we get $a \in \delta(I)$ or $b \in \delta(I)$. Hence, $a \circ c \subseteq \delta(I)$ or $b \circ c \subseteq \delta(I)$. □

The next example is given to explain that the converse of Theorem 3.2 may not be always true.

Example 3.6 *Note that the hyperring in Example 3.4. There, we show that $6\mathbb{Z}$ is 2-absorbing δ_2-primary hyperideal of $(\mathbb{Z}, +, \circ)$. Consider $\delta_2(6\mathbb{Z})$. We have that it is not a prime since $2 \circ 3 \subseteq \delta_2(6\mathbb{Z})$ but $2, 3 \notin \delta_2(6\mathbb{Z})$.*

Theorem 3.3 *Let I be a 2-absorbing δ-primary C-hyperideal of R with $rad(\delta(I)) \subseteq \delta(rad(I))$. Then $rad(I)$ is a 2-absorbing δ-primary C-hyperideal of R.*

Proof It can be proved in a similar manner to Theorem 2.3. □

Theorem 3.4 *Let I, K, and L be proper hyperideals of R with $L \subseteq K \subseteq I$. If I is a δ-primary hyperideal where $\delta(I) = \delta(L)$, then K is a 2-absorbing δ-primary hyperideal.*

Proof Suppose $a, b, c \in R$ with $a \circ b \circ c \subseteq K$ and $a \circ b \nsubseteq K$. We get two cases as $K \subseteq I$. The first case: Let $a \circ b \nsubseteq I$. Then $c \in \delta(I) = \delta(L) \subseteq \delta(K)$ with our assumption. Thus, we get $a \circ c \subseteq \delta(K)$ and $b \circ c \subseteq \delta(K)$. The second case: Let $a \circ b \subseteq I$. It means $a \in I \subseteq \delta(K)$ or $b \in \delta(I) = \delta(L) \subseteq \delta(K)$ by assumption. Consequently, we get $a \circ c \subseteq \delta(K)$ or $b \circ c \subseteq \delta(K)$. In the both cases, we obtain that K is 2-absorbing δ-primary. □

Corollary 3.1 *Let a hyperideal I of R be δ-primary and $K \in \mathcal{I}(\mathcal{R})$ with $K \subseteq I$ and $\delta(I) = \delta(K)$. Then K is 2-absorbing δ-primary.*

Proof The claim is verified by Theorem 3.4. □

Theorem 3.5 *Let δ and η be two expansion functions of $\mathcal{I}(\mathcal{R})$ and $I \in \mathcal{I}(\mathcal{R})$. If $\eta(I)$ is a prime hyperideal, I is a 2-absorbing $\delta \circ \eta$-primary.*

Proof Take $a, b, c \in R$ with $a \circ b \circ c \subseteq I$ and $a \circ b \nsubseteq I$. Thus, it means $a \circ b \nsubseteq \eta(I)$ or $a \circ b \subseteq \eta(I)$. Let $a \circ b \nsubseteq \eta(I)$. Then $c \in \eta(I) \subseteq \delta(\eta(I))$ by Definition **??**. Thus, we acquire $a \circ c \subseteq \delta \circ \eta(I)$ and $b \circ c \subseteq \delta \circ \eta(I)$. Let $a \circ b \subseteq \eta(I)$. By our assumption, we get $a \in \eta(I)$ or $b \in \eta(I)$. Then $a \circ c \subseteq \eta(I) \subseteq \delta(\eta(I))$ or $b \circ c \subseteq \delta(\eta(I))$. □

Theorem 3.6 *Let δ be an expansion function of $\mathcal{I}(\mathcal{R})$ and I, J δ-primary hyperideals of R with $\delta(I \cap J) = \delta(I) \cap \delta(J)$. Then $I \cap J$ is a 2-absorbing δ-primary hyperideal of R.*

Proof Let $a \circ b \circ c \subseteq I \cap J$ and $a \circ b \nsubseteq I \cap J$ where $a, b, c \in R$. It is deduced from either $a \circ b \nsubseteq I$ or $a \circ b \nsubseteq J$. Thus, we consider the following cases.

Case 1: Let $a \circ b \subseteq I$ and $a \circ b \nsubseteq J$. Since $a \circ b \nsubseteq J$, there exists an element $r \in a \circ b$ such that $r \notin J$. Since $r \circ c \subseteq J$ and $r \notin J$, then $c \in \delta(J)$. Also, $a \in I \subseteq \delta(I)$ or $b \in \delta(I)$ as $a \circ b \subseteq I$. Hence, we get $a \circ c \subseteq \delta(I)$ or $b \circ c \subseteq \delta(I)$. Then we obtain $a \circ c \subseteq \delta(I) \cap \delta(J) = \delta(I \cap J)$ or $b \circ c \subseteq \delta(I) \cap \delta(J) = \delta(I \cap J)$.

Case 2: If we assume $a \circ b \nsubseteq I$ and $a \circ b \subseteq J$, in that case we get $a \circ c \subseteq \delta(I)$ or $b \circ c \subseteq \delta(I)$ by a similar way to the proof of Case 1.

Case 3: Let $a \circ b \nsubseteq I$ and $a \circ b \nsubseteq J$. We have elements $r, s \in a \circ b$ with $r \notin I$ and $s \notin J$. Thus, we have $r \circ c \subseteq I$ and $s \circ c \subseteq J$. Hence, $c \in \delta(I)$ and $c \in \delta(J)$ by our assumption. Consequently, $a \circ c \subseteq \delta(I)$ and $b \circ c \subseteq \delta(I)$. □

Theorem 3.7 *Let δ have the property * and $K = I \cap J$ for some δ-primary hyperideals I and J of R. K is a 2-absorbing δ-primary hyperideal.*

Proof It is clear by previous Theorem. □

Proposition 3.1 *Let $(R, +, \circ)$ be a multiplicative hyperring and $I, J, K \in \mathcal{I}*(\mathcal{R})$. If $I \subseteq J \cup K$, then $I \subseteq J$ or $I \subseteq K$.*

Proof Let $I \subseteq J \cup K$, $I \nsubseteq J$ and $I \nsubseteq K$. There are $a, b \in R$ so that $a \in I - J$ and $b \in I - K$. Then $a - b \in I$. Thus, $a \in J$ or $b \in K$ since $I \subseteq J \cup K$, yielding a contradiction.

□

Theorem 3.8 *Let $I \in \mathcal{I}*(\mathcal{R})$. Then we have the following equivalent statements:*

1. *I is a 2-absorbing δ-primary hyperideal.*
2. *$(I : a \circ b) \subseteq (\delta(I) : a) \bigcup (I : b)$ if $a \circ b \nsubseteq \delta(I)$ for some $a, b \in R$.*
3. *$(I : a \circ b) \subseteq (\delta(I) : a)$ or $(I : a \circ b) = (I : b)$ if $a \circ b \nsubseteq \delta(I)$ for some $a, b \in R$.*

Proof (1) $\Rightarrow$ (2): We suppose $a, b \in R$ with $a \circ b \nsubseteq \delta(I)$ and take $x \in (I : a \circ b)$. Then, we have $a \circ b \circ x \subseteq I$. Thus, $b \circ x \subseteq I$ or $a \circ x \subseteq \delta(I)$ by our assumption. Consequently, $x \in (I : b)$ or $x \in (\delta(I) : a)$, that is, $x \in (\delta(I) : a) \bigcup (I : b)$.

(2) $\Leftarrow$ (1): We assume $a \circ b \circ x \subseteq I$, $a \circ b \nsubseteq \delta(I)$ and $b \circ x \nsubseteq I$ for each $a, b, x \in R$. Then we have $x \in (I : a \circ b)$. By our assumption, $x \in (\delta(I) : a) \bigcup (I : b)$. Since $b \circ x \nsubseteq I$, then we obtain $a \circ x \subseteq \delta(I)$.
(2) $\Leftrightarrow$ (3): It is clear from Proposition 3.1 and $(I : b) \subseteq (I : a \circ b)$. □

Theorem 3.9 *Let $f : R \to S$ be $\delta\gamma$-homomorphism. Then:*

1. *Let $J \in \mathcal{I}(\mathcal{S})$ be a 2-absorbing γ-primary. $f^{-1}(J)$ is a 2-absorbing δ-primary hyperideal of R.*

2. *Let f be an epimorphism and $I \in \mathcal{I}(\mathcal{R})$ with $\ker(f) \subseteq I$. $f(I)$ is a 2-absorbing γ-primary in $\mathcal{I}(\mathcal{S})$ if and only if I is 2-absorbing δ-primary in $\mathcal{I}(\mathcal{R})$.*

Proof

1. It can be easily seen that $f^{-1}(J)$ is a proper hyperideal. We assume $a, b, c \in R$ with $a \circ b \circ c \subseteq f^{-1}(J)$. Clearly, $f(a \circ b \circ c) = f(a) \circ f(b) \circ f(c) \subseteq J$. As J is 2-absorbing γ-primary hyperideal, we obtain $f(a) \circ f(b) = f(a \circ b) \subseteq J$ or $f(b) \circ f(c) = f(b \circ c) \subseteq \gamma(J)$ or $f(a) \circ f(c) = f(a \circ c) \subseteq \gamma(J)$. In that case, $a \circ b \subseteq f^{-1}(J)$ or $b \circ c \subseteq f^{-1}(\gamma(J))$ or $a \circ c \subseteq f^{-1}(J)$, that is, $a \circ b \subseteq f^{-1}(J)$ or $b \circ c \subseteq \delta(f^{-1}(J))$ or $a \circ c \subseteq \delta(f^{-1}(J))$

2. Note that $f(I)$ is a proper hyperideal. Suppose $a, b, c \in S$ with $a \circ b \circ c \subseteq f(I)$. By our assumption, we have $a', b', c' \in R$ with $f(a') = a$, $f(b') = b$ and $f(c') = c$. Thus, it has been obtained $f(a') \circ f(b') \circ f(c') = f(a' \circ b' \circ c') \subseteq f(I)$. Let $k \in a' \circ b' \circ c'$. There is a $y \in a' \circ b' \circ c'$ such that $f(y) = x$ for every $x \in f(a' \circ b' \circ c')$. Then $f(k) = x$ for any $x \in f(a' \circ b' \circ c')$. Moreover, there is a $y' \in I$ such that $f(y') = x$ for every $x \in f(a' \circ b' \circ c')$ since $f(a' \circ b' \circ c') \subseteq f(I)$ and so we get $f(k - y') = f(k) - f(y') = 0 \in < 0 >$ since $f(k) = f(y')$. Since f is epimorphism, then $k - y' \in f^{-1} < 0 >= ker(f) \subseteq I$ and so $k \in I$, that is, $a' \circ b' \circ c' \subseteq I$. Therefore, this indicates $a' \circ b' \subseteq I$ or $b' \circ c' \subseteq \delta(I)$ or $a' \circ c' \subseteq \delta(I)$, that is, $f(a' \circ b') = f(a') \circ f(b') \subseteq f(I)$ or $f(b' \circ c') = f(b') \circ f(c') \subseteq f(\delta(I))$ or $f(a' \circ c') = f(a') \circ f(c') \subseteq f(\delta(I))$. Clearly, $a \circ b \subseteq f(I)$ or $b \circ c \subseteq f(\delta(I)) = \gamma(f(I))$ or $a \circ c \subseteq f(\delta(I)) = \gamma(f(I))$. Consequently, $f(I)$ is a 2-absorbing γ-primary. The converse part is verified from (1).

□

Corollary 3.2 *Let $I, K \in \mathcal{I} * (\mathcal{R})$ with $I \subseteq K$. K is 2-absorbing δ-primary if and only if K/I is a 2-absorbing δ_q-primary hyperideal of the quotient hyperring R/I.*

Proof The claim is verified by Theorem 3.9. □

Definition 3.2 *Given expansion function δ of $\mathcal{I}(\mathcal{R})$, $I \in \mathcal{I} * (\mathcal{R})$ is called a strongly 2-absorbing δ-primary hyperideal if $I_1 \circ I_2 \circ I_3 \subseteq I$ for some hyperideals I_1, I_2, I_3 of R implies $I_1 \circ I_2 \subseteq I$ or $I_2 \circ I_3 \subseteq \delta(I)$ or $I_1 \circ I_3 \subseteq \delta(I)$.*

Lemma 3.1 *Let a hyperideal I of R be a 2-absorbing δ-primary. Then for each $a, b \in R$ and $J \in \mathcal{I}(\mathcal{R})$, $a \circ b \circ J \subseteq I$ and $a \circ b \nsubseteq I$ imply $a \circ J \subseteq \delta(I)$ or $b \circ J \subseteq \delta(I)$.*

Proof We suppose $a, b \in R$ and $J \in \mathcal{I}(\mathcal{R})$ with $a \circ b \circ J \subseteq I$ and $a \circ b \nsubseteq I$. Assume $a \circ J = \bigcup_{j_i \in J} a \circ j_i \nsubseteq \delta(I)$ and $b \circ J = \bigcup_{j_i \in J} b \circ j_i \nsubseteq \delta(I)$. Then there are $j_1, j_2 \in J$ such that $a \circ j_1 \nsubseteq \delta(I)$ and $b \circ j_2 \nsubseteq \delta(I)$. Since $a \circ b \circ j_1 \subseteq I$, $a \circ b \nsubseteq I$ and $a \circ j_1 \nsubseteq \delta(I)$, then $b \circ j_1 \subseteq \delta(I)$. In a similar way, we get $a \circ j_2 \subseteq \delta(I)$ since $a \circ b \circ j_2 \subseteq I$, $a \circ b \nsubseteq I$ and $b \circ j_2 \nsubseteq \delta(I)$. We have $a \circ b \circ (j_1 + j_2) \subseteq I$ as $a \circ b \circ j_1 + a \circ b \circ j_2 \subseteq I$. Then $a \circ b \circ (j_1 + j_2) \subseteq I$ and $a \circ b \nsubseteq I$ imply $a \circ (j_1 + j_2) \subseteq \delta(I)$ or $b \circ (j_1 + j_2) \subseteq \delta(I)$. If $a \circ (j_1 + j_2) \subseteq \delta(I)$, then $a \circ j_1 = a \circ (j_1 + j_2 - j_2) \subseteq a \circ (j_1 + j_2) - a \circ j_2 \subseteq \delta(I)$ since $a \circ j_2 \subseteq \delta(I)$. In a similar manner, if $b \circ (j_1 + j_2) \subseteq \delta(I)$, then $b \circ j_2 = b \circ (j_1 + j_2 - j_1) \subseteq b \circ (j_1 + j_2) - (b \circ j_1) \subseteq \delta(I)$ as $b \circ j_1 \subseteq \delta(I)$. Thus, we obtain $a \circ j_1 \subseteq \delta(I)$ or $b \circ j_2 \subseteq \delta(I)$, yielding a contradiction. Consequently, we conclude $a \circ J \subseteq \delta(I)$ or $b \circ J \subseteq \delta(I)$. □

Theorem 3.10 *A hyperideal I of R is 2-absorbing δ-primary if and only if it is a strongly 2-absorbing δ-primary hyperideal of R.*

Proof $(\Leftarrow)$: It is trivial by the definition.
$(\Rightarrow)$: We suppose $I_1 \circ I_2 \circ I_3 \subseteq I$ and $I_1 \circ I_2 \nsubseteq I$ for every hyperideals I_1, I_2, I_3 of R. We show that $I_2 \circ I_3 \subseteq \delta(I)$ or $I_1 \circ I_3 \subseteq \delta(I)$. For this case, we assume $I_2 \circ I_3 \nsubseteq \delta(I)$ and $I_1 \circ I_3 \nsubseteq \delta(I)$. Then there are $q_1 \in I_1, q_2 \in I_2$ with $q_1 \circ I_3 \nsubseteq \delta(I)$ and $q_2 \circ I_3 \nsubseteq \delta(I)$. Since $I_1 \circ I_2 \nsubseteq I$, then there are $a \in I_1, b \in I_2$ with $a \circ b \nsubseteq I$. Then we deduce $a \circ I_3 \subseteq \delta(I)$ or $b \circ I_3 \subseteq \delta(I)$ as $a \circ b \circ I_3 \subseteq I$ and $a \circ b \nsubseteq I$ by Lemma 2.1. We get the following cases:

Case 1: Assume that $a \circ I_3 \subseteq \delta(I)$ and $b \circ I_3 \nsubseteq \delta(I)$. As $q_1 \circ b \circ I_3 \subseteq I$, $b \circ I_3 \nsubseteq \delta(I)$ and $q_1 \circ I_3 \nsubseteq \delta(I)$, then $q_1 \circ b \subseteq I$ by Lemma 2.1. As $a \circ I_3 \subseteq \delta(I)$ and $q_1 \circ I_3 \nsubseteq \delta(I)$, it means that $(a + q_1) \circ I_3 \nsubseteq \delta(I)$. Indeed, if $(a + q_1) \circ I_3 \subseteq \delta(I)$, then we get $(a + q_1) \circ x \subseteq \delta(I)$ for every $x \in I_3$ and so it is obtained $q_1 \circ x = (a + q_1 - a) \circ x \subseteq (a + q_1) \circ x - a \circ x \subseteq \delta(I)$, a contradiction. By Lemma 2.1, we obtain $(a + q_1) \circ b \subseteq I$ as $(a + q_1) \circ b \circ I_3 \subseteq I$, $(a + q_1) \circ I_3 \nsubseteq \delta(I)$ and $b \circ I_3 \nsubseteq \delta(I)$. Then $a \circ b = (a + q_1 - q_1) \circ b \subseteq (a + q_1) \circ b - (q_1 \circ b) \subseteq I$, that is, we get $a \circ b \subseteq I$, yielding a contradiction.

Case 2: Let $a \circ I_3 \nsubseteq \delta(I)$ and $b \circ I_3 \subseteq \delta(I)$. As $a \circ q_2 \circ I_3 \subseteq \delta(I)$, $a \circ I_3 \nsubseteq \delta(I)$ and $q_2 \circ I_3 \nsubseteq \delta(I)$, then $a \circ q_2 \subseteq I$ by Lemma 2.1. As $b \circ I_3 \subseteq \delta(I)$ and $q_2 \circ I_3 \nsubseteq \delta(I)$, we get $(b + q_2) \circ I_3 \nsubseteq \delta(I)$. Indeed if $(b + q_2) \circ I_3 \subseteq \delta(I)$, then we get $(b + q_2) \circ x \subseteq \delta(I)$ for every $x \in I_3$ and then we conclude $q_2 \circ x = (b + q_2 - a) \circ x \subseteq (b + q_2) \circ x - b \circ x \subseteq \delta(I)$, a contradiction. By Lemma 2.1, we obtain $a \circ (b + q_2) \subseteq I$ as $a \circ (b + q_2) \circ I_3 \subseteq I$, $(b + q_2) \circ I_3 \nsubseteq \delta(I)$ and $a \circ I_3 \nsubseteq \delta(I)$. Then $a \circ b = (b + q_2 - q_2) \circ a \subseteq (b + q_2) \circ a - (q_2 \circ a) \subseteq I$, that is, we get $a \circ b \subseteq I$, yielding a contradiction.

Case 3: Let $a \circ I_3 \subseteq \delta(I)$ and $b \circ I_3 \subseteq \delta(I)$. Since $b \circ I_3 \subseteq \delta(I)$ and $q_2 \circ I_3 \nsubseteq \delta(I)$, it is obtained $(b + q_2) \circ I_3 \nsubseteq \delta(I)$. If $(b + q_2) \circ I_3 \subseteq \delta(I)$, then we get $(b + q_2) \circ x \subseteq \delta(I)$ for every $x \in I_3$. Then $q_2 \circ x = (b + q_2 - b) \circ x \subseteq (b + q_2) \circ x - b \circ x \subseteq \delta(I)$, a contradiction. By Lemma 2.1, we obtain $q_1 \circ (b + q_2) \subseteq I$ as $q_1 \circ (b + q_2) \circ I_3 \subseteq I$, $(b + q_2) \circ I_3 \nsubseteq \delta(I)$ and $q_1 \circ I_3 \nsubseteq \delta(I)$. By Lemma 2.1, we have $q_1 \circ q_2 \subseteq I$ since $q_1 \circ q_2 \circ I_3 \subseteq I$, $q_1 \circ I_3 \nsubseteq \delta(I)$ and $q_2 \circ I_3 \nsubseteq \delta(I)$. Also, it is obtained $(a + q_1) \circ I_3 \nsubseteq \delta(I)$ as $a \circ I_3 \subseteq \delta(I)$ and $q_1 \circ I_3 \nsubseteq \delta(I)$ by a similar way to the explain in above. By Lemma 2.1, we obtain $(a + q_1) \circ q_2 \subseteq I$ as $(a + q_1) \circ q_2 \circ I_3 \subseteq I$, $(a + q_1) \circ I_3 \nsubseteq \delta(I)$ and $q_2 \circ I_3 \nsubseteq \delta(I)$. Then it is clear that $(a + q_1) \circ (b + q_2) \subseteq I$ since $(a + q_1) \circ (b + q_2) \circ I_3 \subseteq I$, $(a + q_1) \circ I_3 \nsubseteq \delta(I)$ and $(b + q_2) \circ I_3 \nsubseteq \delta(I)$. Thus, $a \circ b = (a + q_1 - q_1) \circ (b + q_2 - q_2) \subseteq (a + q_1) \circ (b + q_2) - (a + q_1) \circ q_2 - q_1 \circ (b + q_2) - q_1 \circ q_2 \subseteq I$ since $(a + q_1) \circ (b + q_2) \subseteq I$, $(a + q_1) \circ q_2 \subseteq I$, $q_1 \circ (b + q_2) \subseteq I$ and $q_1 \circ q_2 \subseteq I$. Hence, $a \circ b \subseteq I$, a contradiction. Consequently, it must be $I_2 \circ I_3 \subseteq \delta(I)$ or $I_1 \circ I_3 \subseteq \delta(I)$. □

4. Generalization of hyperideals of product of multiplicative hyperrings

Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two multiplicative hyperrings with nonzero identity. Recall $(R = R_1 \times R_2, +, \circ)$ is a multiplicative hyperring with the operation $+$ and the hyperoperation $\circ$ are defined respectively as $(x, y) + (z, t) = (x +_1 y, z +_2 t)$ and $(x, y) \circ (z, t) = \{(a, b) \in R | a \in x \circ_1 z, b \in y \circ_2 t\}$ for all $(x, y), (z, t) \in R$ (for more information, see [4]). Note that each hyperideal of R is the cartesian product of hyperideals of R_1 and R_2, respectively. Suppose that δ_1 and δ_2 are expansion functions of hyperideals of R_1 and R_2, respectively. Let δ_R be a function of hyperideals of R with $\delta_R(I_1 \times I_2) = \delta_1(I_1) \times \delta_2(I_2)$ for every hyperideals I_i of R_i for $i \in \{1, 2\}$. It is seen that the function δ_R is expansion function of hyperideals of R. In this section, it is characterized the structure of (2-absorbing) δ_R-primary hyperideals of R.

Theorem 4.1 *Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two multiplicative hyperrings with nonzero identity and δ_1 and δ_2 be expansion functions of hyperideals of R_1 and R_2, respectively. Let $I_1 \in \mathcal{I}*(R_1)$ and $(R = R_1 \times R_2, +, \circ)$. I_1 is a δ_1-primary hyperideal of R_1 if and only if $I_1 \times R_2$ is a δ_R-primary hyperideal of R.*

Proof $(\Rightarrow)$: We presume $(x, y), (z, t) \in R$ with $(x, y) \circ (z, t) \subseteq I_1 \times R_2$. By the above definition, we deduce $x \circ_1 z \subseteq I_1$. Hence, we have $x \in I_1$ or $z \in \delta_1(I_1)$ and so $(x, y) \in I_1 \times R_2$ or $(z, t) \in \delta_R(I_1 \times R_2)$.
$(\Rightarrow)$: Let I_1 be not a δ_1-primary hyperideal of R_1. We have $x, z \in R_1$ with $x \circ_1 z \subseteq I_1$, $x \notin I_1$ and $z \notin \delta_1(I_1)$. Note that $(x, 1_{R_2}) \circ (z, 1_{R_2}) \subseteq I_1 \times R_2$. By assumption, $(x, 1_{R_2}) \in I_1 \times R_2$ or $(z, 1_{R_2}) \in \delta_R(I_1 \times R_2)$. It means $x \in I_1$ or $z \in \delta_1(I_1)$, a contradiction. Thus, I_1 is δ_1-primary. □

Theorem 4.2 *Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two multiplicative hyperring with nonzero identity, δ_1 and δ_2 be expansion functions of hyperideals of R_1 and R_2, respectively. Let $I = I_1 \times I_2$ be a proper hyperideals of R for some hyperideals I_1 and I_2 of R_1 and R_2, respectively. Then the following are equivalent:*

1. *$I = I_1 \times I_2$ is a δ_R-primary hyperideal of R.*
2. *$I_1 = R_1$ and I_2 is a δ_2-primary hyperideal of R_2 or $I_2 = R_2$ and I_1 is a δ_1-primary hyperideal of R_1.*

Proof $(1) \Rightarrow (2)$: Let $I_1 = R_1$ $(I_2 = R_2)$. Then I_2 is a δ_2-primary hyperideal of R_2(I_1 is a δ_1-primary hyperideal of R_1) by Theorem 4.1.
$(2) \Rightarrow (1)$: It is obvious by Theorem 4.1. □

As a result of Theorem 4.1 and Theorem 4.2, we have that if I_1 and I_2 are δ_1-primary and δ_2-primary hyperideal of R_1 and R_2, respectively, then $I_1 \times I_2$ may not be a δ_R-primary hyperideal of $R = R_1 \times R_2$. For this case, we give the next example.

Example 4.1 *Assume that $(\mathbb{Z}, +, \cdot)$ is the ring of integers. Then $(\mathbb{Z}, +, \circ_1)$ and $(\mathbb{Z}, +, \circ_2)$ are two multiplicative hyperring with a hyperoperation $x \circ_1 y = \{xy, 3xy\}$ and $x \circ_2 y = \{xy, 2xy\}$, respectively. Consider $(R = \mathbb{Z} \times \mathbb{Z}, +, \circ)$ is a multiplicative hyperring with a hyperoperation $(x, y) \circ (z, t) = \{(a, b) | a \in x \circ_1 z, b \in y \circ_2 t\}$. Note that $3\mathbb{Z} = \{3k | k \in \mathbb{Z}\}$ is a δ_0-primary hyperideal of $(\mathbb{Z}, +, \circ_1)$ and $2\mathbb{Z} = \{2k | k \in \mathbb{Z}\}$ is a δ_0-primary hyperideal of $(\mathbb{Z}, +, \circ_2)$. But $3\mathbb{Z} \times 2\mathbb{Z}$ is not a $\delta_R = \delta_0 \times \delta_0$-primary hyperideal since $(2, 0) \circ (0, 3) \in 3\mathbb{Z} \times 2\mathbb{Z}$ but $(2, 0), (0, 3) \notin 3\mathbb{Z} \times 2\mathbb{Z}$ and $(2, 0), (0, 3) \notin \delta_R(3\mathbb{Z} \times 2\mathbb{Z}) = 3\mathbb{Z} \times 2\mathbb{Z}$.*

Theorem 4.3 *Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two multiplicative hyperring with nonzero identity, δ_1 and δ_2 be expansion functions of hyperideals of R_1 and R_2, respectively. Let $I_1 \in \mathcal{I}*(R_1)$ and $(R = R_1 \times R_2, +, \circ)$.*

I_1 is a 2-absorbing δ_1-primary hyperideal of R_1 if and only if $I_1 \times R_2$ is a 2-absorbing δ_R-primary hyperideal of R.

Proof $(\Rightarrow)$: Let $(x,y),(z,t),(u,v) \in R$ with $(x,y) \circ (z,t) \circ (u,v) \subseteq I_1 \times R_2$. Then $x \circ_1 z \circ_1 u \subseteq I_1$ by the previous definition. Hence, we have $x \circ_1 z \in I_1$ or $z \circ_1 u \in \delta_1(I_1)$ or $x \circ_1 u \in \delta_1(I_1)$ and so $(x,y) \circ (z,t) \subseteq I_1 \times R_2$ or $(z,t) \circ (u,v) \subseteq \delta_R(I_1 \times R_2)$ or $(x,y) \circ (u,v) \subseteq \delta_R(I_1 \times R_2)$.
$(\Rightarrow)$: Let $I_1 \in \mathcal{I} * (R_1)$ be not 2-absorbing δ_1-primary. We have $x,z,u \in R_1$ with $x \circ_1 z \circ_1 u \subseteq I_1$, $x \circ_1 z \nsubseteq I_1$, $z \circ_1 u \nsubseteq \delta_1(I_1)$ and $x \circ_1 u \nsubseteq \delta_1(I_1)$. Note that $(x, 1_{R_2}) \circ (z, 1_{R_2}) \circ (u,v) \subseteq I_1 \times R_2$. By assumption, $(x, 1_{R_2}) \circ (z, 1_{R_2}) \subseteq I_1 \times R_2$ or $(z, 1_{R_2}) \circ (u, 1_{R_2}) \subseteq \delta_R(I_1 \times R_2)$ or $(x, 1_{R_2}) \circ (u, 1_{R_2}) \subseteq \delta_R(I_1 \times R_2)$. It means $x \circ_1 z \subseteq I_1$ or $z \circ_1 u \subseteq \delta_1(I_1)$ or $x \circ_1 u \subseteq \delta_1(I_1)$, a contradiction. Therefore, I_1 is 2-absorbing δ_1-primary. □

Theorem 4.4 *Let $(R_1, +_1, \circ_1)$ and $(R_2, +_2, \circ_2)$ be two multiplicative hyperring with nonzero identity, δ_1, δ_2 be expansion functions of hyperideals of R_1, R_2, respectively. Let $I = I_1 \times I_2$ be a proper hyperideals of $R = R_1 \times R_2$ for some hyperideals I_1, I_2 of R_1, R_2, respectively, with for every $i \in \{1,2\}$. If I_i is a proper with $\delta_i(I_i) \neq R_i$ for each $i \in \{1,2\}$, then we have the following equivalent statements:*

1. *$I = I_1 \times I_2$ is a 2-absorbing δ_R-primary hyperideal of R.*
2. *I_i is a δ_i-primary hyperideal of R_i for every $i \in \{1,2\}$ or $I_1 = R_1$ and I_2 is a δ_2-primary hyperideal of R_2 or $I_2 = R_2$ and I_1 is a δ_1-primary hyperideal of R_1.*

Proof $(1) \Rightarrow (2)$: When $I_1 = R_1$ $(I_2 = R_2)$, then I_2 is a 2-absorbing δ_2-primary hyperideal of R_2(I_1 is a δ_1-primary hyperideal of R_1) by Theorem 4.3. Let $I_i \in \mathcal{I} * (R_i)$ for every $i \in \{1,2\}$. Assume that I_1 is not a δ_1-primary hyperideal of R_1. Then there are $x,y \in R_1$ with $xy \in I_1$, $x \notin I_1$ and $y \notin \delta_1(I_1)$. Then $(x, 1_{R_2}) \circ (1_{R_1}, 0_{R_2}) \circ (y, 1_{R_2}) \subseteq I$. By our assumption, $(x, 1_{R_2}) \circ (1_{R_1}, 0_{R_2}) \subseteq I$ or $(1_{R_1}, 0_{R_2}) \circ (y, 1_{R_2}) \subseteq \delta_R(I)$ or $(x, 1_{R_2}) \circ (y, 1_{R_2}) \subseteq \delta_R(I)$, a contradiction. Hence, I_1 is a δ_1-primary hyperideal of R_1. In a similar way, it is seen that I_2 is a δ_2-primary hyperideal of R_2.
$(2) \Rightarrow (1)$: Let I_1 be a 2-absorbing δ_1-primary of R_1. Thus, $I = I_1 \times R_2$ is clearly a 2-absorbing δ_R-primary by Theorem 4.3. Also, $I = R_1 \times I_2$ is so where I_2 is 2-absorbing δ_2-primary of R_2 by Theorem 4.3. Let I_1 be a δ_1-primary hyperideal of R_1 and I_2 be a δ_2-primary hyperideal of R_2. Then we get $J = I_1 \times R_2$ and $K = R_1 \times I_2$ are 2-absorbing δ_R-primary hyperideals of R. Thus, $J \cap K = I_1 \times I_2$ is 2-absorbing δ_R-primary by Theorem 3.6 and Theorem 4.2. □

As a result of Theorems 4.3 and 4.4, we have that if I_1 and I_2 are 2-absorbing δ_1-primary and 2-absorbing δ_2-primary hyperideal of R_1 and R_2, respectively, then $I_1 \times I_2$ may not be a 2-absorbing δ_R-primary hyperideal of $R = R_1 \times R_2$. For this case, see the next example.

Example 4.2 *Assume that $(\mathbb{Z}, +, \cdot)$ is the ring of integers. Then $(\mathbb{Z}, +, \circ_1)$ is a multiplicative hyperring with a hyperoperation $x \circ_1 y = \{2xy, 3xy\}$. Let $(R = \mathbb{Z} \times \mathbb{Z}, +, \circ)$ is a multiplicative hyperring with a hyperoperation $(x,y) \circ (z,t) = \{(a,b) | a \in x \circ_1 z, b \in y \circ_1 t\}$. Note that $6\mathbb{Z} = \{6k | k \in \mathbb{Z}\}$ and $12\mathbb{Z} = \{12k | k \in \mathbb{Z}\}$ are two δ_2-primary hyperideal of $(\mathbb{Z}, +, \circ_1)$. However, $6\mathbb{Z} \times 12\mathbb{Z}$ is not a 2-absorbing $\delta_R = \delta_2 \times \delta_2$-primary hyperideal since $(2,1) \circ (1,3) \circ (3,4) \in 6\mathbb{Z} \times 12\mathbb{Z}$ but $(2,1) \circ (1,3), (2,1) \circ (3,4), (1,3) \circ (3,4) \nsubseteq 6\mathbb{Z} \times 12\mathbb{Z}$ and $(2,1) \circ (1,3), (2,1) \circ (3,4), (1,3) \circ (3,4) \nsubseteq \delta_R(6\mathbb{Z} \times 12\mathbb{Z})$.*

5. Conclusion

In this paper, our purpose is to introduce the concepts δ-primary and 2-absorbing δ-primary hyperideal over multiplicative hyperrings. These structures are the unify prime and primary, 2-absorbing and 2-absorbing primary hyperideals, respectively. We obtain many specific results explaining the structures. For instance, we indicate that a hyperideal I of R is δ-primary if and only if $L \circ K \subseteq I$ for some $L, K \in \mathcal{I}(\mathcal{R})$ implies $L \subseteq I$ or $K \subseteq \delta(I)$. Then we also showed that a similar result is satisfied for 2-absorbing δ-primary hyperideals of R. We characterize δ-primary hyperideals and also 2-absorbing δ-primary hyperideals of cartesian product of multiplicaitve hyperrings. This paper makes a great contribution to classify hyperideals of multiplicative hyperrings.

As a new research subject, we suggest the concept of n-absorbing δ-primary hyperideals of multiplicative hyperrings.

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