

On a new generalization of telephone numbers

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Abstract: In this paper we introduce a new generalization of telephone numbers. We give the generating function, direct formulae, and matrix generators for these numbers. Moreover, we present their interpretations and we prove some properties of these numbers connected with congruences.

Key words: Telephone numbers, matrix generators

1. Introduction

The classical telephone numbers, also known as involution numbers, are given by the recurrence relation

$$T(n) = T(n-1) + (n-1)T(n-2) \quad (1)$$

for $n \geq 2$, with initial conditions $T(0) = T(1) = 1$. Connections of these numbers with symmetric groups were observed for the first time in 1800 by Heinrich August Rothe, who pointed out that $T(n)$ is the number of involutions (self-inverse permutations) in the symmetric group S_n (see, for example, [5, 8]). Because involutions correspond to standard Young tableaux it is clear that the n th involution number is also the number of Young tableaux on the set $\{1, 2, \dots, n\}$ (for details see [3]). It is worth mentioning that a telephone interpretation of recursion (1) is due to John Riordan, who noticed that $T(n)$ is the number of connection patterns in a telephone system with n subscribers (see [10]). Obviously one can find many other interpretations of recurrence relation (1), for example in the mathematics of chess or theory of representation.

Quite recently, the paper [12] introduced generalized telephone numbers $T(p, n)$ defined for integers $n \geq 0$ and $p \geq 1$ by the following recursion:

$$T(p, n) = pT(p, n-1) + (n-1)T(p, n-2),$$

with initial conditions $T(p, 0) = 1$, $T(p, 1) = p$. The authors gave a new interpretation of numbers $T(p, n)$ and proved some of their properties. In this paper, we present a new one-parameter generalization of the classical telephone numbers. We give two interpretations of these numbers, the first one connected with telephones and the second one connected with a special edge coloring of a graph. We also derive direct formulae for these numbers and we show a connection with Hessenberg matrices' determinants. Moreover, we prove some results on congruences being generalizations of the classical results for telephone numbers.

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2. Definition and interpretations

Let $p \geq 1, n \geq 0$ be integers. Generalized telephone numbers $T_p(n)$ are defined recursively as follows:

$$T_p(n) = T_p(n - 1) + p(n - 1)T_p(n - 2) \quad \text{for } n \geq 2, \tag{2}$$

with initial conditions $T_p(0) = T_p(1) = 1$.

The first table presents generalized telephone numbers $T_p(n)$ for some values of n .

Table 1.

n	0	1	2	3	4	5	6
$T_p(n)$	1	1	$1 + p$	$1 + 3p$	$1 + 6p + 3p^2$	$1 + 10p + 15p^2$	$1 + 15p + 45p^2 + 15p^3$

The second table presents generalized telephone numbers $T_p(n)$ for a few fixed values of p and n .

Table 2.

n	0	1	2	3	4	5	6	7	8	9	10
$T_1(n)$	1	1	2	4	10	26	76	232	764	2620	9496
$T_2(n)$	1	1	3	7	25	81	331	1303	5937	26785	133651
$T_3(n)$	1	1	4	10	46	166	586	3574	15880	101656	530416

As we can observe, if $p = 1$, then we obtain telephone numbers $T_1(n) = T(n)$. Moreover, for odd p and $n \geq 2$ generalized telephone numbers $T_p(n)$ are even numbers. In turn, for even p , generalized telephone numbers are odd numbers.

Now we present a few interpretations of generalized telephone numbers $T_p(n)$. We begin with an interpretation connected with telephones.

By Riordan’s interpretation, as mentioned earlier, the classical telephone numbers count connection patterns in a telephone system with n subscribers. It is known that such a network can be modeled by a complete n vertex graph K_n and each pattern of connections in such a network is a matching (a set of independent edges) in K_n . Thus, in the language of graph theory, the n th telephone number is the total number of matchings of a complete graph K_n . It is worth emphasizing that the total number of matchings of a given graph, also known as the Hosoya index, is an important graph parameter used in chemical graph theory (cf. [11]). Note that modern networks give their members many more possibilities than only a classical phone call (it can be a video call, a fax, etc.). If we assume that a network with n subscribers gives its users p possibilities of connections with each member of the network, then such a network can be modeled by a complete multigraph K_n^p . It is clear that each pattern of connections in such a network is a matching in K_n^p . Let $\mu(K_n^p)$ be the total number of connection patterns in the network modeled by a complete multigraph K_n^p . We will prove that $\mu(K_n^p)$ is equal to the n th generalized telephone number $T_p(n)$. For convenience, in our calculation we put $\mu(K_0^p) = 1$.

Theorem 1 *Let $p \geq 1$ and $n \geq 0$ be integers. Then $\mu(K_n^p) = \mu(K_{n-1}^p) + p(n - 1)\mu(K_{n-2}^p)$ with initial values $\mu(K_0^p) = \mu(K_1^p) = 1$.*

Proof The initial conditions are obvious. Let a complete multigraph K_n^p be a model of a network with n subscribers and p possibilities of connections between each two members of the network. Assume that a vertex x corresponds to a fixed subscriber of this network. If x is not calling then the number of connection patterns in the network is equal to $\mu(K_{n-1}^p)$. If x is calling another member of the network then the remaining $n - 2$ subscribers of this network realize their connections in a network modeled by a complete multigraph K_{n-2}^p . A subscriber x chooses one of $n - 1$ members of the network and one of p possibilities of connections. Thus, in the case when x is calling, there are $p(n - 1)\mu(K_{n-2}^p)$ patterns of connections. Finally, we get $\mu(K_n^p) = \mu(K_{n-1}^p) + p(n - 1)\mu(K_{n-2}^p)$. \square

By the definition of generalized telephone numbers $T_p(n)$ and Theorem 1, we obtain:

Corollary 2 *Let $p \geq 1$ and $n \geq 0$ be integers. Then $\mu(K_n^p) = T_p(n)$.*

Now we present a graph interpretation related to a special edge-shade coloring. We use the concept of this coloring introduced and studied in [1, 2].

Let G be an undirected, connected, and simple graph. Let us consider an edge coloring of a graph G , $c : E(G) \rightarrow \{A, 2B_1, \dots, 2B_p\}$, $p \geq 1$. A graph G is $(A, 2B_1, \dots, 2B_p)$ -edge colored if for every monochromatic subgraph H induced by all edges of color $2B_i$, $i = 1, \dots, p$ in G , there is a partition of H into edge disjoint paths of length two. There are no restrictions for a color A . In other words, the edges of G are partitioned into paths and paths of length two can be colored with one of $2B_i$ possible shades, for $i = 1, \dots, p$. We also have to consider all possible partitions into paths of every monochromatic subgraph. Clearly, this edge coloring always exists in an arbitrary graph.

Let $\mathcal{C}(G) = \{c_j(G) : j \geq 1\}$ be a family of all distinct $(A, 2B_1, \dots, 2B_p)$ -edge colored copies of a graph G . Clearly, $c_j(G)$ is an $(A, 2B_1, \dots, 2B_p)$ -edge colored graph G . By $\theta(c_j(G))$ we denote the total number of partitions of all $2B_i$ -monochromatic subgraphs H of $c_j(G)$ into edge disjoint paths of length two. If G is colored only with A , then $\theta(c_j(G)) = 1$.

Let $\sigma_{(A, 2B_1, \dots, 2B_p)}(G)$ denote the total number of $(A, 2B_1, \dots, 2B_p)$ -edge colorings of G . Then

$$\sigma_{(A, 2B_1, \dots, 2B_p)}(G) = \sum_{j=1}^{|\mathcal{C}(G)|} \theta(c_j(G)).$$

Let e be an arbitrary edge of G . By $\sigma_{A(e)}(G)$ and $\sigma_{2B_i(e)}(G)$, for $1 \leq i \leq p$, we denote a number of all $(A, 2B_1, \dots, 2B_p)$ -edge colorings of G , where edge e has a color A or $2B_i$, respectively. Then

$$\sigma_{(A, 2B_1, \dots, 2B_p)}(G) = \sum_{i=1}^p \sigma_{2B_i(e)}(G) + \sigma_{A(e)}(G)$$

is the basic rule of counting the parameter $\sigma_{(A, 2B_1, \dots, 2B_p)}(G)$.

Let $S(m)$ be a star of size m , $m \geq 1$. We will prove that the total number of $(A, 2B_1, \dots, 2B_p)$ -edge colorings of a star $S(m)$ is equal to the m th generalized telephone number $T_p(m)$.

Theorem 3 *Let $p \geq 1$ and $m \geq 1$ be integers. Then $\sigma_{(A, 2B_1, \dots, 2B_p)}(S(m)) = T_p(m)$.*

Proof (by induction on m) If $m = 1, 2$, then obviously the theorem is true. Let $m \geq 2$ and e be an arbitrary edge of $S(m)$. Suppose that $\sigma_{(A, 2B_1, \dots, 2B_p)}(S(k)) = T_p(k)$ for $k < m$. If $c(e) = A$, then a graph $S(m) \setminus e$ is isomorphic to $S(m - 1)$, and by induction hypothesis it holds that $\sigma_{A(e)}(S(m)) = \sigma_{(A, 2B_1, \dots, 2B_p)}(S(m - 1)) = T_p(m - 1)$. If $c(e) = 2B_i, i = 1, \dots, p$, then there exists an edge e' adjacent to e which is also $2B_i$ colored. The edge e' can be chosen in $m - 1$ ways in the star $S(m)$. Moreover, we can choose the color $2B_i$ in p ways. Using the induction hypothesis and the fact that the graph $S(m) \setminus \{e, e'\}$ is isomorphic to the graph $S(m - 2)$ we obtain $\sum_{i=1}^p \sigma_{2B_i(e)}(S(m)) = p(m - 1)\sigma_{(A, 2B_1, \dots, 2B_p)}(S(m - 2)) = p(m - 1)T_p(m - 2)$.

Finally, taking into account the above considerations, we have

$$\sigma_{(A, 2B_1, \dots, 2B_p)}(S(m)) = p(m - 1)T_p(m - 2) + T_p(m - 1) = T_p(m),$$

which ends the proof. □

The following theorem gives the direct formula for generalized telephone numbers.

Theorem 4 *Let $m \geq 2$ and $p \geq 1$ be integers. Then*

$$T_p(m) = 1 + \sum_{2 \leq 2j \leq m} \binom{m}{2j} (2j - 1)!! p^j.$$

Proof To prove this theorem we use the graph interpretation mentioned earlier. Let us consider $(A, 2B_1, \dots, 2B_p)$ -edge colorings of a star $S(m)$. If a star $S(m)$ is colored only by A , then $\sigma_{A(e)}(S(m)) = 1$.

Assume that the colors $2B_i, i = 1, \dots, p$ are used. Then, according to the definition of $(A, 2B_1, \dots, 2B_p)$ -edge colorings, we color an even number of edges of the star $S(m)$. In this way we obtain a subgraph of a star $S(m)$ with even number of edges. Hence, such a subgraph can be chosen in $\binom{m}{2j}$ ways. Moreover, $2j$ edges can be partitioned into paths of length 2 in $(2j - 1)(2j - 3) \dots \cdot 3 \cdot 1$ ways. Taking into account that we have p colors $2B_i$ and in every partition of $2j$ edges there are j parts to color, we obtain p^j possibilities of coloring of this subgraph. Consequently, $\sum_{i=1}^p \sigma_{2B_i(e)}(S(m)) = \sum_{2 \leq 2j \leq m} \binom{m}{2j} (2j - 1)!! p^j$.

Finally, according to Theorem 4, we obtain

$$\sigma_{(A, 2B_1, \dots, 2B_p)}(S(m)) = T_p(m) = 1 + \sum_{2 \leq 2j \leq m} \binom{m}{2j} (2j - 1)!! p^j$$

and the theorem is proved. □

3. Generating function, summation formula, and matrix generator of $T_p(n)$

In the previous section we obtained a direct formula for generalized telephone numbers based on their graph interpretation. To get another direct formula for numbers $T_p(n)$ we use a generating function. It is well known (cf. [5]) that the exponential generating function of the classical telephone numbers is $\sum_{n=0}^{\infty} \frac{T(n)x^n}{n!} =$

$\exp\left(x + \frac{x^2}{2}\right)$ and a summation formula, obtained by the generating function, is $T(n) = n! \sum_{i+2j=n}^{\infty} \frac{1}{2^j i! j!}$. Now we derive the exponential generating function and the summation formula for generalized telephone numbers $T_p(n)$.

Theorem 5 *The generating function of the generalized telephone numbers $T_p(n)$ is*

$$\sum_{n=0}^{\infty} \frac{T_p(n)x^n}{n!} = \exp\left(x + \frac{px^2}{2}\right).$$

Proof Suppose that $T_p(n) = n!a_n$. Substituting to (2) we obtain

$$na_n = a_{n-1} + pa_{n-2} \text{ with } a_0 = a_1 = 1. \tag{3}$$

Let $y = \sum_{n=0}^{\infty} a_n x^n$ be the generating function of the sequence a_n . By properties of generating functions and (3) we have

$$x \frac{dy}{dx} = \sum_{n=1}^{\infty} na_n x^n = x + \sum_{n=2}^{\infty} (a_{n-1} + pa_{n-2})x^n = x + x \sum_{n=1}^{\infty} a_n x^n + px^2 \sum_{n=0}^{\infty} a_n x^n.$$

Thus, the generating function y should satisfy a differential equation of the form

$$x \frac{dy}{dx} = xy + px^2 y.$$

Solving this equation by separating variables, we get

$$y = C \exp\left(x + \frac{px^2}{2}\right).$$

Since $a_0 = 1$, then $C = 1$, which completes the proof. □

As an immediate consequence of Theorem 5 we obtain a summation formula for generalized telephone numbers $T_p(n)$.

Corollary 6 *Let $n \geq 1, p \geq 1$ be integers. Then $T_p(n) = n! \sum_{i+2j=n} \frac{p^j}{2^j i! j!}$.*

Proof By the proof of Theorem 5 it follows that a_n is the coefficient of x^n in the power series of $\exp\left(x + \frac{px^2}{2}\right)$.

Using the fact that

$$\exp\left(x + \frac{px^2}{2}\right) = \exp(x) \cdot \exp(px^2/2) = \sum_{i=0}^{\infty} \frac{x^i}{i!} \cdot \sum_{j=0}^{\infty} \frac{p^j x^{2j}}{2^j j!} = \sum_{i+2j=n} \frac{p^j}{2^j i! j!} x^n,$$

we obtain

$$a_n = \sum_{i+2j=n}^{\infty} \frac{p^j}{2^j i! j!},$$

and by dependence $T_p(n) = n!a_n$ we get

$$T_p(n) = n! \sum_{i+2j=n}^{\infty} \frac{p^j}{2^j i! j!}.$$

□

Now we present the connection between generalized telephone numbers $T_p(n)$ and lower Hessenberg matrices' determinants.

Let us recall that a square matrix $A = [a_{ij}]$ of size n is the lower Hessenberg matrix if $a_{ij} = 0$, for $j > i + 1$, and $a_{i,i+1} \neq 0$, for some i . It is worth mentioning that Hessenberg matrices are often used as matrix generators of numbers defined recursively. There are many such known generators for numbers of the Fibonacci type (for details, see, for instance, [4, 6, 7]). In this paper we define a lower Hessenberg matrix being the generator of the generalized telephone numbers $T_p(n)$.

Let $H_{p,n} = [h_{ij}]$ be the lower Hessenberg matrix of size n defined as follows:

$$h_{ij} = \begin{cases} 0 & \text{if } j > i + 1, \\ -1 & \text{if } j = i + 1, \\ 1 & \text{if } i + j \text{ is even and } j \leq i, \\ p(n + 1 - i) - 1 & \text{if } i + j \text{ is odd and } j < i \text{ and } i < n, \\ p & \text{if } i + j \text{ is odd and } j < i \text{ and } i = n. \end{cases}$$

From the above definition for $i = 1, 2, 3, 4$ we have

$$H_{p,1} = [1], \quad H_{p,2} = \begin{bmatrix} 1 & -1 \\ p & 1 \end{bmatrix}, \quad H_{p,3} = \begin{bmatrix} 1 & -1 & 0 \\ 2p-1 & 1 & -1 \\ 1 & p & 1 \end{bmatrix}, \quad H_{p,4} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 3p-1 & 1 & -1 & 0 \\ 1 & 2p-1 & 1 & -1 \\ p & 1 & p & 1 \end{bmatrix},$$

and generally

$$H_{p,n} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ p(n-1)-1 & 1 & -1 & \cdots & 0 & 0 \\ 1 & p(n-2)-1 & 1 & \cdots & 0 & 0 \\ p(n-3)-1 & 1 & p(n-3)-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 2p-1 & 1 & 2p-1 & \cdots & 1 & -1 \\ 1 & p & 1 & \cdots & p & 1 \end{bmatrix} \text{ for odd } n \text{ and}$$

$$H_{p,n} = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ p(n-1)-1 & 1 & -1 & \cdots & 0 & 0 \\ 1 & p(n-2)-1 & 1 & \cdots & 0 & 0 \\ p(n-3)-1 & 1 & p(n-3)-1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 2p-1 & 1 & \cdots & 1 & 1 \\ p & 1 & p & \cdots & 1 & p \end{bmatrix} \text{ for even } n.$$

Theorem 7 *Let $n \geq 1, p \geq 1$ be integers. Then $\det H_{p,n} = T_p(n)$.*

Proof (by induction on n) If $n = 1, 2$, then the result is obvious. Now assume that $\det H_{p,k} = T_p(k)$, for $k < n$. We shall show that $\det H_{p,n} = T_p(n)$. Computing $\det H_{p,n}$ by Laplace expansion of the determinant with respect to the first row and then applying the basic properties of determinants, we have

$$\det H_{p,n} = \det H_{p,n-1} + p(n-2) \det H_{p,n-2}.$$

Hence, by induction assumption and recurrence relation (2), we obtain

$$\det H_{p,n} = T_p(n-1) + p(n-1)T_p(n-2) = T_p(n),$$

as desired. □

Note that for $p = 1$, by Theorem 7, we get the lower Hessenberg matrix generator $H_{1,n}$ for the classical telephone numbers $T(n)$.

4. Some properties of generalized telephone numbers

In this section we present some properties of generalized telephone numbers $T_p(n)$ connected with congruences. By recursion (2) we can deduce that

$$T_p(n+1) \equiv T_p(n) \pmod{pn},$$

and hence, by properties of congruences, we have

$$T_p(n+1) \equiv T_p(n) \pmod{p}, \quad T_p(n+1) \equiv T_p(n) \pmod{n}.$$

Now we will prove the following property.

Theorem 8 *Let $n \geq 0, p \geq 1$ be integers. Then for any odd integer m ,*

$$T_p(n+m) \equiv T_p(n) \pmod{pm}.$$

Proof We proceed by induction on n . If $n = 0$, then by Theorem 4 and initial conditions for generalized telephone numbers $T_p(n)$ we obtain

$$T_p(m) - T_p(0) = \sum_{2 \leq 2j \leq m} \binom{m}{2j} (2j-1)!! p^j.$$

Because m is odd and $2j$ is even it is clear that pm divides $T_p(m) - T_p(0)$. Thus, $T_p(m) \equiv T_p(0) \pmod{pm}$. Analogously, for $n = 1$, we get $T_p(1+m) \equiv T_p(1) \pmod{pm}$.

Suppose the theorem is true for $n < k$. By recursion (2), for $n = k$, we have

$$T_p(k+m) = T_p(k+m-1) + p(k+m-1)T_p(k+m-2).$$

Hence,

$$T_p(k+m) \equiv T_p(k+m-1) + p(k-1)T_p(k+m-2) \pmod{pm}.$$

Using the induction assumption we obtain

$$T_p(k+m-1) \equiv T_p(k-1) \pmod{pm} \quad \text{and} \quad T_p(k+m-2) \equiv T_p(k-2) \pmod{pm},$$

and then by properties of congruences we have

$$T_p((k + m - 1) + p(k - 1)T_p(k + m - 2) \equiv T_p(k - 1) + p(k - 1)T_p(k - 2) \pmod{pm}).$$

Finally, by the transitivity property we get $T_p(k + m) \equiv T_p(k) \pmod{pm}$, which ends the proof. □

An obvious consequence of Theorem 4 and properties of congruences is the following corollary.

Corollary 9 *If $n \geq 0$, $p \geq 1$ are integers and m is an odd integer, then:*

(i) $T_p(n + m) \equiv T_p(n) \pmod{p}$,

(ii) $T_p(n + m) \equiv T_p(n) \pmod{m}$.

From Theorem 4, for $p = 1$ and odd integer m we obtain $T_1(n + m) \equiv T_1(n) \pmod{m}$, and because $T_1(n) = T(n)$ we have $T(n + m) \equiv T(n) \pmod{m}$. Such a congruence for the classical telephone numbers was proved by Chowla et al. (see [5]). In 1955 Moser and Wyman [9] proved that if we resign from the assumption that m is odd, then the following property for the classical telephone numbers holds:

$$T(n + m) \equiv T(n) \cdot T(m) \pmod{m},$$

where $n \geq 0$ and $m \geq 1$ are integers. It turns out that generalized telephone numbers $T_p(n)$ satisfy analogous congruence.

Theorem 10 *Let $n \geq 0$, $m \geq 1$, $p \geq 1$ be integers. Then*

$$T_p(n + m) \equiv T_p(n) \cdot T_p(m) \pmod{pm}.$$

Proof (by induction on n) For $n = 0$ the theorem is obvious and for $n = 1$ it follows immediately from recursion (2). Now assume that the theorem is true for $n < k$. We shall prove it is also true for $n = k$. By recurrence relation (2) we have

$$T_p(k + m) = T_p(k + m - 1) + p(k + m - 1)T_p(k + m - 2).$$

Hence,

$$T_p(k + m) \equiv T_p(k + m - 1) + p(k - 1)T_p(k + m - 2) \pmod{pm}.$$

By induction assumption we have

$$T_p(k + m - 1) \equiv T_p(k - 1) \cdot T_p(m) \pmod{pm}$$

and

$$T_p(k + m - 2) \equiv T_p(k - 2) \cdot T_p(m) \pmod{pm}.$$

Then, using properties of congruences, we can write

$$T_p(k + m - 1) + p(k - 1)T_p(k + m - 2) \equiv T_p(k) \cdot T_p(m) \pmod{pm},$$

and finally we obtain

$$T_p(k + m) \equiv T_p(k) \cdot T_p(m) \pmod{pm}.$$

Thus, the proof is complete. □

As an immediate consequence of Theorem 10 and properties of congruences, we get the following corollary.

Corollary 11 *If $n \geq 0$, $m \geq 1$, and $p \geq 1$ are integers, then:*

$$(i) \quad T_p(n+m) \equiv T_p(n) \cdot T_p(m) \pmod{p},$$

$$(ii) \quad T_p(n+m) \equiv T_p(n) \cdot T_p(m) \pmod{m}.$$

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