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On the solvability of the main boundary value problems for a nonlocal Poisson equation

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Abstract: Solvability of the main boundary value problems for the nonlocal Poisson equation is studied. Existence and uniqueness theorems for the considered problems are obtained. The necessary and sufficient solvability conditions for all problems are given and integral representations for the solutions are constructed.

Key words: Nonlocal equation, Poisson equation, existence and uniqueness, Green's function

1. Introduction

The concept of a nonlocal operator and the related concept of a nonlocal differential equation appeared in mathematics quite recently. For example, in [20], the notion of nonlocal differential equations incorporated the loaded equations, equations with fractional derivatives of the unknown function, and equations with deviating arguments, or in other words all equations in which the unknown functions and/or their derivatives enter with different values of arguments.

A specific type of nonlocal differential equation is formed by equations in which the deviation of arguments has an involutive character. A mapping S is called an involution if $S^2(x) \equiv S(S(x)) = x$ for all x. It is well known that differential equations containing an involution in the unknown function or its derivative give model equations with alternating deviation of the argument. Generally speaking, these equations can be attributed to the class of functional-differential equations.

The study of equations with involution has a long history. In 1816 Babbage considered in [5] some algebraic and differential equations with involution. A treatment of solvability for various differential equations with involution was given by Przeworska-Rolewicz in [21] and Wiener in [33]. Spectral problems for a first-order differential equation with involution were studied in [6, 7]. In [16–19, 23, 29], further problems in the spectral theory of the first- and second-order differential operators with involution were discussed. The results of studying the spectral properties for differential equations with involution were used in [1, 14, 31] to solve the related inverse problems. The series of papers by Cabada and Tojo (see [8, 30] for an expanded list of citations) pioneered in creating a comprehensive theory of Green's functions for the one-dimensional differential equations with involution.

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Solvability issues for certain partial differential equations with involution were covered in [2–4]. Furthermore, in [12, 27, 28, 32], boundary value problems for second- and fourth-order elliptic equations were studied in the case when an involution appears in the boundary conditions.

This paper studies the principal boundary value problems for a nonlocal Poisson equation. The paper is organized as follows. After the introduction of the key notions and main problems, Section 2 proceeds with auxiliary statements from the theory of systems of algebraic equations. Section 3 proves Theorem 3.5 on the uniqueness of solutions to the main problems. In Section 4, the existence of the solution to the Dirichlet problem is obtained. Similar statements for the Neumann and Robin problems are presented in Sections 5 and 6.

Let us proceed with the statement of the considered problems.

Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball, $n \geq 2$, $\partial\Omega$ be the unit sphere, and S be a real orthogonal matrix: $S \cdot S^T = E$. Suppose also that there exists a natural number $l \in \mathbb{N}$ such that $S^l = E$.

Note that, since any orthogonal transform is isometric, any $x \in \Omega$ and any $x \in \partial\Omega$ satisfy the inclusions $S^k x \in \Omega$, and $S^k x \in \partial\Omega$ respectively for any positive integer k.

Let us give some simple examples of such mappings S.

Example 1.1 *Let, for any $x \in \Omega$, the mapping S be defined by the relation $Sx = -x$, i.e. $S = -E$. Obviously, one has $S \cdot S^T = -E\,(-E) = E$, $S^2 = E$, and therefore l equals 2.*

Example 1.2 *The mapping S can clearly be a rotation in the space $\mathbb{R}^n$, e.g., S is the product of rotations $S = C^1_{\varphi_1} C^2_{\varphi_2} \cdots C^{n-2}_{\varphi_{n-2}}$ where C^i_φ corresponds to the matrix*

$$\begin{pmatrix} E_i & 0 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi & 0 \\ 0 & \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 0 & E_{n-i-2} \end{pmatrix},$$

E_i is the $i \times i$ unit matrix, and $i = \overline{1, n-2}$. Indeed, one has $S^T = C^{n-2}_{-\varphi_{n-2}} \ldots C^2_{-\varphi_2} C^1_{-\varphi_1}$, $C^i_\varphi C^i_\psi = C^i_{\varphi+\psi}$, and therefore

$$SS^T = C^1_{\varphi_1} C^2_{\varphi_2} \ldots C^{n-2}_{\varphi_{n-2}} \cdot C^{n-2}_{-\varphi_{n-2}} \ldots C^2_{-\varphi_2} C^1_{-\varphi_1} = E.$$

Moreover, it is necessary to suppose that there exists a natural number $l \in \mathbb{N}$ such that $S^l = E$.

Let $a_1, a_2, \ldots, a_l$ be some real numbers, and let $f(x)$ and $g(s)$ be functions defined on Ω and $\partial\Omega$, respectively. Introduce the operator

$$Lu(x) \equiv -\sum_{k=1}^{l} a_k \Delta u\left(S^{k-1}x\right).$$

We consider the following boundary value problems in Ω.

Dirichlet problem. *Find a function $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$ that satisfies the equation*

$$Lu(x) = f(x),\ x \in \Omega, \tag{1.1}$$

and the boundary condition

$$u(s) = g(s),\ s \in \partial\Omega. \tag{1.2}$$

Neumann problem. *Find a function* $u(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *that satisfies equation* (1.1) *and the boundary condition*

$$\frac{\partial u(s)}{\partial \nu} = g(s), \ s \in \partial\Omega, \tag{1.3}$$

where ν *is the external normal to the sphere* $\partial\Omega$.

Robin problem. *Find a function* $u(x) \in C^2(\Omega) \cap C^1(\bar{\Omega})$ *that satisfies equation* (1.1) *and the boundary condition*

$$\frac{\partial u(s)}{\partial \nu} + cu(s) = g(s), \ s \in \partial\Omega, \tag{1.4}$$

with a given positive real c.

In the case when $a_1 \neq 0, \ a_k = 0, \ k = 2, 3, \dots, l$ we obtain the classical Dirichlet, Neumann, and Robin problems for the conventional Poisson equation. Note that in [22] nonlocal boundary value problems for the classical two-dimensional Laplace equation with the mapping S from Example 1.2 in the boundary condition are studied.

2. Auxiliary statements

In this section we present some auxiliary statements from the theory of systems of algebraic equations.

Consider the following matrix A, which depends on the real numbers $a_1, a_2, \dots, a_l$:

$$A = \begin{pmatrix} a_1 & a_2 & \dots & a_l \\ a_l & a_1 & \dots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}.$$

Lemma 2.1 *Let* $\lambda_1 = \exp(i\frac{2\pi}{l})$ *be the primitive* l*th root of unity. Then*

$$\det A = \prod_{k=1}^{l} \left(a_1\lambda_0^k + \dots + a_l\lambda_{l-1}^k\right),$$

where $\lambda_k = \exp(i\frac{2\pi k}{l}), \ k = 0, \dots, l-1$.

Proof Clearly, $\lambda_k = \lambda_1^k$ and $\lambda_l = \lambda_0 = 1$. Let us prove that the number

$$\mu_k = a_1\lambda_0^k + \dots + a_l\lambda_{l-1}^k = \sum_{q=1}^{l} a_q\lambda_{q-1}^k \tag{2.1}$$

is an eigenvalue and the vector $B_k = \left(1, \lambda_1^k, \dots, \lambda_{l-1}^k\right)^T$ is its eigenvector of the matrix A for any $k = 1, \dots, l$. Since indices of the numbers λ_k can be changed modulo l, in what follows we may consider indices of numbers a_k also modulo l. Then, for example, $a_0 = a_l$, $a_{-1} = a_{l-1}$, and $a_{l+1} = a_1$. Since the mth row of the matrix A has the form $(a_{2-m}, a_{3-m}, \dots, a_{l-m+1})$, the element in the mth row of the vector $C_k = AB_k$ $(m = 1, \dots, l)$ equals

$$(AB_k)_m = \sum_{j=1}^{l} a_{j-m+1}\lambda_{j-1}^k = \lambda_{m-1}^k \sum_{j=1}^{l} a_{j-m+1}\lambda_{j-m}^k = \mu_k\lambda_{m-1}^k;$$

here we apply the relation $\lambda_m^k = \lambda_s^k \lambda_{m-s}^k$. Therefore, $AB_k = C_k = \mu_k B_k$.

Hence, the equality

$$\det A = \prod_{k=1}^{l} \mu_k = \prod_{k=1}^{l} \left(a_1 \lambda_0^k + \ldots + a_l \lambda_{l-1}^k\right)$$

yields the lemma's statement. The lemma is proved. □

Example 2.2 *If $l = 3$, then $\lambda_1 = \exp(i\frac{2\pi}{3})$, and therefore $\lambda_k = \exp(i\frac{2\pi k}{3})$. In this case, we have*

$$\det A = \det \begin{pmatrix} a_1 & a_2 & a_3 \\ a_3 & a_1 & a_2 \\ a_2 & a_3 & a_1 \end{pmatrix}$$

$$= \left(a_1 + a_2 \exp(i\frac{2\pi}{3}) + a_3 \exp(i\frac{4\pi}{3})\right) \left(a_1 + a_2 \exp(i\frac{4\pi}{3}) + a_3 \exp(i\frac{8\pi}{3})\right) (a_1 + a_2 + a_3)$$

$$= (a_1 + a_2 + a_3)\,(a_1^2 + a_2^2 + a_3^2 - a_2 a_3 - a_1 a_2 - a_1 a_3) = a_1^3 + a_2^3 + a_3^3 - 3a_1 a_2 a_3.$$

If $l = 4$, then $\lambda_1 = \exp(i\frac{2\pi}{4}) = i$, and we have

$$\det A = (a_1 + a_2 i - a_3 - a_4 i)\,(a_1 - a_2 + a_3 - a_4)\,(a_1 - a_2 i - a_3 + a_4 i)\,(a_1 + a_2 + a_3 + a_4)$$

$$= \left((a_1 + a_3)^2 - (a_2 + a_4)^2\right) \left((a_1 - a_3)^2 + (a_4 - a_2)^2\right).$$

Lemma 2.3 *Let the numbers μ_k in (2.1) be nonzero. Then there exists an inverse to the matrix A, which is given by the following formula:*

$$A^{-1} \equiv \begin{pmatrix} a_1 & a_2 & \ldots & a_l \\ a_l & a_1 & \ldots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \cdots & a_1 \end{pmatrix}^{-1} = \frac{1}{l} \mathrm{M}_+ \mathrm{diag}^{-1}\,(\mu_1, \ldots, \mu_l)\, {\mathrm{M}_-}^T,$$

where

$$\mathrm{M}_+ = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1 & \lambda_1^2 & \ldots & \lambda_1^l \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{l-1} & \lambda_{l-1}^2 & \cdots & \lambda_{l-1}^l \end{pmatrix}, \quad \mathrm{M}_- = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \lambda_1^{-1} & \lambda_1^{-2} & \ldots & \lambda_1^{-l} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_{l-1}^{-1} & \lambda_{l-1}^{-2} & \cdots & \lambda_{l-1}^{-l} \end{pmatrix}.$$

Proof Obviously, $\mathrm{M}_+ = (B_1, \ldots, B_l)$ where $B_k = \left(1, \lambda_1^k, \ldots, \lambda_{l-1}^k\right)^T$ is the eigenvector of A corresponding to the eigenvalue μ_k (see Lemma 2.1). Then $A\mathrm{M}_+ = (\mu_1 B_1, \ldots, \mu_l B_l)$, and hence

$$A\mathrm{M}_+ \mathrm{diag}^{-1}\,(\mu_1, \ldots, \mu_l) = (\mu_1 B_1, \ldots, \mu_l B_l)\,\mathrm{diag}\left(\mu_1^{-1}, \ldots, \mu_l^{-1}\right) = (B_1, \ldots, B_l) = \mathrm{M}_+.$$

This relation yields the equality

$$A\mathrm{M}_+ \mathrm{diag}^{-1}\,(\mu_1, \ldots, \mu_l)\, {\mathrm{M}_-}^T = \mathrm{M}_+ {\mathrm{M}_-}^T.$$

In order to calculate the latter product

$$\mathrm{M}_+\mathrm{M}_-{}^T \equiv (m_{i,j})_{i,j=1,\dots,l},$$

we start with the obvious relation

$$m_{i,j} = \sum_{k=1}^{l} \lambda_{i-1}^k \lambda_{j-1}^{-k} = \sum_{k=1}^{l} \left(\frac{\lambda_{i-1}}{\lambda_{j-1}}\right)^k = \sum_{k=1}^{l} \lambda_{i-j}^k, \tag{2.2}$$

where it was taken into account that $\lambda_k/\lambda_j = \lambda_{k-j}$ and $\lambda_0 = 1$. Clearly, the number $\lambda_{i-j} = \lambda_1^{i-j}$ is the lth root of unity for any integers i and j.

It is known that if λ is the lth root of unity then

$$\sum_{k=1}^{l} \lambda^k = \begin{cases} l, & \lambda = 1, \\ 0, & \lambda \neq 1. \end{cases} \tag{2.3}$$

Indeed, for $\lambda = 1$ it is obvious, and for $\lambda \neq 1$ one has the relations

$$\lambda + \lambda^2 + \dots + \lambda^{l-1} + \lambda^l = \frac{1}{1-\lambda}\left(\lambda + \lambda^2 + \dots + \lambda^{l-1} + \lambda^l\right)(1-\lambda) = \frac{1}{1-\lambda}\left(\lambda - \lambda^{l+1}\right) = 0.$$

Therefore,

$$m_{i,j} = \begin{cases} l, & i = j, \\ 0, & i \neq j, \end{cases}$$

and consequently

$$A\mathrm{M}_+\mathrm{diag}\left(\mu_1^{-1}, \dots, \mu_l^{-1}\right)\mathrm{M}_-{}^T = lE.$$

The latter relation yields the lemma's statement. □

Theorem 2.4 *Let* $\mu_k = a_1\lambda_0^k + \dots + a_l\lambda_{l-1}^k \neq 0,\ k = 1, \dots, l$, *where* $\{\lambda_k\}$ *are the* l*th roots of unity. Then the solution of the system of algebraic equations* $Ab = g$ *can be written in the form*

$$b = (b_i)_{i=1,\dots,l} = \frac{1}{l}\left(\sum_{k=1}^{l} \frac{1}{\mu_k} \sum_{j=1}^{l} \lambda_k^{i-j} g_j\right)_{i=1,\dots,l}.$$

Proof Let us find elements of the inverse matrix that exists by Lemma 2.3. Similar to (2.2) we have

$$\left(A^{-1}\right)_{ij} = \frac{1}{l}\Big(\mathrm{M}_+\mathrm{diag}^{-1}(\mu_1, \dots, \mu_l)\,\mathrm{M}_-{}^T\Big)_{ij} = \frac{1}{l}\sum_{k=1}^{l} \frac{\lambda_{i-1}^k}{\mu_k}\lambda_{j-1}^{-k} = \frac{1}{l}\sum_{k=1}^{l} \frac{\lambda_{i-j}^k}{\mu_k} = \frac{1}{l}\sum_{k=1}^{l} \frac{\lambda_k^{i-j}}{\mu_k}. \tag{2.4}$$

It follows from (2.4) that

$$b_i = \left(A^{-1}g\right)_i = \frac{1}{l}\sum_{j=1}^{l} g_j \sum_{k=1}^{l} \frac{\lambda_k^{i-j}}{\mu_k} = \frac{1}{l}\sum_{k=1}^{l} \frac{1}{\mu_k}\sum_{j=1}^{l} \lambda_k^{i-j} g_j.$$

The theorem is proved. □

3. Uniqueness

In order to study the uniqueness of the solution to the Dirichlet problem (1.1), (1.2), we start with the following statement.

Lemma 3.1 *The operator $I_S u(x) = u(Sx)$ and the Laplace operator Δ commute: $\Delta I_S u(x) = I_S \Delta u(x)$. Operators $\Lambda = \sum_{i=1}^n x_i u_{x_i}(x)$ and I_S also commute: $\Lambda I_S u(x) = I_S \Lambda u(x)$, and the equality $\nabla I_S = I_S S^T \nabla$ holds.*

Proof We write the orthogonal matrix S in the form $S = (s_{ij})_{i,j=1,\dots,l}$. Since

$$\frac{\partial}{\partial x_i} I_S u(x) = \frac{\partial}{\partial x_i} u(Sx) = \frac{\partial}{\partial x_i} u((S^1_{row}, x), \dots, (S^n_{row}, x)) = \sum_{j=1}^n s_{ji} I_S u_{x_j}(x) = (S^i_{col}, I_S \nabla u(x)) = I_S (S^i_{col}, \nabla) u(x),$$

we obtain

$$\Lambda I_S u(x) = \Lambda u(Sx) = \sum_{i=1}^n x_i \frac{\partial}{\partial x_i} u(Sx) = \sum_{i=1}^n x_i \left(S^i_{col}, I_S \nabla u(x)\right) = \left(\sum_{i=1}^n x_i S^i_{col}, I_S \nabla u(x)\right)$$

$$= (Sx, I_S \nabla u(x)) = I_S (x, \nabla u(x)) = I_S \Lambda u(x).$$

Furthermore, one has the relation

$$\frac{\partial^2}{\partial x_i^2} I_S u(x) = \frac{\partial}{\partial x_i} I_S (S^i_{col}, \nabla) u(x) = I_S (S^i_{col}, \nabla)^2 u(x),$$

and therefore,

$$\Delta I_S u(x) = \sum_{i=1}^n I_S (S^i_{col}, \nabla)^2 u(x) = I_S \left| \left((S^1_{col}, \nabla), \dots, (S^n_{col}, \nabla) \right) \right|^2 u(x)$$

$$= I_S \left| S^T \nabla \right|^2 u(x) = I_S (S^T \nabla, S^T \nabla) u(x) = I_S (S S^T \nabla, \nabla) u(x) = I_S \Delta u(x).$$

Finally, the following relations hold:

$$\nabla I_S u(x) = I_S ((S^1_{col}, \nabla), \dots, (S^n_{col}, \nabla)) u(x) = I_S (S^T \nabla) u(x).$$

The lemma is proved completely. □

Corollary 3.2 *If a function $u(x)$ is harmonic in Ω, then the function $u(Sx) = I_S u(x)$ is also harmonic in Ω.*

Indeed, by Lemma 3.1, the equality $\Delta u(x) = 0$ yields the relation $\Delta I_S u(x) = I_S \Delta u(x) = 0$.

Corollary 3.3 *If a function $u(x)$ is harmonic in Ω, then it satisfies the homogeneous equation (1.1) in Ω.*

Indeed, according to Lemma 3.1, for any $x \in \Omega$, we have

$$Lu(x) = -\sum_{k=1}^l a_k \Delta u \left(S^{k-1} x\right) = \sum_{k=1}^l a_k \Delta I_{S^{k-1}} u\,(x) = \sum_{k=1}^l a_k I_{S^{k-1}} \Delta u\,(x) = 0.$$

The converse statement is also true.

Lemma 3.4 *Suppose that a function* $u \in C^2(\Omega)$ *satisfies the homogeneous equation* (1.1). *Then, under the condition* $\det A \neq 0$, *the function* $u(x)$ *is harmonic in the domain* Ω.

Proof Let a function $u \in C^2(\Omega)$ satisfy the homogeneous equation (1.1). Denote

$$v(x) = \sum_{k=1}^{l} a_k u(S^{k-1}x). \tag{3.1}$$

It is obvious that $v(x) \in C^2(\Omega)$ and $\Delta v(x) = 0$, $x \in \Omega$, i.e. the function $v(x)$ is harmonic in the domain Ω. It follows from Corollary 3.2 that functions $v(S^k x)$ are also harmonic in Ω. On the other hand, due to the condition $S^l = E$, equation (3.1) yields the following relations:

$$\begin{aligned} v(Sx) &= a_l u(x) + a_1 u(Sx) + \ldots + a_{l-1} u(S^{l-1}x), \\ v(S^2x) &= a_{l-1} u(x) + a_l u(Sx) + \ldots + a_{l-2} u(S^{l-1}x), \\ &\cdots \quad \cdots \quad \cdots \\ v(S^{l-1}x) &= a_2 u(x) + a_3 u(Sx) + \ldots + a_1 u(S^{l-1}x). \end{aligned} \tag{3.2}$$

Therefore, the functions $u(x), u(Sx), \ldots, u(S^{l-1}x)$ satisfy the system of algebraic equations (3.1), (3.2) with matrix A:

$$\begin{pmatrix} v(x) \\ v(Sx) \\ \vdots \\ v(S^{l-1}x) \end{pmatrix} = \begin{pmatrix} a_1 & a_2 & \ldots & a_l \\ a_l & a_1 & \ldots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \ldots & a_1 \end{pmatrix} \begin{pmatrix} u(x) \\ u(Sx) \\ \vdots \\ u(S^{l-1}x) \end{pmatrix}.$$

As the determinant of this system does not vanish, we may apply Theorem 2.4 with

$$b = \left(u(x), u(Sx), \ldots, u(S^{l-1}x)\right)^T$$

and

$$g = \left(v(x), v(Sx), \ldots, v(S^{l-1}x)\right)^T.$$

It follows from Theorem 2.4 with $i = 1$ that

$$u(x) = b_1 = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\mu_k} \sum_{j=1}^{l} \lambda_k^{1-j} g_j = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\mu_k} \sum_{j=1}^{l} \lambda_k^{1-j} v(S^{j-1}x) = \sum_{j=1}^{l} v(S^{j-1}x) \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\lambda_k^{j-1} \mu_k},$$

where, according to (2.1), $\mu_k = a_1 \lambda_0^k + \ldots + a_l \lambda_{l-1}^k$ and $\lambda_k = \exp\left(i \frac{2\pi}{l} k\right) = \lambda_1^k$. If we denote

$$b_j = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\lambda_k^{j-1} \mu_k},$$

for $j = 1, 2, \ldots, l$, then we get

$$u(x) = \sum_{j=1}^{l} b_j v(S^{j-1}x) = b_1 v(x) + b_2 v(Sx) + \ldots + b_l v(S^{l-1}x). \tag{3.3}$$

As was noted above, the functions $v(S^k x)$ are harmonic in Ω for all $k = 0, 1, \dots, l-1$, and thus the function $u(x)$ in (3.3) is also harmonic on the domain Ω. The lemma is proved. □

Lemma 3.4 yields the following statement.

Theorem 3.5 *Suppose that the inequalities $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$ hold for all $k = 1, \dots, l$ and solutions of Dirichlet, Neumann, and Robin problems exist. Then:*

1) the solution of the Dirichlet problem (1.1), (1.2) is unique;

2) the solution of the Neumann problem (1.1), (1.3) is unique up to a constant term;

3) for any $c > 0$, the solution of the Robin problem (1.1), (1.4) is unique.

Proof 1) Let us prove that the homogeneous problem (1.1), (1.2) has only the trivial solution, and hence the solution of the nonhomogeneous problem (1.1), (1.2) is unique. Let $u(x)$ be a solution of the homogeneous problem (1.1), (1.2). If $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$ for all $k = 1, \dots, l$, then by Lemma 2.1, $\det A \neq 0$. It follows from Lemma 3.4 that the function $u(x)$ is harmonic in the domain Ω and satisfies the homogeneous condition (1.2). Therefore, the function $u(x)$ is a solution to the following Dirichlet problem:

$$\Delta u(x) = 0,\ x \in \Omega; \quad u(x)\big|_{\partial\Omega} = 0.$$

As the solution of this Dirichlet problem is unique, we have $u(x) \equiv 0$.

2) If $u(x)$ is a solution of the homogeneous problem (1.1), (1.3), then by Lemma 3.4, the function $u(x)$ is harmonic in the domain Ω and hence satisfies the boundary condition of the following Neumann problem:

$$\Delta u(x) = 0,\ x \in \Omega; \quad \frac{\partial u(x)}{\partial \nu}\Big|_{\partial\Omega} = 0.$$

This problem has only constant solutions: $u(x) \equiv C$.

3) In this case the function $u(x)$ satisfies the homogeneous Robin problem:

$$\Delta u(x) = 0, x \in \Omega; \left.\frac{\partial u(x)}{\partial \nu} + cu(x)\right|_{\partial\Omega} = 0.$$

If $c > 0$ then the unique solution to this problem is the function $u(x) \equiv 0$. The theorem is proved. □

4. Existence of the solution to the Dirichlet problem

In this section we study the existence of solutions to the Dirichlet problem. Let

$$P(x,y) = \frac{1}{\omega_n}\frac{1-|x|^2}{|x-y|^n}$$

be the Poisson kernel, ω_n be an area of the unit sphere $\partial\Omega$, and $G(x,y)$ be the Green's function of the Dirichlet problem, which can be represented in the following form (see, e.g., [9]):

$$G(x,y) = \frac{1}{\omega_n}\left[E(x,y) - E\left(x|y|, \frac{y}{|y|}\right)\right],\ E(x,y) = \left\{\begin{array}{l} -\ln|x-y|,\ n=2, \\ (n-2)^{-1}|x-y|^{2-n},\ n \geq 3. \end{array}\right.$$

Let us start with the following auxiliary statements.

Lemma 4.1 *Assume that $g(x)$ is an arbitrary continuous function on $\partial\Omega$ or on Ω. Then the equalities*

$$\int_{\partial\Omega} g(S^k y)\, ds_y = \int_{\partial\Omega} g(y)\, ds_y, \quad \int_{\Omega} g(S^k y)\, dy = \int_{\Omega} g(y)\, dy$$

hold for all $k \in \mathbb{N}$.

Proof Let the function $w(x)$ be a solution of the Dirichlet problem for the Laplace equation in Ω with the condition $w(x) = g(x)$ on the boundary $\partial\Omega$. Then the function $w(S^k x)$ is a solution of the Dirichlet problem for the Laplace equation in Ω (Corollary 3.3) with the condition $w(S^k x) = g(S^k x)$ on $\partial\Omega$. It is known that solutions of these problems could be represented via the Poisson integrals

$$w(x) = \int_{\partial\Omega} P(x,y) g(y)\, ds_y, \quad w(S^k x) = \int_{\partial\Omega} P(x,y) g(S^k y)\, ds_y.$$

Since

$$P(0,y) = \frac{1}{\omega_n} \frac{1}{|y|^n} = \frac{1}{\omega_n},$$

for $y \in \partial\Omega$, we have the relation

$$\frac{1}{\omega_n} \int_{\partial\Omega} g(y)\, ds_y = w(0) = \frac{1}{\omega_n} \int_{\partial\Omega} g(S^k y)\, ds_y,$$

which implies the first equality.

The second equality appears after the change of variables in the multiple integral

$$\int_{\Omega} g(Sy)\, dy = \int_{\Omega} g(z) |\det S^T|\, dz = \int_{\Omega} g(y)\, dy.$$

The lemma is proved. □

Lemma 4.2 *Let $\mu_k = a_1\lambda_0^k + \ldots + a_l \lambda_{l-1}^k \neq 0$ for all $k = 1, \ldots l$. Then the matrix A^{-1} has a structure similar to matrix A:*

$$\begin{pmatrix} a_1 & a_2 & \ldots & a_l \\ a_l & a_1 & \ldots & a_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \ldots & a_1 \end{pmatrix}^{-1} = \begin{pmatrix} b_1 & b_2 & \ldots & b_l \\ b_l & b_1 & \ldots & b_{l-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_2 & b_3 & \ldots & b_1 \end{pmatrix},$$

where, as in (3.3),

$$b_j = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\lambda_k^{j-1} \mu_k} \tag{4.1}$$

for $j = 1, 2, \ldots, l$, and μ_k are defined in (2.1). Moreover, for $k = 1, 2, \ldots, l$, the equality $\mu_k(b) = 1/\mu_k(a)$ holds where $\mu_k(a) = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k$ and $\mu_k(b) = b_1\lambda_0^k + \ldots + b_l\lambda_{l-1}^k$.

Proof If we consider the indexes of numbers a_k modulo l then it is easy to see that the matrix A can be written as $A = (a_{j-i+1})_{i,j=1,\dots,l}$. By formula (2.4) from Theorem 2.4, we find the inverse of A explicitly:

$$(A^{-1})_{i,j} = \frac{1}{l}\sum_{k=1}^{l}\frac{\lambda_k^{i-j}}{\mu_k} = \frac{1}{l}\sum_{k=1}^{l}\frac{1}{\mu_k\lambda_k^{j-i}},$$

and, since the indexes i and j of elements in the inverse matrix are powers of the numbers λ_k, then they can be calculated modulo l and the following equality holds:

$$(A^{-1})_{i,j} = \frac{1}{l}\sum_{k=1}^{l}\frac{1}{\mu_k\lambda_k^{j-i+1-1}} = b_{j-i+1},$$

where

$$b_j = \frac{1}{l}\sum_{k=1}^{l}\frac{1}{\lambda_k^{j-1}\mu_k}.$$

Thus, we have $A^{-1} = (b_{j-i+1})_{i,j=1,\dots,l}$.

Let us now calculate $\mu_k(b)$. Taking into account equality (2.3), we obtain the following relations:

$$\mu_k(b) = \sum_{j=1}^{l} b_j\lambda_{j-1}^k = \frac{1}{l}\sum_{j=1}^{l}\lambda_{j-1}^k\sum_{p=1}^{l}\frac{1}{\lambda_p^{j-1}\mu_p(a)} = \frac{1}{l}\sum_{p=1}^{l}\frac{1}{\mu_p(a)}\sum_{j=1}^{l}\frac{\lambda_{j-1}^k}{\lambda_p^{j-1}} = \sum_{p=1}^{l}\frac{1}{\mu_p(a)}\frac{1}{l}\sum_{j=1}^{l}\frac{\lambda_{j-1}^k}{\lambda_{j-1}^p}$$

$$= \sum_{p=1}^{l}\frac{1}{\mu_p(a)}\frac{1}{l}\sum_{j=1}^{l}\lambda_{j-1}^{k-p} = \frac{1}{\mu_k(a)}.$$

The lemma is proved. □

Remark 4.3 *Obviously, one has the equalities*

$$\mu_l(a) = a_1\lambda_0^l + \ldots + a_l\lambda_{l-1}^l = \sum_{j=1}^{l} a_j, \quad \mu_l(b) = b_1\lambda_0^l + \ldots + b_l\lambda_{l-1}^l = \sum_{i=1}^{l} b_i,$$

and therefore, $\sum_{j=1}^{l} a_j \sum_{i=1}^{l} b_i = 1$.

The following theorem is the main one for the Dirichlet problem (1.1), (1.2).

Theorem 4.4 *Let numbers* $\{a_k : k = 1,\dots,l\}$ *satisfy the inequalities* $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$ *for all* $k = 1,\dots,l$, *where* $\{\lambda_k\}$ *are* l*th roots of unity. Then, for arbitrary functions* $f \in C^1(\bar{\Omega})$, $g \in C(\partial\Omega)$, *the solution of the problem* (1.1), (1.2) *exists, is unique, and can be represented in the following integral form:*

$$u(x) = \int_{\Omega} G_S(x,y)f(y)\,dy + \int_{\partial\Omega} P_S(x,y)\sum_{k=1}^{l} a_k g(S^{k-1}y)\,ds_y, \tag{4.2}$$

where

$$G_S(x,y)=\sum_{q=1}^{l} b_q G(S^{q-1}x,y),\quad P_S(x,y)=\sum_{q=1}^{l} b_q P(S^{q-1}x,y), \tag{4.3}$$

and b_q *is defined by* (4.1) *for all* $q=1,\dots,l$.

Proof Consider the following Dirichlet problem for the function $v(x)$ in the domain Ω:

$$-\Delta v(x)=f(x),\ x\in\Omega;\quad v(s)=\sum_{k=1}^{l} a_k g(S^{k-1}s)\equiv h(s),\ s\in\partial\Omega. \tag{4.4}$$

It is clear that $g\in C(\partial\Omega)$ implies $h\in C(\partial\Omega)$, and therefore, the solution of the Dirichlet problem (4.4) exists and is unique. It is also known (see, e.g., [9], p. 35) that for the given functions $f(x)$ and $h(s)$ in (4.4), the solution of problem (4.4) can be represented in the following form:

$$v(x)=\int_\Omega G(x,y)f(y)\,dy+\sum_{k=1}^{l} a_k\int_{\partial\Omega} P(x,y)g(S^{k-1}y)\,ds_y. \tag{4.5}$$

Consider the vector $V=\big(v(x),v(Sx),\dots,v(S^{l-1}x)\big)^T$. It follows from Lemma 4.2 that the matrix A^{-1} repeats the structure of the matrix A. Thus, similarly to equalities (3.1) and (3.2), we define the vector $U=\big(u(x),u(Sx),\dots,u(S^{l-1}x)\big)^T$ by the equality $U=A^{-1}V$. As $\mu_k=a_1\lambda_0^k+\dots+a_l\lambda_{l-1}^k\neq 0$, then, by Lemma 2.1, A is nonsingular and thus $\det A^{-1}\neq 0$. Since $AU=V$, it follows from (3.3) that the function $u(x)$ is uniquely defined via the function $v(x)$ from (4.5) by the formula

$$u(x)=\sum_{q=1}^{l} b_q v(S^{q-1}x), \tag{4.6}$$

where the numbers b_q are defined in (4.1).

Let us check that the function $u(x)$ in (4.6) is the solution of the problem (1.1), (1.2). On one hand, we have the implications

$$f\in C^1(\bar\Omega),\ g\in C(\partial\Omega)\Rightarrow v\in C^2(\Omega)\cap C(\bar\Omega)\Rightarrow u\in C^2(\Omega)\cap C(\bar\Omega).$$

On the other hand, due to Lemma 3.1 and equality (4.4), we get the equalities

$$\Delta u(x)=\sum_{q=1}^{l} b_q\Delta v(S^{q-1}x)=\sum_{q=1}^{l} b_q I_{S^{q-1}}\Delta v(x)=\sum_{q=1}^{l} b_q I_{S^{q-1}}\Delta v(x)=-\sum_{q=1}^{l} b_q I_{S^{q-1}}f(x)=-\sum_{q=1}^{l} b_q f(S^{q-1}x)$$

for all $x\in\Omega$.

Hence, taking into account the relations $b_0=b_l$ and $S^l=E$, we obtain

$$-I_S\Delta u(x)=\Delta u(Sx)=\sum_{q=1}^{l} b_q f(S^q x)=\sum_{q=1}^{l-1} b_q f(S^q x)+b_l f(x)=b_0 f(x)+\sum_{q=2}^{l} b_{q-1}f(S^{q-1}x)=\sum_{q=1}^{l} b_{q-1}f(S^{q-1}x).$$

Similarly, assuming $b_{-1} = b_{l-1}$, we get the relations

$$-\Delta u(S^2 x) = \sum_{q=1}^{l} b_{q-1} f(S^q x) = \sum_{q=1}^{l-1} b_{q-1} f(S^q x) + b_{l-1} f(x) = b_{-1} f(x) + \sum_{q=2}^{l} b_{q-2} f(S^{q-1} x)$$

$$= \sum_{q=1}^{l} b_{q-2} f(S^{q-1} x).$$

Continuing this process and assuming further that $b_{-p} = b_{l-p}$, we come to the general relation

$$\Delta u(S^{p-1} x) = -\sum_{q=1}^{l} b_{q-p+1} f(S^{q-1} x)$$

for any $p = 1, \ldots, l$. Taking into account these equalities, we can calculate the left-hand side of (1.1):

$$Lu(x) = -\sum_{p=1}^{l} a_p \Delta u(S^{p-1} x) = \sum_{p=1}^{l} a_p \sum_{q=1}^{l} b_{q-p+1} f(S^{q-1} x) = \sum_{q=1}^{l} f(S^{q-1} x) \sum_{p=1}^{l} a_p b_{q-p+1}.$$

Inserting here the values b_p from (4.1) and μ_k from (2.1), one can simplify the sum:

$$\sum_{p=1}^{l} a_p b_{q-p+1} = \frac{1}{l} \sum_{p=1}^{l} a_p \sum_{k=1}^{l} \frac{1}{\lambda_k^{q-p} \mu_k} = \frac{1}{l} \sum_{k=1}^{l} \frac{1}{\lambda_k^{q-1} \mu_k} \sum_{p=1}^{l} \lambda_{p-1}^{k} a_p = \frac{1}{l} \sum_{k=1}^{l} \frac{\mu_k}{\lambda_k^{q-1} \mu_k}$$

$$= \frac{1}{l} \sum_{k=1}^{l} \lambda_k^{1-q} = \frac{1}{l} \sum_{k=1}^{l} \lambda_{1-q}^{k}.$$

It follows from (2.3) that

$$\sum_{p=1}^{l} a_p b_{q-p+1} = \begin{cases} 1, & q = 1, \\ 0, & q \neq 1, \end{cases} \tag{4.7}$$

and thus, equation (1.1) holds true:

$$-\sum_{p=1}^{l} a_p \Delta u(S^{p-1} x) = f(x).$$

Let us now verify the boundary condition (1.2). For $s \in \partial\Omega$, we have

$$h(s) = \sum_{k=1}^{l} a_k g(S^{k-1} s),$$

and hence

$$h(Ss) = \sum_{k=1}^{l} a_k g(S^k s) = \sum_{k=1}^{l-1} a_k g(S^k s) + a_l g(s) = a_l g(s) + \sum_{k=2}^{l} a_{k-1} g(S^{k-1} s) = \sum_{k=1}^{l} a_{k-1} g(S^{k-1} s),$$

and, similarly by induction, we come to the relations

$$h(S^{p-1}s) = I_S h(S^{p-2}s) = \sum_{k=1}^{l} a_{k-p+2} g(S^k s) = \sum_{k=1}^{l-1} a_{k-p+2} g(S^k s) + a_{l-p+2} g(s) = a_{l-p+2} g(s) + \sum_{k=2}^{l} a_{k-p+1} g(S^{k-1}s)$$

$$= \sum_{k=1}^{l} a_{k-p+1} g(S^{k-1}s).$$

Therefore,

$$u(s)|_{\partial\Omega} = \sum_{p=1}^{l} b_p v(S^{p-1}s)|_{\partial\Omega} = \sum_{p=1}^{l} b_p h(S^{p-1}s) = \sum_{p=1}^{l} b_p \sum_{k=1}^{l} a_{k-p+1} g(S^{k-1}s) = \sum_{k=1}^{l} g(S^{k-1}s) \sum_{p=1}^{l} a_{k-p+1} b_p.$$

In order to calculate the latter internal sum, we change the index $q = k-p+1 \Rightarrow p = k-q+1$ and apply equality (4.7):

$$\sum_{p=1}^{l} a_{k-p+1} b_p = \sum_{q=1}^{l} a_q b_{k-q+1} = \begin{cases} 1, & k = 1, \\ 0, & k \neq 1. \end{cases}$$

Hence, we get the equality

$$u(s)|_{\partial\Omega} = \sum_{k=1}^{l} g(S^{k-1}s) \sum_{p=1}^{l} a_{k-p+1} b_p = g(s),$$

and therefore the boundary condition (1.2) for the function $u(x)$ holds true.

Furthermore, it follows from (4.5) that

$$v(S^{q-1}x) = \int_\Omega G(S^{q-1}x, y) f(y)\, dy + \int_{\partial\Omega} P(S^{q-1}x, y) \sum_{k=1}^{l} a_k g(S^{k-1}y)\, ds_y.$$

Substituting this expression in (4.6) instead of $v(S^{q-1}x)$ and taking into account (4.3), we obtain the relation

$$u(x) = \int_\Omega \left[\sum_{q=1}^{l} b_q G(S^{q-1}x, y)\right] f(y)\, dy + \int_{\partial\Omega} \left[\sum_{q=1}^{l} b_q P(S^{q-1}x, y)\right] \sum_{k=1}^{l} a_k g(S^{k-1}y)\, ds_y$$

$$= \int_\Omega G_S(x, y) f(y)\, dy + \int_{\partial\Omega} P_S(x, y) \sum_{k=1}^{l} a_k g(S^{k-1}y)\, ds_y.$$

Thus, representation (4.2) for function $u(x)$ is verified. The theorem is proved. □

Remark 4.5 *Applying Lemma 4.1, one can rewrite the last term on the right-hand side of (4.2) in the following* *form:*

$$\int_{\partial\Omega} P_S(x,y)\sum_{k=1}^{l} a_k g(S^{k-1}y)\,ds_y = \sum_{k=1}^{l} a_k \int_{\partial\Omega} P_S(x,y)g(S^{k-1}y)\,ds_y = \sum_{k=1}^{l} a_k \int_{\partial\Omega} P_S(x,\left(S^{k-1}\right)^T z)g(z)\,ds_z$$

$$= \int_{\partial\Omega} \left(\sum_{k=1}^{l} a_k P_S(x,\left(S^{k-1}\right)^T z)\right) g(z)\,ds_z = \int_{\partial\Omega} \left(\sum_{k=1}^{l} a_k \sum_{q=1}^{l} b_q P(S^{q-1}x,\left(S^{k-1}\right)^T z))\right) g(z)\,ds_z.$$

Example 4.6 *Let S be a symmetric matrix satisfying $S^2 = E$ $(l = 2)$. In this case the problem (1.1), (1.2) takes the form*

$$-a_1\Delta u(x) - a_2\Delta u(Sx) = f(x),\ x \in \Omega; \quad u(s) = g(s),\ s \in \partial\Omega.$$

Here we have $\lambda_1 = e^{i\pi} = -1$, $\lambda_2 = e^{2i\pi} = 1$, $\mu_1 = a_1 - a_2$, $\mu_2 = a_1 + a_2$, and

$$A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_1 \end{pmatrix}, \quad \det A = \mu_1 \cdot \mu_2 = a_1^2 - a_2^2.$$

Suppose that $a_1^2 - a_2^2 \neq 0 \Leftrightarrow a_1 \neq \pm a_2$. Then, by (4.1), we obtain the relations

$$b_1 = \frac{1}{2}\sum_{k=1}^{2} \frac{1}{\lambda_k^0 \mu_k} = \frac{1}{2}\left(\frac{1}{\mu_1} + \frac{1}{\mu_2}\right) = \frac{1}{2}\left(\frac{1}{a_1 - a_2} + \frac{1}{a_2 + a_1}\right) = \frac{a_1}{a_1^2 - a_2^2},$$

$$b_2 = \frac{1}{2}\sum_{k=1}^{2} \frac{1}{\lambda_k^1 \mu_k} = \frac{1}{2}\left(\frac{1}{-\mu_1} + \frac{1}{\mu_2}\right) = \frac{1}{2}\left(-\frac{1}{a_1 - a_2} + \frac{1}{a_2 + a_1}\right) = \frac{-a_2}{a_1^2 - a_2^2}.$$

According to formula (4.3) of Theorem 4.4, we find the functions $G_S(x,y)$ and $P_S(x,y)$ by the following equalities:

$$G_S(x,y) = \sum_{q=1}^{2} b_q G(S^{q-1}x,y) = b_1 G(x,y) + b_2 G(Sx,y) = \frac{a_1 G(x,y) - a_2 G(Sx,y)}{a_1^2 - a_2^2},$$

$$P_S(x,y) = \sum_{q=1}^{2} b_q P(S^{q-1}x,y) = \frac{a_1 P(x,y) - a_2 P(Sx,y)}{a_1^2 - a_2^2},$$

and it follows from (4.2) that the solution takes the form

$$u(x) = \int_{\Omega} G_S(x,y)f(y)\,dy + \int_{\partial\Omega} P_S(x,y)\sum_{k=1}^{l} a_k g(S^{k-1}y)\,ds_y = \int_{\Omega} \frac{a_1 G(x,y) - a_2 G(Sx,y)}{a_1^2 - a_2^2} f(y)\,dy$$

$$+ \int_{\partial\Omega} \frac{a_1 P(x,y) - a_2 P(Sx,y)}{a_1^2 - a_2^2}\left(a_1 g(y) + a_2 g(Sy)\right) ds_y.$$

Applying Lemma 4.1 and the symmetry of S, the last integral on the right-hand side of this representation can be rewritten in the following form:

$$I = \int_{\partial\Omega} \frac{a_1 P(x,y) - a_2 P(Sx,y)}{a_1^2 - a_2^2}\left(a_1 g(y) + a_2 g(Sy)\right) ds_y$$

$$= \int_{\partial\Omega} \frac{\left(a_1\left(a_1 P(x,y) - a_2 P(Sx,y)\right) + a_2\left(a_1 P(x,Sy) - a_2 P(Sx,Sy)\right)\right)}{a_1^2 - a_2^2} g(y)\,ds_y.$$

If we note that, for $y \in \partial\Omega$, *the equalities*

$$P(Sx,y)=\frac{1}{\omega_n}\frac{1-|Sx|^2}{|Sx-y|^n}=\frac{1}{\omega_n}\frac{1-|x|^2}{|x-Sy|^n}=P(x,Sy),$$

$$P(Sx,Sy)=\frac{1}{\omega_n}\frac{1-|Sx|^2}{|Sx-Sy|^n}=\frac{1}{\omega_n}\frac{1-|x|^2}{|x-y|^n}=P(x,y)$$

hold, then the integral I *equals*

$$I=\frac{1}{a_2^2-a_1^2}\int_{\partial\Omega}\left(a_1^2-a_2^2\right)P(x,y)g(y)\,ds_y=\int_{\partial\Omega}P(x,y)g(y)\,ds_y,$$

and finally

$$u(x)=\int_{\Omega}\frac{a_1G(x,y)-a_2G(Sx,y)}{a_1^2-a_2^2}f(y)\,dy+\int_{\partial\Omega}P(x,y)g(y)\,ds_y.$$

Consider a particular case of the above problem when $f(x)=-x_i$ *and* $g(s)=s_j^2$ *with some* i,j: $1\le i,j\le n$. *Then the auxiliary problem* (4.4) *takes the form*

$$\Delta v(x)=x_i,\ x\in\Omega;\ v\big|_{\partial\Omega}=(a_1+a_2)s_j^2.$$

Here it is more convenient to use the results of [10] instead of the integral representation for $v(x)$. *The straightforward calculation shows that*

$$v(x)=(a_1+a_2)x_j^2+\left(1-|x|^2\right)\left(\frac{a_1+a_2}{n}-\frac{x_i}{2(n+2)}\right),$$

and therefore the problem's solution is given explicitly:

$$\begin{aligned}u(x)=\frac{a_1}{a_1^2-a_2^2}&\left((a_1+a_2)x_j^2+\left(1-|x|^2\right)\left(\frac{a_1+a_2}{n}-\frac{x_i}{2(n+2)}\right)\right)\\&-\frac{a_2}{a_1^2-a_2^2}\left((a_1+a_2)x_j^2+\left(1-|x|^2\right)\left(\frac{a_1+a_2}{n}+\frac{x_i}{2(n+2)}\right)\right)\\&=x_j^2+\left(1-|x|^2\right)\left(\frac{1}{n}-\frac{x_i}{2(n+2)(a_1-a_2)}\right).\end{aligned}$$

It is easy to verify this representation. Obviously, the boundary condition holds true and the validity of the equation follows from the relations

$$\begin{aligned}a_1\Delta u(x)+a_2\Delta u(-x)&=(a_1+a_2)\Delta x_j^2-\frac{a_1+a_2}{n}\Delta|x|^2\\&+(a_1-a_2)\Delta\left(\frac{x_i|x|^2}{2(n+2)(a_1-a_2)}\right)=(a_1+a_2)(2-2)+2(n+2)\frac{x_i}{2(n+2)}=x_i.\end{aligned}$$

Here we applied the equality $\Delta\left(|x|^{2m}P_s(x)\right)=2m(2m+2s+n-2)|x|^{2m-2}P_s(x)$ *[11] where* $P_s(x)$ *is any homogeneous harmonic polynomial of degree* s.

5. Existence of the solution to the Neumann problem

Let us find necessary and sufficient conditions for the solvability of the Neumann problem (1.1), (1.3). Let $G_N(x,y)$ denote the Green's function of the classical Neumann problem. Note that the explicit form of this Green's function in the ball Ω for the cases $n=2$ and $n=3$ is well known (see, e.g., [9, 15]), while in the case of dimensions $n \geq 4$ it is constructed in [25, 26].

Theorem 5.1 *Let* $detA \neq 0$, $f \in C^1(\bar{\Omega})$, $g \in C(\partial\Omega)$. *Then the condition*

$$\int_\Omega f(x)\,dx + \left(\sum_{k=1}^{l} a_k\right)\int_{\partial\Omega} g(x)\,dS_x = 0 \tag{5.1}$$

is necessary and sufficient for the solvability of the problem (1.1), (1.3). *If the solution of the problem exists then it is unique up to a constant term and can be represented in the form*

$$u(x) = \int_\Omega G_{N,S}(x,y)f(y)\,dy + \int_{\partial\Omega} G_{N,S,l}(x,y)g(y)\,dS_y, \tag{5.2}$$

where

$$G_{N,S}(x,y) = \sum_{m=0}^{l-1} b_m G_N(S^{m-1}x,y), \quad G_{N,S,l}(x,y) = \sum_{m=1}^{l} b_m \sum_{k=1}^{l} a_k G_N\left(S^{m-1}x, \left(S^{k-1}\right)^T y\right),$$

and the coefficients b_m *are given by formula* (4.1).

Proof Suppose that solution $u(x)$ of the problem (1.1), (1.3) exists, and the operator $\Lambda = \sum_{j=1}^n x_j \frac{\partial}{\partial x_j}$ is defined in Lemma 3.1. For convenience, we assume that the function $u(x)$ belongs to the class $C^3(\Omega)\cap C^1(\bar{\Omega})$ (it is sufficient to require that $f(x) \in C^\lambda(\bar{\Omega})$, $g(x) \in C^{\lambda+1}(\partial\Omega)$ with $\lambda > 1$). We apply the operator Λ to the function $u(x)$ and introduce $w(x) = \Lambda u(x)$. Then, taking into account the equality $\Delta\Lambda u(x) = (\Lambda+2)\,\Delta u(x)$, $x \in \Omega$, and the commutability of the operators Λ and I_S (by Lemma 3.1), we obtain the following relations:

$$Lw(x) = -\sum_{k=1}^{l} a_k \Delta I_{S^{k-1}} \Lambda u\,(x) = -\sum_{k=1}^{l} a_k \Delta\Lambda I_{S^{k-1}} u\,(x)$$
$$= -\,(\Lambda+2)\sum_{k=1}^{l} a_k \Delta u\left(S^{k-1}x\right) = (\Lambda+2)\,Lu(x) = (\Lambda+2)\,f(x).$$

It follows from boundary condition (1.3) and the property of Λ that

$$w(x)\big|_{\partial\Omega} = \Lambda u(x)\big|_{\partial\Omega} = \frac{\partial u(x)}{\partial\nu}\Big|_{\partial\Omega} = g(x).$$

Thus, if $u(x)$ solves the problem (1.1), (1.3), then the function $w(x) = \Lambda u(x)$ satisfies the Dirichlet-type boundary value problem (1.1), (1.2):

$$Lw(x) = (\Lambda+2)\,f(x),\; x \in \Omega; \quad w(x)\big|_{\partial\Omega} = g(x). \tag{5.3}$$

Moreover, the equality $w(x) = \Lambda u(x)$ implies the necessity of the condition $w(0) = 0$.

Furthermore, if the functions $F(x) = (\Lambda + 2)\, f(x)$ and $g(x)$ are smooth enough, then, by Theorem 4.4, the solution of the problem (5.3) exists, is unique, and can be represented in the form (4.2), i.e.

$$w(x) = \int_\Omega G_S(x,y)F(y)\,dy + \int_{\partial\Omega} P_S(x,y)\sum_{k=1}^{l} a_k g(S^{k-1}y)\,ds_y, \tag{5.4}$$

where functions $G_S(x,y)$ and $P_S(x,y)$ are defined in (4.3).

Let us find the conditions under which the equality $w(0) = 0$ holds. It follows from the representation (5.4) that

$$w(0) = \int_\Omega G_S(0,y)F(y)\,dy + \int_{\partial\Omega} P_S(0,y)\sum_{k=1}^{l} a_k g(S^{k-1}y)\,ds_y.$$

Furthermore, in the case when $n > 2$, we obtain the relations

$$G_S(0,y) = \sum_{q=1}^{l} b_q G(0,y) = G(0,y)\sum_{q=1}^{l} b_q = C_1\left(|y|^{2-n} - 1\right),$$

$$P_S(0,y) = \sum_{q=1}^{l} b_q P(0,y) = P(0,y)\sum_{q=1}^{l} b_q = C_2,$$

where

$$C_1 = \frac{1}{(n-2)\omega_n}\mu_l(b), \quad C_2 = \frac{1}{\omega_n}\mu_l(b),$$

and, due to Remark 4.3, $\mu_l(b) = \sum_{q=1}^{l} b_q$.

Let us consider the following integral:

$$I_j \equiv \int_\Omega \left(|y|^{2-n} - 1\right) y_j \frac{\partial}{\partial y_j} f(y)\,dy.$$

Taking it by parts, we obtain the following equalities:

$$I_j = \int_\Omega y_j\left(|y|^{2-n} - 1\right)\frac{\partial}{\partial y_j} f(y)\,dy = \int_{\partial\Omega} y_j^2\left(|y|^{2-n} - 1\right) f(y)\,dy - \int_\Omega \frac{\partial}{\partial y_j}\left[y_j\left(|y|^{2-n} - 1\right)\right] f(y)\,dy$$
$$= -\int_\Omega \left[|y|^{2-n} - 1 + (2-n)y_j^2|y|^{-n}\right] f(y)\,dy = \int_\Omega \left[1 - |y|^{2-n} + (n-2)y_j^2|y|^{-n}\right] f(y)\,dy.$$

Since

$$F(y) = (\Lambda + 2)f(y) = \left(2 + \sum_{j=1}^{n} y_j \frac{\partial}{\partial y_j}\right) f(y)$$

and

$$\sum_{j=1}^{n}\left[1 - |y|^{2-n} + (n-2)y_j^2|y|^{-n}\right] = n\left(1 - |y|^{2-n}\right) + (n-2)|y|^{-n}\sum_{j=1}^{n} y_j^2$$
$$= n\left(1 - |y|^{2-n}\right) + (n-2)|y|^{2-n} = n - 2|y|^{2-n},$$

we get the relations

$$\int_\Omega G_S(0,y)F(y)\,dy = \int_\Omega G_S(0,y)\left(2+\sum_{j=1}^n y_j\frac{\partial}{\partial y_j}\right)f(y)\,dy$$
$$= C_1\left(\int_\Omega \left[n-2|y|^{2-n}\right]f(y)\,dy + 2\int_\Omega \left[|y|^{2-n}-1\right]f(y)\,dy\right)$$
$$= (n-2)C_1\int_\Omega f(y)\,dy = (n-2)\frac{1}{(n-2)\omega_n}\mu_l(b)\int_\Omega f(y)\,dy = \frac{\mu_l(b)}{\omega_n}\int_\Omega f(y)\,dy.$$

On the other hand, keeping in mind that $\mu_l(a)=\sum_{q=1}^l a_q$, and applying the equality $\mu_l(a)\mu_l(b)=1$ from Lemma 4.2, we get the relations

$$\int_{\partial\Omega} P_S(0,y)\sum_{k=1}^l a_k g(S^{k-1}y)\,ds_y = C_2\int_{\partial\Omega}\sum_{k=1}^l a_k g(S^{k-1}y)\,ds_y$$
$$= C_2\mu_l(a)\int_{\partial\Omega} g(y)\,ds_y = \mu_l(a)\mu_l(b)\frac{1}{\omega_n}\int_{\partial\Omega} g(y)\,ds_y = \frac{1}{\omega_n}\int_{\partial\Omega} g(y)\,ds_y.$$

Thus, $w(0)$ vanishes if and only if the following equality holds true:

$$w(0)=\mu_l(b)\frac{1}{\omega_n}\int_\Omega f(y)\,dy + \frac{1}{\omega_n}\int_{\partial\Omega} g(y)\,ds_y = 0.$$

Therefore, by Lemma 4.2, we finally obtain the condition

$$\int_\Omega f(y)\,dy + \mu_l(a)\int_{\partial\Omega} g(y)\,ds_y = 0.$$

Hence, the necessity of condition (5.1) for existence of the solution to the problem (1.1), (1.3) is verified.

Now let us show that the condition (5.1) is also sufficient for existence of the solution of (1.1), (1.3). For that purpose, we consider the following Neumann problem with respect to the function $v(x)$ in the domain Ω:

$$\begin{gathered} -\Delta v(x) = f(x),\ x\in\Omega; \\ \frac{\partial v(x)}{\partial\nu}\Big|_{\partial\Omega} = a_1 g(s) + a_2 g(Ss) + \ldots + a_l g(S^{l-1}s) \equiv h(s). \end{gathered} \tag{5.5}$$

It is known that the solvability condition for this problem is given by the following relation:

$$\int_\Omega f(x)\,dx + \int_{\partial\Omega} h(x)\,ds_x = 0 \Leftrightarrow \int_\Omega f(x)\,dx + \sum_{k=1}^l a_k\int_{\partial\Omega} g(S^{k-1}x)\,ds_x = 0. \tag{5.6}$$

Since, by Lemma 4.1,

$$\int_{\partial\Omega} g(S^k x)\,ds_x = \int_{\partial\Omega} g(x)\,ds_x,$$

we get

$$\sum_{k=1}^{l} a_k \int_{\partial\Omega} g(S^{k-1}x)\, ds_x = \mu_l(a) \int_{\partial\Omega} g(x)\, ds_x,$$

and therefore, condition (5.6) can be rewritten in the form of (5.1). If this condition holds true then the solution of problem (5.5) exists up to a constant term and can be represented in the following form (see, e.g., [15]):

$$v(x) = \int_{\Omega} G_N(x,y) f(y)\, dy + \sum_{k=1}^{l} a_k \int_{\partial\Omega} G_N(x,y) g(S^{k-1}y)\, ds_y. \tag{5.7}$$

Similarly to the Dirichlet problem, the solution of the Neumann problem can be obtained by the formula

$$u(x) = \sum_{m=1}^{l} b_m v(S^{m-1}x), \tag{5.8}$$

where the coefficients b_m are given by (4.1).

Indeed, if we consider the vector $V = \big(v(x), v(Sx), \ldots, v(S^{l-1}x)\big)^T$, then, under the theorem's conditions, one can determine the vector $U = \big(u(x), u(Sx), \ldots, u(S^{l-1}x)\big)^T$ via the equality $U = A^{-1}V$. Due to the relation $AU = V$ and the formula (3.3), the function $u(x)$ is uniquely determined via the function $v(x)$ from (5.8) and (5.7). Mimicking the reasoning for the Dirichlet problem, it can be shown that the function (5.8) satisfies all conditions of the problem (1.1), (1.3). Furthermore, inserting the function $v(x)$ from (5.7) into the right-hand side of (5.8), we have the relations

$$u(x) = \sum_{m=1}^{l} b_m \int_{\Omega} G_N(S^{m-1}x, y) f(y)\, dy + \sum_{m=1}^{l} b_m \sum_{k=1}^{l} a_k \int_{\partial\Omega} G_N(S^{m-1}x, y) g(S^{k-1}y)\, ds_y$$

$$= \int_{\Omega} \left[\sum_{m=1}^{l} b_m G_N(S^{m-1}x, y)\right] f(y)\, dy + \int_{\partial\Omega} \left[\sum_{m=1}^{l} b_m \sum_{k=1}^{l} a_k G_N\left(S^{m-1}x, \left(S^{k-1}\right)^T y\right)\right] g(y)\, ds_y$$

$$= \int_{\Omega} G_{N,S}(x,y) f(y)\, dy + \int_{\partial\Omega} G_{N,S,l}(x,y) g(y)\, ds_y,$$

where we used the notations

$$G_{N,S}(x,y) = \sum_{m=0}^{l-1} b_m G_N(S^{m-1}x, y), \quad G_{N,S,l}(x,y) = \sum_{m=1}^{l} b_m \sum_{k=1}^{l} a_k G_N\left(S^{m-1}x, \left(S^{k-1}\right)^T y\right).$$

Hence, representation (5.2) for the solution of the problem (1.1), (1.3) is verified. The theorem is proved. □

6. Existence of the solution to the Robin problem

Now we give the main statement for the problem (1.1), (1.4). Let $G_R(x,y)$ be the Green's function of the classical Robin problem. Note that the explicit form of $G_R(x,y)$ is constructed in [13, 24].

The following statement holds true.

Theorem 6.1 *Let $c > 0$ and the numbers $\{a_k : k = 1, \ldots, l\}$ be such that $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$ for all $k = 1, \ldots, l$, where $\{\lambda_k\}$ are lth roots of unity. Then, for any $f \in C^1(\bar{\Omega})$, $g \in C(\partial\Omega)$, the solution of the problem (1.1), (1.4) exists, is unique, and can be represented in the following form:*

$$u(x) = \int_\Omega G_{R,S}(x,y)f(y)\,dy + \int_{\partial\Omega} G_{R,S,l}(x,y)g(y)\,ds_y, \tag{6.1}$$

where

$$\begin{aligned} G_{R,S}(x,y) &= \sum_{q=1}^{l} b_q G_R(S^{m-1}x, y), \\ G_{R,S,l}(x,y) &= \sum_{q=1}^{l} b_q \sum_{k=1}^{l} a_k G_R\left(S^{m-1}x, \left(S^{k-1}\right)^T y\right), \end{aligned} \tag{6.2}$$

and the coefficients b_q are given by (4.1).

Proof Consider the following Robin problem for the function $v(x)$ in the domain Ω:

$$\begin{aligned} -\Delta v(x) &= f(x),\ x \in \Omega; \\ \frac{\partial v(s)}{\partial \nu} + cu(s) &= \sum_{k=1}^{l} a_k g(S^{k-1}s) \equiv h(s),\ s \in \partial\Omega. \end{aligned} \tag{6.3}$$

It is known that, for any given functions $f(x)$ and $h(s) = \sum_{k=1}^{l} a_k g(S^{k-1}s)$, the solution of the problem (6.3) exists, is unique, and can be represented in the following form (see, e.g., [15]):

$$v(x) = \int_\Omega G_R(x,y)f(y)\,dy + \sum_{k=1}^{l} a_k \int_{\partial\Omega} G_R(x,y)g(S^{k-1}y)\,ds_y. \tag{6.4}$$

Let us further consider the vector $V = \left(v(x), v(Sx), \ldots, v(S^{l-1}x)\right)^T$. It follows from the vector equality $U = A^{-1}V$ and Lemma 4.2 that the structure of the matrix A^{-1} is similar to A. Therefore, similarly to equalities (3.1) and (3.2), we define the vector $U = \left(u(x), u(Sx), \ldots, u(S^{l-1}x)\right)^T$ from the equality $U = A^{-1}V$. Since $\mu_k = a_1\lambda_0^k + \ldots + a_l\lambda_{l-1}^k \neq 0$, it follows from Lemma 2.1 that A is nonsingular and $\det A^{-1} \neq 0$. As $AU = V$, formula (3.3) implies that the function $u(x)$ is uniquely determined via the function $v(x)$ in (6.4) by the formula

$$u(x) = \sum_{q=1}^{l} b_q v(S^{q-1}x), \tag{6.5}$$

where the coefficients b_q are defined in (4.1). Now, as in the case of the Dirichlet problem, it can be easily shown that the function $u(x)$ in (6.5) is the solution of the problem (1.1), (1.4).

Furthermore, the representation (6.4) yields the following equality:

$$v(S^{q-1}x) = \int_\Omega G_R(S^{q-1}x, y)f(y)\,dy + \int_{\partial\Omega} G_R(S^{q-1}x, y)\sum_{k=1}^{l} a_k g(S^{k-1}y)\,ds_y,$$

and hence, substituting this expression instead of $v(S^{q-1}x)$ in equality (6.5) and taking into account the formula (6.2), we get the relations

$$u(x) = \int_{\Omega} \left[\sum_{q=1}^{l} b_q G_R(S^{q-1}x, y) \right] f(y)\, dy + \int_{\partial\Omega} \left[\sum_{q=1}^{l} b_q G_R(S^{q-1}x, y) \right] \sum_{k=1}^{l} a_k g(S^{k-1}y)\, ds_y$$

$$= \int_{\Omega} G_{R,S}(x,y) f(y)\, dy + \int_{\partial\Omega} \left[\sum_{q=1}^{l} b_q \sum_{k=1}^{l} a_k G_R(S^{q-1}x, \left(S^{k-1}\right)^T y) \right] g(y)\, ds_y = \int_{\Omega} G_{R,S}(x,y) f(y)\, dy$$

$$+ \int_{\partial\Omega} G_{R,S,l}(x,y) g(y)\, ds_y.$$

Thus, representation (6.1) for function $u(x)$ is verified. The theorem is proved. □

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