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On the exponential Diophantine equation $P_n^x + P_{n+1}^x = P_m$

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Abstract: In this paper, we find all the solutions of the title Diophantine equation in nonnegative integer variables (m, n, x), where P_k is the kth term of the Pell sequence.

Key words: Pell numbers, linear form in logarithms, reduction method

1. Introduction

Let $(P_n)_{n\geq 0}$ be the Pell sequence given by

$$P_0 = 0,\ P_1 = 1 \text{ and } P_{n+2} = 2P_{n+1} + P_n, \quad \text{for all } n \geq 0.$$

It is well known that

$$P_n^2 + P_{n+1}^2 = P_{2n+1}, \ \text{for all } n \geq 0.$$

In particular, this identity tells us that the sum of the squares of two consecutive Pell numbers is still a Pell number. This raises the following natural question: can we find all triples of nonnegative integers (m, n, x) such that

$$P_n^x + P_{n+1}^x = P_m? \tag{1.1}$$

We prove the following theorem:

Theorem 1.1 *All the solutions of the Diophantine equation (1.1) in nonnegative integers (m, n, x) are*

$$(m, n, x) \in \{(1, 0, x), (2n + 1, n, 2), (2, n, 0)\}. \tag{1.2}$$

Namely, we have

$$P_0^x + P_1^x = P_1, \quad P_n^2 + P_{n+1}^2 = P_{2n+1}, \quad P_n^0 + P_{n+1}^0 = P_2.$$

The Diophantine equation (1.1) was studied when we replace the Pell numbers by the Fibonacci numbers in [5] and [6] and when we replace the Pell numbers by k-generalized Fibonacci numbers in [8].

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2. Auxiliary results

2.1. Pell sequence

Let $(\alpha, \beta) = (1+\sqrt{2}, 1-\sqrt{2})$ be the roots of the characteristic equation $x^2 - 2x - 1 = 0$ of the Pell sequence $(P_n)_{n\geq 0}$. The Binet formula for P_n,

$$P_n = \frac{\alpha^n - \beta^n}{2\sqrt{2}}, \quad \text{holds for all } n \geq 0. \tag{2.1}$$

This implies easily that the inequality

$$\alpha^{n-2} \leq P_n \leq \alpha^{n-1} \tag{2.2}$$

holds for all positive integers n. It is easy to prove that

$$\frac{P_n}{P_{n+1}} \leq \frac{3}{7} \tag{2.3}$$

holds for all $n \geq 2$.

2.2. Linear forms in logarithms

The proof of our main theorem uses lower bounds for linear forms in logarithms of algebraic numbers and a version of the Baker–Davenport reduction method. Let us recall some results.

For any nonzero algebraic number γ of degree d over $\mathbb{Q}$, whose minimal polynomial over $\mathbb{Z}$ is $a\prod_{i=1}^{d}\left(X - \gamma^{(i)}\right)$ (with $a > 0$), we denote by

$$h(\gamma) = \frac{1}{d}\left(\log a + \sum_{i=1}^{d} \log \max\left(1, \left|\gamma^{(i)}\right|\right)\right)$$

the usual absolute logarithmic height of γ.

With this notation, Matveev proved the following theorem (see [7]):

Theorem 2.1 *Let $\gamma_1, \ldots, \gamma_s$ be real algebraic numbers and let $b_1, \ldots, b_s$ be nonzero rational integer numbers. Let D be the degree of the number field $\mathbb{Q}(\gamma_1, \ldots, \gamma_s)$ over $\mathbb{Q}$ and let A_j be a positive real number satisfying*

$$A_j = \max\{Dh(\gamma_j), |\log \gamma_j|, 0.16\} \quad \textit{for} \quad j = 1, \ldots, s.$$

Assume that

$$B \geq \max\{|b_1|, \ldots, |b_s|\}.$$

If $\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1 \neq 0$, then

$$|\gamma_1^{b_1} \cdots \gamma_s^{b_s} - 1| \geq \exp(-1.4 \cdot 30^{s+3} \cdot s^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 \cdots A_s).$$

2.3. Reduction method

In 1998, Dujella and Pethő in [4, Lemma 5(a)] gave a version of the reduction method based on the Baker–Davenport lemma [1]. We next present the following lemma from [3], which is an immediate variation of the result due to Dujella and Pethő from [4] and will be one of the key tools used in this paper to reduce the upper bounds on n of the Diophantine equation (1.1).

Lemma 2.2 *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let*

$$\varepsilon = ||\mu q|| - M \cdot ||\gamma q||,$$

where $||\cdot||$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then there is no solution of the inequality

$$0 < r\gamma - s + \mu < AB^{-k}$$

in positive integers r, s, and k with

$$r \leq M \quad \textit{and} \quad k \geq \frac{\log(Aq/\varepsilon)}{\log B}.$$

3. The proof of Theorem 1.1

3.1. An inequality for x in terms of m and n

We assume that $n \geq 1$, as the solution with $n = 0$ is obvious. Observe that when $x = 0$, then $P_m = 2 = P_2$. Since $P_{n+1} < P_{n+1} + P_n < P_{n+2}$, the Diophantine equation (1.1) has no solution when $x = 1$. Furthermore, when $n = 1$ we get $P_m = 1 + 2^x$ and all solutions of this Diophantine equation are $(m, x) = (2, 0)$ or $(3, 2)$ (see [2, Theorem 2.2]). We can assume that $n \geq 2$ and $x \geq 3$. Therefore, we have

$$P_m \geq P_2^3 + P_3^3 = 133,$$

which implies that $m \geq 7$.

Using inequality (2.2), we get

$$\alpha^{m-1} > P_m = P_n^x + P_{n+1}^x \geq P_{n+1}^x > \alpha^{(n-1)x},$$

and

$$\alpha^{m-2} < P_m = P_n^x + P_{n+1}^x < (P_n + P_{n+1})^x < P_{n+2}^x < \alpha^{(n+1)x}.$$

Thus, we have

$$(n-1)x + 1 < m < (n+1)x + 2. \tag{3.1}$$

Estimate (3.1) is essential for our purpose.

Now, we rewrite the equation (1.1) as

$$\frac{\alpha^m}{2\sqrt{2}} - P_{n+1}^x = P_n^x + \frac{\beta^m}{2\sqrt{2}}. \tag{3.2}$$

Dividing both sides of equation (3.2) by P_{n+1}^x and using the inequality (2.3), we obtain

$$\left|\alpha^m (2\sqrt{2})^{-1} P_{n+1}^{-x} - 1\right| < 2\left(\frac{P_n}{P_{n+1}}\right)^x < \frac{2}{2.3^x}. \tag{3.3}$$

Put

$$\Lambda_1 := \alpha^m (2\sqrt{2})^{-1} P_{n+1}^{-x} - 1. \tag{3.4}$$

If $\Lambda_1 = 0$, then $\alpha^m = 2\sqrt{2}P_{n+1}^x$, so $\alpha^{2m} \in \mathbb{Z}$, which is false for all positive integers m. Therefore, one sees that $\Lambda_1 \neq 0$.

We will use Matveev's theorem to get a lower bound for Λ_1. Put

$$s := 3,\ \gamma_1 := \alpha,\ \gamma_2 := 2\sqrt{2},\ \gamma_3 := P_{n+1}, \quad b_1 := m, \quad b_2 := -1,\ b_3 := -x.$$

Note that $\gamma_1, \gamma_2, \gamma_3 \in \mathbb{Q}(\sqrt{2})$. Thus, we take $D := 2$. Since

$$h(\gamma_1) = (\log\alpha)/2,\ h(\gamma_2) = (\log 8)/2 \ \text{ and } h(\gamma_3) = \log P_{n+1} < n\log\alpha,$$

we take

$$A_1 := \log\alpha,\ A_2 := \log 8,\ A_3 := 2n\log\alpha.$$

Finally, inequality (3.1) implies that $m > (n-1)x \geq x$, so we take $B := m$. It is also the case that $B := m \leq (n+1)x + 2 < (n+2)x$. Hence, Matveev's theorem implies that

$$\begin{aligned} \log|\Lambda_1| &\geq -1.4 \times 30^6 \times 3^{4.5} \times 2^2 \times (1+\log 2)(\log\alpha)(\log 8)(2n\log\alpha)(1+\log m) \\ &\geq -3.14 \times 10^{12} n(1+\log m). \end{aligned} \tag{3.5}$$

Thus, inequalities (3.3) and (3.5) together with (3.4) imply that

$$x < 3.8 \times 10^{12} n(1+\log m) < 6.1 \times 10^{12} n\log m,$$

where we used the fact that $1+\log m < 1.6\log m$ for $m \geq 7$. Together with the fact that $m < (n+2)x$, we get that

$$x < 6.1 \times 10^{12} n\log((n+2)x). \tag{3.6}$$

3.2. The case when $2 \leq n \leq 85$

In this case

$$x < 6.1 \times 10^{12} n\log((n+2)x) < 5 \times 10^{14}\log(87x),$$

giving $x < 2.2 \times 10^{16}$. Thus,

$$m < (n+2)x \leq 87x \leq 2 \times 10^{18}.$$

We consider again Λ_1 given by expression (3.4). Put

$$\Gamma_1 := m\log\alpha - \log(2\sqrt{2}) - x\log P_{n+1}.$$

Thus, $\Lambda_1 = e^{\Gamma_1} - 1$. It is easy to see that the right-hand side of (3.2) is a number in the interval $[P_n^x - 1, P_n^x + 1]$. In particular, Λ_1 is positive, which implies that Γ_1 is positive. Thus,

$$0 < \Gamma_1 < e^{\Gamma_1} - 1 = \Lambda_1 < \frac{2}{2.3^x},$$

so

$$0 < m\left(\frac{\log\alpha}{\log P_{n+1}}\right) - x - \left(\frac{\log(2\sqrt{2})}{\log P_{n+1}}\right) < \frac{2}{2.3^x\log P_{n+1}} < \frac{2}{2.3^x} < \frac{2}{(2.3^{1/87})^m}. \tag{3.7}$$

For us, inequality (3.7) is

$$0 < m\gamma - x + \mu < AB^{-m},$$

where

$$\gamma := \frac{\log\alpha}{\log P_{n+1}}, \quad \mu = -\frac{\log(2\sqrt{2})}{\log P_{n+1}}, \quad A = 2, \quad B = 1.009 < 2.3^{1/87}.$$

We take $M := 2 \times 10^{18}$.

For each n in the interval $[2, 85]$, we take $q = q_{89}$ to be the denominator of the 89^{th} convergent to γ. For all $n \in [2, 85]$, we have $q > 6M$ and $\varepsilon > 0$, so we may apply Lemma 2.2. Furthermore, since the minimal value of ε is at least 3×10^{-25} and the maximal value of q is 8×10^{51}, Lemma 2.2 tells us that all solutions (m, x) of inequality (3.7) have

$$m < \frac{\log(8 \times 10^{51}/(3 \times 10^{-25}))}{\log 1.009} < 19650.$$

For example, if $n = 85$, then the terms of the continued fraction of γ are

$$[0, 84, 1, 4, 1, 1, 3, 3, 1, 1, 7, 3, 1, 1, 2, 12, 1, 1, 4, 2, 1, 11, 2, 1, 1, 1, 1, 2, 17, 4, 1, 66, \ldots],$$

its 89th convergent is

$$q_{89} = 41219479303567561160989604443297308424784207571 9,$$

and the corresponding ε is

$$\varepsilon = 8.0172343856806690497663453758579692502637207189220 \cdot 10^{-25}.$$

Therefore, the corresponding bound is 18430.

Next, since $(n-1)x \le m$, we have

$$x \le m/(n-1) < 19650/(n-1).$$

A computer search with Maple revealed that there are no solutions to the equation (1.1) in the range $n \in [2, 85]$, $m \in [7, 19650]$, and $x \in [3, 19650/(n-1)]$. A few minutes of computations confirm the result contained in the main theorem.

From now on, we assume that $n \ge 86$.

3.3. An upper bound on x in terms of n

Recall that, by (3.6), we have

$$x < 6.1 \times 10^{12} n \log((n+2)x).$$

Next we give an upper bound on x depending only on n. If

$$x \le n + 2, \tag{3.8}$$

then we are through. Otherwise, i.e. if $n + 2 < x$, we then have

$$x < 6.1 \times 10^{12} n \log x^2 = 1.22 \times 10^{13} n \log x,$$

which can be rewritten as

$$\frac{x}{\log x} < 1.22 \times 10^{13} n. \tag{3.9}$$

Using the fact that for all $A \geq 3$

$$\frac{x}{\log x} < A \quad \text{yields} \quad x < 2A \log A,$$

and the fact that $\log(1.22 \times 10^{13} n) < 8 \log n$ holds for all $n \geq 86$, we get that

$$\begin{aligned} x &< 2(1.22 \times 10^{13} n) \log(1.22 \times 10^{13} n) \\ &< 2.44 \times 10^{13} n(8 \log n) \\ &< 2 \times 10^{14} n \log n. \end{aligned}$$

From (3.8) and the last inequality above we conclude that

$$x < 2 \times 10^{14} n \log n \tag{3.10}$$

holds for all $n \geq 86$.

3.4. An absolute upper bound on x

Let us look at the element

$$y := \frac{x}{\alpha^{2n}}.$$

The inequality (3.10) implies that

$$y < \frac{2 \times 10^{14} n \log n}{\alpha^{2n}} < \frac{1}{\alpha^n}, \tag{3.11}$$

where the last inequality holds for all $n \geq 44$. In particular, $y < \alpha^{-86} < 10^{-32}$. We now write

$$P_n^x = \frac{\alpha^{nx}}{8^{x/2}} \left(1 - \frac{(-1)^n}{\alpha^{2n}}\right)^x,$$

and

$$P_{n+1}^x = \frac{\alpha^{(n+1)x}}{8^{x/2}} \left(1 - \frac{(-1)^{n+1}}{\alpha^{2(n+1)}}\right)^x.$$

If n is odd, then

$$1 < \left(1 - \frac{(-1)^n}{\alpha^{2n}}\right)^x = \left(1 + \frac{1}{\alpha^{2n}}\right)^x < e^y < 1 + 2y,$$

because $y < 10^{-32}$ is very small, while if n is even, then

$$1 > \left(1 - \frac{(-1)^n}{\alpha^{2n}}\right)^x = \exp\left(x \log\left(1 - \frac{1}{\alpha^{2n}}\right)\right) > e^{-2y} > 1 - 2y,$$

again because $y < 10^{-32}$ is very small. Thus,

$$\left| P_n^x - \frac{\alpha^{nx}}{8^{x/2}} \right| < \frac{2y\alpha^{nx}}{8^{x/2}},$$

and of course, a similar inequality holds if we replace n by $n+1$. We now return to our equation (1.1) and rewrite it as

$$\frac{\alpha^m-\beta^m}{2\sqrt{2}}=P_m=P_n^x+P_{n+1}^x=\frac{\alpha^{nx}}{8^{x/2}}+\frac{\alpha^{(n+1)x}}{8^{x/2}}+\left(P_n^x-\frac{\alpha^{nx}}{8^{x/2}}\right)+\left(P_{n+1}^x-\frac{\alpha^{(n+1)x}}{8^{x/2}}\right),$$

or

$$\begin{aligned}\left|\frac{\alpha^m}{8^{1/2}}-\frac{\alpha^{nx}}{8^{x/2}}(1+\alpha^x)\right| &= \left|\frac{\beta^m}{8^{1/2}}+\left(P_n^x-\frac{\alpha^{nx}}{8^{x/2}}\right)+\left(P_{n+1}^x-\frac{\alpha^{(n+1)x}}{8^{x/2}}\right)\right|\\ &< \frac{1}{\alpha^m}+\left|P_n^x-\frac{\alpha^{nx}}{8^{x/2}}\right|+\left|P_{n+1}^x-\frac{\alpha^{(n+1)x}}{8^{x/2}}\right|\\ &< \frac{1}{\alpha^m}+2y\left(\frac{\alpha^{nx}(1+\alpha^x)}{8^{x/2}}\right).\end{aligned}$$

Multiplying both sides of it by $\alpha^{-(n+1)x}8^{x/2}$, we obtain that

$$\left|\alpha^{m-(n+1)x}8^{(x-1)/2}-(1+\alpha^{-x})\right|<\frac{8^{x/2}}{\alpha^{m+(n+1)x}}+2y(1+\alpha^{-x})<\frac{1}{2\alpha^n}+\frac{15y}{7}<\frac{3}{\alpha^n}, \tag{3.12}$$

where we used the fact that $8^{x/2}/(\alpha^{(n+1)x})\leq(2\sqrt{2}/\alpha^{86})^x<1/2$, $m\geq(n-1)x\geq n$ and $\alpha^x\geq\alpha^3>14$, as well as inequality (3.11). Hence, we conclude that

$$\left|\alpha^{m-(n+1)x}8^{(x-1)/2}-1\right|<\frac{1}{\alpha^x}+\frac{3}{\alpha^n}\leq\frac{4}{\alpha^l}, \tag{3.13}$$

where $l:=\min\{n,x\}$. We now set

$$\Lambda_2:=\alpha^{m-(n+1)x}8^{(x-1)/2}-1 \tag{3.14}$$

and observe that $\Lambda_2\neq0$. Indeed, if $\Lambda_2=0$, then $\alpha^{2((n+1)x-m)}=8^{x-1}\in\mathbb{Z}$, which is possible only when $(n+1)x=m$. However, if this were so, then we would get $0=\Lambda_2=8^{(x-1)/2}-1$, which leads to the conclusion that $x=1$, which is not possible. Hence, $\Lambda_2\neq0$. Next, let us notice that since $x\geq3$ and $n\geq86$, we have that

$$|\Lambda_2|\leq\frac{1}{\alpha^3}+\frac{3}{\alpha^{86}}<\frac{1}{2}, \tag{3.15}$$

so that $\alpha^{m-(n+1)x}8^{(x-1)/2}\in[1/2,3/2]$. In particular,

$$(n+1)x-m<\frac{1}{\log\alpha}\left(\frac{(x-1)\log8}{2}+\log2\right)<x\left(\frac{\log8}{2\log\alpha}\right)<1.2x, \tag{3.16}$$

and

$$(n+1)x-m>\frac{1}{\log\alpha}\left(\frac{(x-1)\log8}{2}-\log2\right)>1.1x-2>0. \tag{3.17}$$

We lower-bound the left-hand side of inequality (3.13) using again Matveev's theorem. We take

$$s:=2,\ \gamma_1:=\alpha,\ \gamma_2:=2\sqrt{2},\ b_1:=m-(n+1)x,\ b_2:=x-1.$$

As in the previous application of Matveev's result, we can take

$$D := 2,\ A_1 := \log\alpha,\ A_2 := \log 8.$$

Also, we can take $B := 1.2x$. We thus get that

$$\log|\Lambda_2| > -1.4 \times 30^5 \times 2^{4.5} \times 2^2(1+\log 2)(\log\alpha)(\log 8)(1+\log(1.2x)). \tag{3.18}$$

Then inequalities (3.13) and (3.18) lead to

$$\begin{aligned} l &< \frac{\log 4}{\log\alpha} + 1.4 \times 30^5 \times 2^{4.5} \times 2^2(1+\log 2)(\log 8)(1+\log(1.2x)) \\ &< 1.1 \times 10^{10}(1+\log(1.2x)) \\ &< 1.1 \times 10^{10}(2.2\log x) \\ &< 2.5 \times 10^{10}\log x. \end{aligned}$$

Here, we used the fact that $1+\log(1.2x) < 2.2\log x$ for all $x \geq 3$.

We next distinguish two cases.

Case 1. If $l = x$, we then obtain that $x < 2.5 \times 10^{10}\log x$, so

$$x < 10^{12}.$$

Case 2. If $l = n$, then using (3.10), we obtain that

$$n < 2.5 \times 10^{10}\log(2 \times 10^{14} n\log n).$$

This last inequality above leads to $n < 1.7 \times 10^{12}$, so, by (3.10) once again, we obtain that

$$x < 2 \times 10^{14} \times (1.7 \times 10^{12})\log(1.7 \times 10^{12}) < 10^{28}.$$

In conclusion, we have that

$$x < 10^{28}. \tag{3.19}$$

3.5. A better upper bound on x

Next, we take

$$\Gamma_2 := (x-1)\log(2\sqrt{2}) - ((n+1)x - m)\log\alpha.$$

Observe that $\Lambda_2 = e^{\Gamma_2} - 1$, where Λ_2 is given by (3.14). Since $|\Lambda_2| < \frac{1}{2}$, we have that $e^{|\Gamma_2|} < 2$, and using inequality (3.13) we obtain

$$|\Gamma_2| \leq e^{|\Gamma_2|}\left|e^{\Gamma_2} - 1\right| < 2|\Lambda_2| < \frac{2}{\alpha^x} + \frac{6}{\alpha^n}.$$

This leads to

$$\left|\frac{\log(2\sqrt{2})}{\log\alpha} - \frac{(n+1)x - m}{x-1}\right| < \frac{1}{(x-1)\log\alpha}\left(\frac{2}{\alpha^x} + \frac{6}{\alpha^n}\right). \tag{3.20}$$

Note first that $\alpha^n \geq \alpha^{86} > 10^{32} > 10^4 x$ by estimate (3.19). Assume next that $x > 100$. Then $\alpha^x > 10^4 x$. Hence, we obtain

$$\frac{1}{(x-1)\log\alpha}\left(\frac{2}{\alpha^x}+\frac{6}{\alpha^n}\right) < \frac{8}{(x-1)10^4 x\log\alpha} < \frac{1}{1100(x-1)^2}. \tag{3.21}$$

Estimates (3.20) and (3.21) lead to

$$\left|\frac{\log(2\sqrt{2})}{\log\alpha} - \frac{(n+1)x-m}{x-1}\right| < \frac{1}{1100(x-1)^2}. \tag{3.22}$$

By a criterion of Legendre, inequality (3.22) implies that the rational number

$$\frac{(n+1)x-m}{x-1}$$

is a convergent to $\gamma := \log(2\sqrt{2})/\log\alpha$. Let

$$[a_0, a_1, a_2, a_3, a_4, a_5, a_6, \ldots] = [1, 5, 1, 1, 3, 3, \ldots]$$

be the continued fraction of γ, and let p_k/q_k be its kth convergent. Assume that $((n+1)x-m)/(x-1) = p_k/q_k$ for some k. Then $x-1 = dq_k$ for some positive integer d, which in fact is the greatest common divisor of $(n+1)x-m$ and $x-1$. We have the inequality

$$q_{54} > 1.08 \times 10^{28} > x - 1.$$

Thus, $k \in \{0, \ldots, 53\}$. Furthermore, $a_k \leq 66$ for all $k = 0, 1, \ldots, 53$. From the known properties of the continued fraction, we have that

$$\left|\gamma - \frac{(n+1)x-m}{x-1}\right| = \left|\gamma - \frac{p_k}{q_k}\right| > \frac{1}{(a_k+2)q_k^2} \geq \frac{d^2}{68(x-1)^2} \geq \frac{1}{68(x-1)^2},$$

which contradicts inequality (3.22). Hence, $x \leq 100$.

3.6. The final step

To finish, we go back to inequality (3.12) and rewrite it as

$$\left|\alpha^{m-(n+1)x}8^{(x-1)/2}(1+\alpha^{-x})^{-1} - 1\right| < \frac{3}{\alpha^n(1+\alpha^{-x})} < \frac{3}{\alpha^n}.$$

Recall that $x \in [3, 100]$, and using (3.16) and (3.17) we have

$$1.1x - 2 < (n+1)x - m < 1.2x.$$

Put $t := (n+1)x - m$. We computed all the numbers $\left|\alpha^{-t}8^{(x-1)/2}(1+\alpha^{-x})^{-1} - 1\right|$ for all $x \in [3, 100]$ and all $t \in [\lfloor 1.1x - 2\rfloor, \lfloor 1.2x\rfloor]$. None of them ended up being zero and the smallest of these numbers is $> 10^{-2}$. Thus, $1/10^2 < 3/\alpha^n$, or $\alpha^n < 3 \times 10^2$, so $n \leq 7$, which is false.

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