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A new general subclass of m-fold symmetric bi-univalent functions given by subordination

Arzu AKGÜL*
Department of Mathematics, Faculty of Arts and Science, Kocaeli University, Kocaeli, İzmit, Turkey

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Abstract: In a recent work, Orhan et al. (Afrika Matematika, 2016) defined a subclass of analytic bi-univalent one-fold symmetric functions. The main purpose of this paper is to generalize and improve the results of Orhan et al.

Key words: Analytic functions, m-fold symmetric bi-univalent functions, coefficient bounds, subordination

1. Introduction

Let A denote the family of analytic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

in $U = \{z : |z| < 1\}$ and $S = \{f \in A : f \text{ is univalent in } U\}$.

Theorem 1 *[8] The range of every function of class S contains a disk of radius $\frac{1}{4}$.*

By Theorem 1, every function $f \in A$ has an inverse f^{-1} defined by

$$f^{-1}(f(z)) = z \quad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w \quad \left(|w| < r_0(f)\ ,\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots . \tag{2}$$

A function $f \in A$ is called bi-univalent if both f and f^{-1} are univalent in U. We say that f is in the class Σ for such functions. The function $f \in A$ is said to be subordinate to another analytic function g, shown as

$$f(z) \prec g(z), \tag{3}$$

*Correspondence: akgul@kocaeli.edu.tr
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provided that there is an analytic function w defined on U with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1$$

fulfilling the following condition:

$$f(z) = g\left(w(z)\right).$$

For each function $f \in S$, the function

$$h(z) = \sqrt[m]{f(z^m)} \qquad (z \in U, \quad m \in \mathbb{N}) \tag{4}$$

is univalent and maps the unit disk U into a region with m-fold symmetry. A function is called m-fold symmetric (see [18]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \qquad (z \in U, \quad m \in \mathbb{N}). \tag{5}$$

We say that f is in the class S_m for such functions, which are normalized by series expansion (5). The series expansion for f^{-1} is given as follows:

$$\begin{aligned} g\left(w\right) &= w - a_{m+1} w^{m+1} + \left[(m+1)a_{m+1}^2 - a_{2m+1}\right] w^{2m+1} \\ &\quad - \left[\frac{1}{2}(m+1)(3m+2)a_{m+1}^3 - (3m+2)a_{m+1}a_{2m+1} + a_{3m+1}\right] w^{3m+1} + \cdots, \end{aligned} \tag{6}$$

where $f^{-1} = g$. This form was recently given in [18]. Σ_m denotes the class of such functions. Recently, many authors investigated bounds for various subclasses of m-fold symmetric bi-univalent functions (see [1, 2, 4–6, 15, 16, 19]).

In the present work, we also use the symbol $\mathcal{P}$ denoting the class of analytic functions of the form:

$$p\left(z\right) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

such that

$$R\left(p\left(z\right)\right) > 0 \qquad (z \in U).$$

According to the study in [14], the m-fold symmetric function p in the class $\mathcal{P}$ has the following form:

$$p\left(z\right) = 1 + p_m z + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \ . \tag{7}$$

Assume that φ is an analytic function together with positive real part in U such that

$$\varphi(0) = 1 \qquad \text{and} \qquad \varphi'(0) > 0,$$

and $\varphi(U)$ is symmetric with respect to the real axis. The function φ has a series expansion of the following form:

$$\varphi\left(z\right) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots \ \left(B_1 > 0\right). \tag{8}$$

Let $u(z)$ and $v(z)$ be two analytic functions in U with

$$u(0) = v(0) = 0 \quad \text{and} \quad max\{|u(z)|, |v(w)|\} < 1.$$

Observe that

$$u(z) = b_m z^m + b_{2m} z^{2m} + b_{3m} z^{3m} + \cdots \tag{9}$$

and

$$v(w) = c_m w^m + c_{2m} w^{2m} + c_{3m} w^{3m} + \cdots. \tag{10}$$

Also, we notice that

$$|b_m| \leq 1, |b_{2m}| \leq 1 - |b_m|^2 \,, \; |c_m| \leq 1, \quad |c_{2m}| \leq 1 - |c_m|^2. \tag{11}$$

By some simple calculations we can state that

$$\varphi\left(u\left(z\right)\right) = 1 + B_1 b_m z^m + \left(B_1 b_{2m} + B_2 b_m^2\right) z^{2m} + \cdots \left(|z| < 1\right) \tag{12}$$

and

$$\varphi\left(v\left(w\right)\right) = 1 + B_1 c_m w^m + \left(B_1 c_{2m} + B_2 c_m^2\right) w^{2m} + \cdots \left(|w| < 1\right). \tag{13}$$

The aim of this paper is to introduce a new subclasses $\mathcal{S}_{\Sigma_m}^{\varphi}(\lambda, \mu)$ of Σ_m and derive estimates on the initial coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ for functions in $\mathcal{S}_{\Sigma_m}^{\varphi}(\lambda, \mu)$, motivated essentially by the work of Ma and Minda [11].

2. Coefficient estimates for $\mathcal{S}_{\Sigma_m}^{\varphi}(\lambda, \mu)$

Definition 2 *For a function $f \in \Sigma_m$, we say that $f \in \mathcal{S}_{\Sigma_m}^{\varphi}(\lambda, \mu)$ if the following conditions are satisfied:*

$$\left((1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu} + \lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}\right) \prec \varphi(z) \quad (\lambda \geq 1, \quad \mu \geq 0, \; z \in U)$$

and

$$\left((1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu} + \lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}\right) \prec \varphi(w) \quad (\lambda \geq 1, \quad \mu \geq 0, \; w \in U),$$

where the function $g = f^{-1}$.

Remark 3

1. Letting $\mu = 1$, we have the class

$$\mathcal{S}_{\Sigma_m}^{\varphi}(\lambda, 1) = B_{\Sigma_m}(\lambda, \varphi)$$

(see Def. 4 in [19]).

2. For $\mu = 1$ and $\lambda = 1$ we have the class

$$\mathcal{S}_{\Sigma_m}^{\varphi}(1,1) = \mathcal{H}_{\Sigma,m}(\varphi)$$

(see Def. 4 in [19]).

3. For $\mu = 0$ and $\lambda = 1$, we have the class of m-fold symmetric bi-starlike functions (see [10]).

In the case of one-fold symmetric functions, we have the following classes:

4. For $m = 1$, we have the class

$$\mathcal{S}_{\Sigma_1}^{\varphi}(\lambda,\mu) = N_{\Sigma}^{\mu,\lambda}(\varphi)$$

(see [12]).

5. For $m = 1, \lambda = 1$, and $\mu = 1$, we have the following class:

$$\mathcal{S}_{\Sigma_1}^{\varphi}(1,1) = H_{\Sigma}^{\varphi}$$

(see [3]).

For $\varphi = \left(\frac{1+z}{1-z}\right)^{\beta}$ and for $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$, we have the following classes:

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+z}{1-z}\right)^{\beta}}(1,1) = \mathcal{H}_{\Sigma}^{\beta}(0 < \beta \leq 1)$$

and

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+(1-2\alpha)z}{1-z}\right)}(1,1) = \mathcal{H}_{\Sigma}^{\alpha}(0 \leq \alpha < 1),$$

respectively (see [17]).

6. For $m = 1$ and $\mu = 1$, we have the following class:

$$\mathcal{S}_{\Sigma_1}^{\varphi}(1,\lambda) = R_{\Sigma}(\lambda,\varphi), \qquad \lambda \geq 0$$

(see [3]).

For $m = 1, \mu = 1$, and $\varphi = \left(\frac{1+z}{1-z}\right)^{\beta}$ and for $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$, we have the following classes:

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+z}{1-z}\right)^{\beta}}(\lambda,1) = \mathcal{B}_{\Sigma}(\beta,\lambda), \quad (0 < \beta \leq 1, \lambda \geq 1)$$

and

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+(1-2\alpha)z}{1-z}\right)}(\lambda,1) = \mathcal{B}_{\Sigma}(\alpha,\lambda), \quad (0 \leq \alpha < 1, \lambda \geq 1),$$

respectively (see [17]).

7. For $m = 1$ and $\lambda = 1$, we have the class

$$\mathcal{S}_{\Sigma_1}^{\varphi}(1,\mu) = \mathcal{F}_{\Sigma}^{\mu}(\varphi)\ (\mu \geq 0)$$

(see [19]).

8. For $m = 1$, $\mu = 0$, $\lambda = 1$, and $\varphi = \left(\frac{1+z}{1-z}\right)^{\beta}$ and for $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$, we have the following classes:

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+z}{1-z}\right)^{\beta}}(1,0) = \mathcal{S}_{\Sigma,\beta}^{*}, \quad (0 < \beta \leq 1)$$

and

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+(1-2\alpha)z}{1-z}\right)}(1,0) = \mathcal{S}_{\Sigma}^{*}(\alpha), \quad (0 \leq \alpha < 1),$$

respectively (see [12]).

9. For $m = 1, \varphi = \left(\frac{1+z}{1-z}\right)^{\beta}$ and for $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$, we have the following classes:

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+z}{1-z}\right)^{\beta}}(\lambda,\mu) = N_{\Sigma}^{\mu,\lambda}(\beta), \quad (0 < \beta \leq 1, \lambda \geq 1, \mu \geq 0)$$

and

$$\mathcal{S}_{\Sigma_1}^{\left(\frac{1+(1-2\alpha)z}{1-z}\right)}(\lambda,\mu) = N_{\Sigma}^{\mu,\lambda}(\alpha), \quad (0 \leq \alpha < 1, \lambda \geq 1, \mu \geq 0),$$

respectively (see [7]).

Theorem 4 *Let f given by (5) be in the class $\mathcal{S}_{\Sigma_m}^{\varphi}(\lambda,\mu)$. Then*

$$|a_{m+1}| \leq \frac{B_1\sqrt{2B_1}}{\sqrt{\left|B_1^2\left(\mu+2m\lambda\right)\left(\mu+m\right) - 2B_2\left(\mu+m\lambda\right)^2\right| + 2B_1\left(\mu+m\lambda\right)^2}} \tag{14}$$

and

$$|a_{2m+1}| \leq \begin{cases} \left(\frac{m+1}{2} - \frac{(\mu+m\lambda)^2}{(\mu+2m\lambda)B_1}\right)\frac{2B_1^3}{\left|B_1^2(\mu+2m\lambda)(\mu+m)-2B_2(\mu+m\lambda)^2\right|+2B_1(\mu+m\lambda)^2} + \frac{B_1}{\mu+2m\lambda} & , \quad B_1 \geq \frac{2(\mu+m\lambda)^2}{(m+1)(\mu+2m\lambda)} \\ \frac{B_1}{\mu+2m\lambda} & , \quad B_1 < \frac{2(\mu+m\lambda)^2}{(m+1)(\mu+2m\lambda)} \end{cases} \tag{15}$$

where $\lambda \geq 1$, $\mu \geq 0$.

Proof Let $f \in \mathcal{S}_{\Sigma_m}^{\varphi}(\lambda,\mu)$. Then there are two analytic functions $u: U \to U$ and $v: U \to U$, with

$$u(0)=v(0)=0,$$

satisfying the following conditions:

$$(1-\lambda)\left(\frac{f(z)}{z}\right)^{\mu}+\lambda f'(z)\left(\frac{f(z)}{z}\right)^{\mu-1}=\varphi(u(z)) \quad \text{and} \quad (1-\lambda)\left(\frac{g(w)}{w}\right)^{\mu}+\lambda g'(w)\left(\frac{g(w)}{w}\right)^{\mu-1}=\varphi(v(w)). \tag{16}$$

Using the equalities (12) and (13) in (16) and by equating the coefficients in equation (16), we have

$$(\mu+m\lambda)\,a_{m+1}=B_1 b_m, \tag{17}$$

$$(\mu+2m\lambda)\left(\frac{\mu-1}{2}a_{m+1}^2+a_{2m+1}\right)=B_1 b_{2m}+B_2 b_m^2, \tag{18}$$

and

$$-(\mu+m\lambda)\,a_{m+1}=B_1 c_m, \tag{19}$$

$$(\mu+2m\lambda)\left[\left(m+\frac{\mu+1}{2}\right)a_{m+1}^2-a_{2m+1}\right]=B_1 c_{2m}+B_2 c_m^2. \tag{20}$$

From (17) and (19) we obtain

$$b_m=-c_m \tag{21}$$

and

$$(\mu+m\lambda)^2\,a_{m+1}^2=B_1^2(b_m^2+c_m^2). \tag{22}$$

Also, from (19), (20), and (22), we have

$$(\mu+2m\lambda)\,(\mu+m)a_{m+1}^2=B_1\left(b_{2m}+c_{2m}\right)+\frac{2B_2\left(\mu+m\lambda\right)^2}{B_1^2}a_{m+1}^2.$$

Therefore, we have

$$\left[(\mu+2m\lambda)\,(\mu+m)B_1^2-2B_2\left(\mu+m\lambda\right)^2\right]a_{m+1}^2=B_1^3\left(b_{2m}+c_{2m}\right). \tag{23}$$

By using the inequalities in (11) for the coefficients b_{2m} and c_{2m}, we obtain

$$\left|(\mu+2m\lambda)\,(\mu+m)B_1^2-2B_2\left(\mu+m\lambda\right)^2\right|\left|a_{m+1}\right|^2\le 2B_1^3\left(1-\left|b_m^2\right|\right), \tag{24}$$

and by using (17) in (24) we have

$$\left|a_{m+1}\right|^2\le\frac{2B_1^3}{\left|B_1^2\left(\mu+2m\lambda\right)\left(\mu+m\right)-2B_2\left(\mu+m\lambda\right)^2\right|+2B_1\left(\mu+m\lambda\right)^2}, \tag{25}$$

which implies assertion (14). Similarly, in order to find the bound on $|a_{2m+1}|$, we subtract (20) from (18). Then we obtain

$$2(\mu+2m\lambda)\,a_{2m+1}=(\mu+2m\lambda)(m+1)a_{m+1}^{2}+B_{1}(b_{2m}-c_{2m}). \tag{26}$$

Thus, in view of (17), (21), and (26), and applying (11) for the coefficients b_{2m},b_m and c_{2m},c_m , we have

$$\begin{aligned} 2(\mu+2m\lambda)|a_{2m+1}| &\leq (\mu+2m\lambda)(m+1)|a_{m+1}|^{2}+2B_{1}\left(1-\left|b_{m}^{2}\right|\right), \\ |a_{2m+1}| &\leq \left(\frac{m+1}{2}-\frac{(\mu+m\lambda)^{2}}{(\mu+2m\lambda)B_{1}}\right)|a_{m+1}|^{2}+\frac{B_{1}}{\mu+2m\lambda}. \end{aligned} \tag{27}$$

Upon substituting the value of $|a_{m+1}|^2$ from inequality (25) and putting it in (27), we have the desired result. □

3. Corollaries of the main theorem

For the case of m-fold symmetric functions, we have following:

Remark 5 *For $\mu=1$, Theorem 4 reduces to the corresponding results of Tang et al. (Theorem 7, p. 11086 in [19]), which we recall here as Corollary 6 below.*

Corollary 6 *(See [19]) Let f given by (5) be in the class $\mathcal{S}_{\Sigma_m}^{\varphi}(\lambda,1)$. Then*

$$|a_{m+1}|\leq\frac{B_{1}\sqrt{2B_{1}}}{\sqrt{\left|B_{1}^{2}(1+2m\lambda)(1+m)-2B_{2}(1+m\lambda)^{2}\right|+2B_{1}(1+m\lambda)^{2}}} \tag{28}$$

and

$$|a_{2m+1}|\leq\begin{cases}\left(\frac{m+1}{2}-\frac{(1+m\lambda)^{2}}{(1+2m\lambda)B_{1}}\right)\frac{2B_{1}^{3}}{\left|B_{1}^{2}(1+2m\lambda)(1+m)-2B_{2}(1+m\lambda)^{2}\right|+2B_{1}(1+m\lambda)^{2}}+\frac{B_{1}}{1+2m\lambda} & ,\quad B_{1}\geq\frac{2(1+m\lambda)^{2}}{(m+1)(1+2m\lambda)}\\ \frac{B_{1}}{1+2m\lambda} & ,\quad B_{1}<\frac{2(1+m\lambda)^{2}}{(m+1)(1+2m\lambda)}\end{cases}, \tag{29}$$

where $0\leq\lambda<1$.

Remark 7 *For $\lambda=1$ and $\mu=1$, Theorem 4 reduces to the corresponding results of Tang et al. (Theorem 1 in [19]), which we recall here as Corollary 8 below.*

Corollary 8 *(See [19]) Let f given by (5) be in the class $\mathcal{S}_{\Sigma_m}^{\varphi}(1,1)$. Then*

$$|a_{m+1}|\leq\frac{B_{1}\sqrt{2B_{1}}}{\sqrt{(1+m)\left[\left|B_{1}^{2}(1+2m)-2B_{2}(1+m)\right|+2B_{1}(1+m)\right]}} \tag{30}$$

and

$$|a_{2m+1}|\leq\begin{cases}\left(1-\frac{2(1+m)}{(1+2m)B_{1}}\right)\frac{2B_{1}^{3}}{(1+m)\left[\left|B_{1}^{2}(1+2m)-2B_{2}(1+m)\right|+2B_{1}(1+m)\right]}+\frac{B_{1}}{1+2m} & ,\quad B_{1}\geq\frac{2(1+m)}{1+2m}\\ \frac{B_{1}}{1+2m} & ,\quad B_{1}<\frac{2(1+m)}{1+2m}\end{cases}. \tag{31}$$

Remark 9 *For* $\lambda=1$ *and* $\mu=0$, *Theorem 4 reduces to the following corollary:*

Corollary 10 *Let* f *given by (5) be in the class* $\mathcal{S}_{\Sigma_m}^{\varphi}(1,0)$. *Then*

$$|a_{m+1}| \leq \frac{B_1\sqrt{B_1}}{m\sqrt{|B_1^2-B_2|+B_1}} \tag{32}$$

and

$$|a_{2m+1}| \leq \begin{cases} \left(\frac{m+1}{2}-\frac{m}{2B_1}\right)\frac{B_1^3}{m^2\left[\left|B_1^2-B_2\right|+B_1\right]}+\frac{B_1}{2m} & , \quad B_1 \geq \frac{m}{1+m} \\ \frac{B_1}{2m} & , \quad B_1 < \frac{m}{1+m} \end{cases}. \tag{33}$$

For the case of one-fold symmetric functions, we have following:

Remark 11 *Theorem 4 reduces to Corollary 12:*

Corollary 12 *Let* f *given by (5) be in the class* $\mathcal{S}_{\Sigma_1}^{\varphi}(\lambda,\mu)$. *Then*

$$|a_2| \leq \frac{B_1\sqrt{2B_1}}{\sqrt{\left|B_1^2(\mu+2\lambda)(\mu+1)-2B_2(\mu+\lambda)^2\right|+2B_1(\mu+\lambda)^2}} \tag{34}$$

and

$$|a_3| \leq \begin{cases} \left(1-\frac{(\mu+\lambda)^2}{(\mu+2\lambda)B_1}\right)\frac{2B_1^3}{\left|B_1^2(\mu+2\lambda)(\mu+1)-2B_2(\mu+\lambda)^2\right|+2B_1(\mu+\lambda)^2} \\ \quad +\frac{B_1}{\mu+2\lambda} & , \quad B_1 \geq \frac{(\mu+\lambda)^2}{\mu+2\lambda} \\ \frac{B_1}{\mu+2\lambda} & , \quad B_1 < \frac{(\mu+\lambda)^2}{\mu+2\lambda} \end{cases} \tag{35}$$

where $0 \leq \lambda < 1, \mu \geq 0$.

From among the many choices of λ,μ and the function φ that would provide the following corollaries:

i) In the case of $\varphi=\left(\frac{1+z}{1-z}\right)^{\beta}$, we have

$$|a_2| \leq \frac{2\beta}{\sqrt{\left|(\mu+2\lambda)(\mu+1)\beta-(\mu+\lambda)^2\beta\right|+(\mu+\lambda)^2}} \tag{36}$$

and

$$|a_3| \leq \begin{cases} 2\beta\left[\left(1-\frac{(\mu+\lambda)^2}{2\beta(\mu+2\lambda)}\right)\frac{2\beta}{\left|(\mu+2\lambda)(\mu+1)\beta-(\mu+\lambda)^2\beta\right|+(\mu+\lambda)^2}+\frac{1}{\mu+2\lambda}\right] & , \quad \beta \geq \frac{(\mu+\lambda)^2}{2(\mu+2\lambda)} \\ \frac{2\beta}{\mu+2\lambda} & , \quad \beta < \frac{(\mu+\lambda)^2}{2(\mu+2\lambda)} \end{cases} \tag{37}$$

and

ii) for $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$, we have

$$|a_2| \le \frac{2(1-\alpha)}{\sqrt{\left|(\mu+2\lambda)(\mu+1)(1-\alpha)-(\mu+\lambda)^2\right|+(\mu+\lambda)^2}} \tag{38}$$

and

$$|a_3| \le \begin{cases} 2(1-\alpha)\left[\left(1-\frac{(\mu+\lambda)^2}{2(1-\alpha)(\mu+2\lambda)}\right)\frac{2(1-\alpha)}{\left|(\mu+2\lambda)(\mu+1)(1-\alpha)-(\mu+\lambda)^2\right|+(\mu+\lambda)^2}+\frac{1}{\mu+2\lambda}\right] & , \quad 1-\alpha \ge \frac{(\mu+\lambda)^2}{2(\mu+2\lambda)} \\ \frac{2(1-\alpha)}{\mu+2\lambda} & , \quad 1-\alpha < \frac{(\mu+\lambda)^2}{2(\mu+2\lambda)} \end{cases}. \tag{39}$$

Remark 13 *For $\mu = 1$, Theorem 4 reduces the corresponding results of Peng and Han [13], which we recall here as Corollary 14.*

Corollary 14 *[13] Let f given by (5) be in the class $\mathcal{S}_{\Sigma_1}^{\varphi}(\lambda, 1)$. Then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{\left|B_1^2(1+2\lambda)-B_2(1+\lambda)^2\right|+B_1(1+\lambda)^2}} \tag{40}$$

and

$$|a_3| \le \begin{cases} \left(1-\frac{(1+\lambda)^2}{(1+2\lambda)B_1}\right)\frac{B_1^3}{\left|B_1^2(1+2\lambda)-B_2(1+\lambda)^2\right|+B_1(1+\lambda)^2}+\frac{B_1}{1+2\lambda} & , \quad B_1 \ge \frac{(1+\lambda)^2}{1+2\lambda} \\ \frac{B_1}{1+2\lambda} & , \quad B_1 < \frac{(1+\lambda)^2}{1+2\lambda} \end{cases} \tag{41}$$

From among the many choices of λ and the function φ that would provide the following corollaries:

For the class of strongly starlike functions, the function φ is given by $\varphi = \left(\frac{1+z}{1-z}\right)^{\beta}$. Theorem 4 reduces the corresponding corollary of Tang et al. [19] (see Corollary 27, p. 1087 in [19]), which we recall here as Corollary 15:

Corollary 15 *[19] Let f given by (5) be in the class $\mathcal{S}_{\Sigma_1}^{\left(\frac{1+z}{1-z}\right)^{\beta}}(\lambda, 1)$. Then*

$$|a_2| \le \frac{2\beta}{\sqrt{(1+\lambda)^2+|1+2\lambda-\lambda^2|\,\beta}}$$

and

$$|a_3| \le \begin{cases} \frac{2\beta^2\left[2(1+2\lambda)+\left|1+2\lambda-\lambda^2\right|\right]}{(1+2\lambda)\left(\left|1+2\lambda-\lambda^2\right|\beta+(1+\lambda)^2\right)} & , \quad \frac{(1+\lambda)^2}{2(1+2\lambda)} < \beta \le 1 \\ \frac{2\beta}{1+2\lambda} & , \quad 0 < \beta \le \frac{(1+\lambda)^2}{2(1+2\lambda)} \end{cases}.$$

Remark 16 *The estimates for* $|a_2|$ *and* $|a_3|$ *given by Corollary 15 are more accurate than those given by Theorem 2.2 of Frasin and Aouf [9].*

ii) For the class of starlike functions of order α, the function φ is given by $\varphi = \left(\frac{1+(1-2\alpha)z}{1-z}\right)$. Theorem 4 reduces the corresponding corollary of Tang et al. [19] (see Corollary 28, p. 1087 in [19], which we recall here as Corollary 17:

Corollary 17 *[19] Let* f *given by (5) be in the class* $\mathcal{S}_{\Sigma_1}^{\left(\frac{1+(1-2\alpha)z}{1-z}\right)}(\lambda, 1)$. *Then*

$$|a_2| \le \frac{2(1-\alpha)}{\sqrt{(1+\lambda)^2 + \left|2(1+2\lambda)(1-\alpha) - (1+\lambda)^2\right|}}$$

and

$$|a_3| \le \begin{cases} 2(1-\alpha)\,\frac{\left|2(1+2\lambda)(1-\alpha)-(1+\lambda)^2\right|+2(1-\alpha)(1+2\lambda)}{(1+2\lambda)\left[\left|2(1+2\lambda)(1-\alpha)-(1+\lambda)^2\right|+(1+\lambda)^2\right]} & , 0 \le \alpha < \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \\ \frac{2(1-\alpha)}{1+2\lambda} & , \frac{1+2\lambda-\lambda^2}{2(1+2\lambda)} \le \alpha < 1 \end{cases}.$$

Remark 18 *The estimates for* $|a_2|$ *and* $|a_3|$ *given by Corollary 17 are more accurate than those given by Theorem 2.2 of Frasin and Aouf [9].*

Remark 19 *For* $\mu = 1 = \lambda$ *Theorem 4 reduces the corresponding corollary of Peng and Han [13], which we recall here as Corollary 20:*

Corollary 20 *[13] Let* f *given by (5) be in the class* $\mathcal{S}_{\Sigma_1}^{\varphi}(1,1)$. *Then*

$$|a_2| \le \frac{B_1\sqrt{B_1}}{\sqrt{|3B_1^2 - 4B_2| + 4B_1}}$$

and

$$|a_3| \le \begin{cases} \left(1 - \frac{4}{3B_1}\right)\frac{B_1^3}{\left|3B_1^2-4B_2\right|+4B_1} + \frac{B_1}{3} & , \quad B_1 \ge \frac{4}{3} \\ \frac{B_1}{3} & , \quad B_1 < \frac{4}{3} \end{cases}.$$

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