Uncountably many nonoscillatory bounded solutions to second-order nonlinear neutral dynamic equations

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Abstract: This work is devoted to the study of the existence of uncountably many nonoscillatory bounded solutions to second-order nonlinear neutral dynamic equations by means of the Darbo fixed point theorem. We construct assumptions without sign conditions on the nonlinear part of the equation. Moreover, we prove the necessary condition for the existence of an asymptotically zero solution to the problem under consideration.

Key words: Nonlinear dynamic equation, neutral type equation, nonoscillation, Darbo fixed point theorem

1. Introduction and preliminaries

During the last decades, the nonoscillation and the boundedness for various nonlinear neutral differential difference equations and their generalization, dynamic equations, have attracted some attention. See, for example, [1–3, 7, 8, 13, 16, 19–23, 25, 26, 28] and the references cited therein.

The fixed point approach is a standard technique in order to provide the sufficient conditions for the existence of nonoscillatory bounded solutions to second-order or higher-order nonlinear neutral dynamic equations. For example, the following problem was considered by Deng and Wang in [10] via the Krasnoselskii fixed point theorem:

$$\left(r(t)\left(x(t) + q(t)x(\tau(t))\right)^{\Delta}\right)^{\Delta} = f(t, x^{\sigma}(t), x(\tau_1(t)), \ldots, x(\tau_m(t)),$$

$$x(\xi_1(t)), \ldots, x(\xi_n(t))).$$

To get their results the authors used sign conditions on the continuous function f and the divergence of the improper integral of function $t \to \frac{1}{r(t)}$. Moreover, sign conditions on the nonlinear part of the equation are crucial in the consideration of Karpuz in [18] for proving sufficient conditions for the existence of a solution to the problem

$$(x(t) + q(t)x(\alpha(t)))^{\Delta^2} + B(t)F(x(\beta(t))) - C(t)F(x(\gamma(t))) = \varphi(t).$$

The Krasnoselskii fixed point theorem was used by many authors; see, for example, [2, 11, 14, 17, 27, 30–32]. On the other hand, Zhengou et el. in [29] used the Banach fixed point theorem to get the existence of nonoscillatory

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solutions of the following higher-order neutral dynamic equation:

$$(x(t) + q(t)x(\tau(t)))^{\Delta^n} + f(t, x(t - \tau_1(t)), \ldots, x(t - \tau_m(t))) = 0,$$

under Lipschitz continuity and monotonicity conditions on the nonlinear part.

In this paper we shall consider a second-order nonlinear neutral dynamic equation of the following form:

$$\left(r(t) \left(x(t) + q(t)x(\tau(t)) \right)^{\Delta} \right)^{\Delta} = a(t)f(x(\delta(t))) + b(t). \tag{1}$$

Some of the results presented in this paper cover the case $q \equiv 0$, which means that equation (1) is the generalization of the Sturm–Louville equation on time scales. Many authors studied such equations; see, for example, [6], [15], and [24] and references therein.

To prove the existence of uncountably many nonoscillatory bounded solutions to the above equation we apply the Darbo fixed point theorem on the space of bounded continuous functions defined on a noncompact Hausdorff space S, denoted by $\mathrm{BC}(S)$. Most of the above-mentioned papers required various types of sign conditions on the nonlinear part of the equation. These conditions are a consequence of the usage of sufficient conditions for the relative compactness of a bounded subset of $\mathrm{BC}(S)$. In this article, we can avoid sign conditions on the nonlinear function f, because we use the necessary and sufficient condition of the relative compactness of a bounded subset of $\mathrm{BC}(S)$, which is a consequence of the Hausdorff theorem. Moreover, our approach allows us to prove the existence of a bounded solution to the considered problem, which satisfies the equation on the maximal interval. In the last theorem, we present the necessary condition for the existence of an asymptotically zero solution to (1).

For basic facts on time scales and dynamic equations, one may consult [5]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ with the topology and ordering inherited from $\mathbb{R}$. In this paper we assume that $\mathbb{T}$ is a time scale such that $\sup \mathbb{T} = +\infty$. In the whole paper we consider only delta differentiable functions on $\mathbb{T}$, which we call shortly differentiable functions. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is denoted by $\mathrm{C}_{\mathrm{rd}}(\mathbb{T})$. The set of functions $f : \mathbb{T} \to \mathbb{R}$ that are differentiable and whose derivative is rd-continuous is denoted by $\mathrm{C}_{\mathrm{rd}}^1(\mathbb{T})$. We define the time scale interval $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$. Let $z(t) := x(t) + q(t)x(\tau(t))$. By a solution to equation (1) we mean a function $x : \mathbb{T} \to \mathbb{R}$ such that $z \in \mathrm{C}_{\mathrm{rd}}^1[t_x, \infty)_{\mathbb{T}}$ and $r \cdot z^{\Delta} \in \mathrm{C}_{\mathrm{rd}}^1[t_x, \infty)_{\mathbb{T}}$ and satisfying (1) for all $t \geq t_x$, for some $t_x \in \mathbb{T}$. A solution to (1) is said to be nonoscillatory if it is eventually positive or eventually negative. By an asymptotically zero solution to equation (1) we mean a solution to (1) that is convergent to zero as $t \to \infty$.

To get the main results of this paper we apply the Darbo fixed point theorem of the following form.

Theorem 1.1 *([4, 9]) Let X be a Banach space; let M be a nonempty, bounded, closed, convex subset of X; and let $A : M \to M$ be an $\alpha-$ condensing operator. Then A has a fixed point in M.*

By $[A]_{\alpha}$ we denote the α-norm of an operator $A : X \to Y$ between Banach spaces and α denotes the Kuratowski measure of noncompactness. This means

$$[A]_{\alpha} = \inf\{k > 0 : \alpha(A(N)) \leq k\alpha(N) \text{ for bounded } N \subset X\}.$$

We say that an operator $A : X \to Y$ between Banach spaces is an α-condensing operator if $[A]_{\alpha} < 1$ (see [4], pp. 164, 195).

Proposition 1.1 *([4, p. 197]) Let X be a Banach space. Suppose that $A : X \to X$ is a continuous operator that admits a representation as a sum $A = A_1 + A_2$, A_1 is a contraction, and A_2 is compact. Then A is the α-condensing operator.*

Let S be an topological space. We consider the Banach space $\mathrm{BC}(S)$ of all bounded continuous functions $f : S \to \mathbb{R}$ equipped with the standard supremum norm, i.e.

$$||f|| = \sup_{s \in S} |f(s)|, \text{ for } f \in \mathrm{BC}(S).$$

Theorem 1.2 *[12, p. 266] Let S be an arbitrary topological space and K a bounded subset of $\mathrm{BC}(S)$. Then K is relatively compact if and only if for every $\varepsilon > 0$ there is a finite collection $\{E_1, \ldots, E_n\}$ of sets with union S and points $s_i \in E_i$, $i = 1, \ldots, n$, such that*

$$\sup_{f \in K} \sup_{s \in E_i} |f(s) - f(s_i)| < \varepsilon, \ i = 1, \ldots, n.$$

2. The main results

We start this section with the presentation of sufficient conditions for the existence of uncountably many nonoscillatory bounded solutions to dynamic equation (1). We recall that $\mathbb{T}$ is a time scale such that $\sup \mathbb{T} = +\infty$.

Theorem 2.1 *Assume that:*

(H_1) $\tau : \mathbb{T} \to \mathbb{T}$ is continuous such that $\tau(t) \le t$, $t \in \mathbb{T}$ and $\lim_{s \to \infty} \tau(s) = +\infty$, $\delta : \mathbb{T} \to \mathbb{T}$ is rd-continuous such that $\delta(t) \le t$, $t \in \mathbb{T}$ and $\lim_{s \to \infty} \delta(s) = +\infty$;

(H_2) $a, b : \mathbb{T} \to \mathbb{R}$ are rd-continuous and $r : \mathbb{T} \to \mathbb{R} \setminus \{0\}$, $q : \mathbb{T} \to \mathbb{R}$ are continuous;

(H_3) $f : \mathbb{R} \to \mathbb{R}$ is continuous;

(H_4) $\int\limits_{t_0}^{\infty} \left(\frac{1}{|r(u)|} \int\limits_{t_1}^{u} |a(s)| \Delta s \right) \Delta u < +\infty$, $\int\limits_{t_0}^{\infty} \left(\frac{1}{|r(u)|} \int\limits_{t_1}^{u} |b(s)| \Delta s \right) \Delta u < +\infty$, *for some* $t_0, t_1 \in \mathbb{T}$, $t_0 \ge t_1$;

(H_5) $q(t) \ge 0$ *for* $t \in \mathbb{T}$, $\sup_{t \in \mathbb{T}} q(t) = q^\star < 1$.

Then equation (1) possesses uncountably many nonoscillatory bounded solutions.

Proof Let $M > 0$ and $L \in (\frac{1+q^\star}{2}M, M)$. From the continuity of f on $[\frac{1-q^\star}{2}M, M]$ we get the existence of $Q > 0$ such that

$$|f(y)| \le Q, \text{ for } y \in \left[\tfrac{1-q^\star}{2}M, M \right].$$

From (H_4) it is clear that there exists $t_2 \in \mathbb{T}$, $t_2 > t_0$, such that

$$\int_{t_2}^{\infty} \left(\frac{1}{|r(u)|} \int_{t_2}^{u} (|a(s)|Q + |b(s)|) \, \Delta s \right) \Delta u < \min\{L - \tfrac{1+q^\star}{2}M, M - L\}. \tag{2}$$

We consider $\mathrm{BC}(\mathbb{T})$, the Banach space of bounded continuous functions on $\mathbb{T}$, and its subset $B_M := \overline{B}(f^M, \frac{M}{4}(1+q^\star))$, where f^M is a constant function $f^M(t) = \frac{M}{4}(3-q^\star)$, $t \in \mathbb{T}$. It is obvious that B_M is a nonempty, bounded, convex, and closed subset of $\mathrm{BC}(\mathbb{T})$. Notice that $x \in B_M$ if and only if $x \in \mathrm{BC}(\mathbb{T})$ and

$$\tfrac{M}{2}(1-q^\star) \le x(t) \le M, \ t \in \mathbb{T}.$$

Define mappings $T^L, T_1^L, T_2 \colon B_M \to \mathrm{BC}(\mathbb{T})$ as follows: $T^L = T_1^L + T_2$ and

$$T_1^L(x)(t) = \begin{cases} L - q(t)x(\tau(t)), & \text{for } t \in [t_2, \infty)_{\mathbb{T}} \\ T_1^L(x)(t_2), & \text{for } t \in (-\infty, t_2)_{\mathbb{T}}, \end{cases}$$

$$T_2(x)(t) = \begin{cases} -\int\limits_t^\infty \left(\frac{1}{r(u)} \int\limits_{t_2}^u \left(a(s)f(x(\delta(s))) + b(s) \right) \Delta s \right) \Delta u, & \text{for } t \in [t_2, \infty)_{\mathbb{T}} \\ T_2(x)(t_2), & \text{for } t \in (-\infty, t_2)_{\mathbb{T}}. \end{cases}$$

First we notice that the continuity of q, (H_1), and (H_5) implies that $T_1^L(x) \in \mathrm{BC}(\mathbb{T})$ for any $x \in B_M$. The following provides that $T_2(x) \in \mathrm{BC}(\mathbb{T})$ for $x \in B_M$. The function $h_x \colon \mathbb{T} \to \mathbb{R}$ given by the formula

$$h_x(u) = \int\limits_{t_2}^u \left(a(s)f(x(\delta(s))) + b(s) \right) \Delta s$$

is the antiderivative of the rd-continuous function $\mathbb{T} \ni s \mapsto a(s)f(x(\delta(s)))+b(s)$. This means that $h_x \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T})$; see [5], p. 27. Now we define function $H_x \colon \mathbb{T} \to \mathbb{R}$ by the formula

$$H_x(t) = \begin{cases} \frac{h_x(t)}{r(t)}, & \text{for } t \in [t_2, \infty)_{\mathbb{T}} \\ 0, & \text{for } t \in (-\infty, t_2)_{\mathbb{T}}. \end{cases}$$

From the above relation, $H_x(t_2) = 0$, and from the continuity of r we get that $H_x \colon \mathbb{T} \to \mathbb{R}$ is rd-continuous on $\mathbb{T}$. Moreover,

$$T_2(x)(t) = -\int\limits_t^\infty H_x(u)\Delta u, \ t \in \mathbb{T}$$

and the function $|H_x|$, for $x \in B_M$, is majorized by $H \colon \mathbb{T} \to \mathbb{R}$ defined by

$$H(t) = \begin{cases} \frac{1}{|r(t)|} \left(\int_{t_2}^t \left(|a(s)|Q + |b(s)| \right) \Delta s \right), & \text{for } t \in [t_2, \infty)_{\mathbb{T}} \\ 0, & \text{for } t \in (-\infty, t_2)_{\mathbb{T}}. \end{cases}$$

From assumption (H_4) the integral in $T_2(x)$ is well defined for any $x \in B_M$. Moreover,

$$T_2(x)(t) = \int_{t_2}^t H_x(u)\Delta u - T_2(x)(t_2), \ t \in \mathbb{T}.$$

It means that T_2 is differentiable on $\mathbb{T}$ with

$$(T_2(x))^\Delta(t) = \begin{cases} H_x(t), & \text{for } t \in [t_2, \infty)_\mathbb{T} \\ 0, & \text{for } t \in (-\infty, t_2)_\mathbb{T}. \end{cases}$$

This implies that $T_2(x)$ is continuous on $\mathbb{T}$. In order to see that $T_2(x)$ is bounded on $\mathbb{T}$, we notice that from (2) we get $|T_2(x)(t)| \leq \min\{L - \frac{1+q^\star}{2}M, M - L\}$ for $t \in \mathbb{T}$. This means that the operator $T^L : B_M \to \mathrm{BC}(\mathbb{T})$ is well defined. Our next goal is to check assumptions of Theorem 1.1 – the Darbo fixed point for the operator $T^L = T_1^L + T_2$.

We show that $T^L(B_M) \subset B_M$. Let $x \in B_M$ and $t \in [t_2, \infty)_\mathbb{T}$. Assumption (H_5) and (2) imply that

$$T^L(x)(t) = L - q(t)x(\tau(t)) - \int_t^\infty H_x(u)\Delta u \leq L + \int_{t_2}^\infty H(u)\Delta u \leq M$$

and

$$T^L(x)(t) = L - q(t)x(\tau(t)) - \int_t^\infty H_x(u)\Delta u \geq L - q(t)x(\tau(t)) - \int_{t_2}^\infty H(u)\Delta u$$

$$\geq L - q^\star M - \min\{L - \tfrac{1+q^\star}{2}M, M - L\} \geq L - q^\star M - L + \tfrac{1+q^\star}{2}M = \tfrac{1-q^\star}{2}M.$$

It is easy to see that

$$\|T_1^L x - T_1^L y\| \leq q^\star \|x - y\|, \text{ for } x, y \in B_M,$$

so that T_1^L is a contraction.

Now we prove the continuity of T_2. Let $x \in B_M$, $\varepsilon > 0$. Since $[\frac{1-q^\star}{2}M, M]$ is a compact set, f is uniformly continuous on $[\frac{1-q^\star}{2}M, M]$. Hence, there exists $\eta > 0$ such that

$$|f(w) - f(v)| < \frac{\varepsilon}{1 + \int_{t_2}^\infty \left(\frac{1}{|r(u)|} \int_{t_2}^u |a(s)|\Delta s\right)\Delta u}, \tag{3}$$

for $|w - v| < \eta$ and $w, v \in [\frac{1-q^\star}{2}M, M]$. For $y \in B_M$ such that $\|x - y\| < \eta$ we have from (3) that

$$|T_2(x)(t) - T_2(y)(t)| \leq \int_t^\infty \left(\frac{1}{|r(u)|} \int_{t_2}^u |a(s)||f(x(\delta(s))) - f(y(\delta(s)))|\Delta s\right)\Delta u < \varepsilon$$

for $t \geq t_2$. Hence, $\|T_2 x - T_2 y\| < \varepsilon$ for $y \in B_M$, $\|x - y\| < \eta$, which proves the continuity of T_2 on B_M. Now we apply the relative compactness criterion in space $\mathrm{BC}(\mathbb{T})$ – Theorem 1.2, for $T_2(B_M)$. From (2) we get that $T_2(B_M) \subset B_M$, which means that $T_2(B_M)$ is bounded. Let $\varepsilon > 0$. From (H_4) we get the existence of t_ε such that $t_\varepsilon > t_2$ and

$$2\int_{t_\varepsilon}^\infty H(u)\Delta u < \varepsilon. \tag{4}$$

Since H is rd-continuous on compact set $[t_2, t_\varepsilon]_\mathbb{T}$, it is bounded by some K, $K > 0$; see [5], p. 23. Moreover, the compactness of $[t_2, t_\varepsilon]_\mathbb{T}$ implies that there exist $s_1, \ldots, s_{n_\varepsilon} \in [t_2, t_\varepsilon]_\mathbb{T}$ such that $s_1 < \ldots < s_{n_\varepsilon}$ and sets $(s_i - \frac{\varepsilon}{K}, s_i + \frac{\varepsilon}{K})_\mathbb{T}$ for $i = 1, \ldots, n_\varepsilon$ provide $\frac{\varepsilon}{K}-$ net of $[t_2, t_\varepsilon]_\mathbb{T}$. We prove that sets $E_0 = (-\infty, t_2)_\mathbb{T}$, $E_{n_\varepsilon + 1} = (t_\varepsilon, \infty)_\mathbb{T}$, $E_i = (s_i - \frac{\varepsilon}{K}, s_i + \frac{\varepsilon}{K})_\mathbb{T}$, for $i = 1, \ldots, n_\varepsilon$ with $s_0 = t_0$, $s_1, \ldots, s_{n_\varepsilon}$, and $s_{n_\varepsilon + 1}$ any element of $(t_\varepsilon, \infty)_\mathbb{T}$ satisfied the condition

$$\sup_{x \in B_M} \sup_{s \in E_i} |T_2(x)(s) - T_2(x)(s_i)| < \varepsilon, \ i = 0, \ldots, n_\varepsilon + 1. \tag{5}$$

For $i = 0$ condition (5) is evident, because T_2 is constant on E_0. For $i = n_\varepsilon + 1$ from (4) we have

$$\sup_{x \in B_M} \sup_{s \in (t_\varepsilon, \infty)_\mathbb{T}} |T_2(x)(s) - T_2(x)(s_{n_\varepsilon + 1})| \leq 2 \int_{t_\varepsilon}^{\infty} H(u) \Delta u < \varepsilon.$$

For $i \in \{1, \ldots, n_\varepsilon\}$ we get

$$\sup_{x \in B_M} \sup_{s \in E_i} |T_2(x)(s) - T_2(x)(s_i)| \leq \sup_{x \in B_M} \sup_{s \in E_i} \int_{\min\{s, s_i\}}^{\max\{s, s_i\}} |H_x(u)| \Delta u \leq$$

$$\sup_{s \in E_i} \int_{\min\{s, s_i\}}^{\max\{s, s_i\}} H(u) \Delta u \leq \sup_{s \in E_i} \int_{\min\{s, s_i\}}^{\max\{s, s_i\}} K \Delta u = \sup_{s \in E_i} K|s - s_i| < \varepsilon.$$

It proves that $T_2(B_M)$ is a relatively compact subset of $\mathrm{BC}(\mathbb{T})$.

Since T_1^L is a contraction and since T_2 is compact, it follows from Proposition 1.1 that $T^L : B_M \to B_M$ is an α-condensing operator. From the Darbo theorem we get that there exists $x \in B_M$, a fixed point of T^L on B_M. Thus, for $t \geq t_2$ we have

$$x(t) + q(t)x(\tau(t)) = L - \int_t^{\infty} \left(\frac{1}{r(u)} \int_{t_2}^{u} (a(s)f(x(\delta(s))) + b(s)) \, \Delta s \right) \Delta u.$$

Since the right side of the above equality is a differentiable function, the left side possesses this property, as well. Hence, for $t \geq t_2$,

$$r(t) \left(x(t) + q(t)x(\tau(t)) \right)^{\Delta} = \int_{t_2}^{t} (a(s)f(x(\delta(s))) + b(s)) \, \Delta s.$$

In an analogous way, we have for $t \geq t_2$

$$\left(r(t) \left(x(t) + q(t)x(\tau(t)) \right)^{\Delta} \right)^{\Delta} = a(t)f(x(\delta(t))) + b(t).$$

Hence, for $x : \mathbb{T} \to \mathbb{R}$ we have that $x(\cdot) + q(\cdot)x(\tau(\cdot)) \in \mathrm{C}_{\mathrm{rd}}^1[t_2, \infty)$, $r(\cdot) \cdot (x(\cdot) + q(\cdot)x(\tau(\cdot)))^{\Delta} \in \mathrm{C}_{\mathrm{rd}}^1[t_2, \infty)$, and x satisfies (1) for $t \geq t_2$ with $x(t) \in [\frac{1-q^*}{2}M, M]$ for $t \in \mathbb{T}$. It means that x is a bounded positive solution to (1).

Now we prove the existence of uncountably many positive solutions to (1) lying in $[\frac{1-q^\star}{2}M, M]$. Let $L_1, L_2 \in [\frac{1+q^\star}{2}M, M]$, and $L_1 < L_2$. From the previous part of the proof, it is easy to see that there exist $t_2^1, t_2^2 \geq t_1$ and $x^1, x^2 \in \mathrm{BC}(\mathbb{T})$, each a fixed point of the operator T^{L_i} on B_M, $i = 1, 2$, respectively, where

$$T^{L_i}(x)(t) = \begin{cases} L_i - q(t)x(\tau(t)) - T_2(x)(t) & \text{for } t \in [t_2^i, \infty)_{\mathbb{T}} \\ T^{L_i}(x)(t_2^i), & \text{for } t \in (-\infty, t_2^i)_{\mathbb{T}}. \end{cases}$$

Thus, x^i are solutions to (1) satisfying this equation for $t \geq \max\{t_2^1, t_2^2\}$. By (H_4) there exists $t_3 \geq \max\{t_2^1, t_2^2\}$ such that

$$|T_2(x^1)(t)| + |T_2(x^2)(t)| \leq \frac{L_2 - L_1}{2}, \ t \geq t_3.$$

From this it is clear that, for $t \geq t_3$,

$$|x^1(t) - x^2(t) + q(t)(x^1(\tau(t)) - x^2(\tau(t)))| \geq L_2 - L_1 - (|T_2(x^1)(t)| + |T_2(x^2)(t)|) > 0,$$

which means that x^1 and x^2 are different solutions to (1) lying in $[\frac{1-q^\star}{2}M, M]$.

In the analogous way, we can prove that (1) possesses uncountably many negative solutions, so the proof of this part is omitted. □

Now we are in a position to formulate and prove sufficient conditions for the existence of a bounded solution to (1) that satisfies (1) on the maximal interval. To achieve our goal we have to make stronger assumptions on the time scale.

Theorem 2.2 *Assume that:*

(H_0) *the number of left-scattered points in every compact subset of $\mathbb{T}$ is finite;*

(H_1') *$\tilde\tau : \mathbb{T} \to \mathbb{T}$ is a surjective function of the form $\tilde\tau(t) = t - \tau$, and $\tilde\delta : \mathbb{T} \to \mathbb{T}$ is a surjective function of the form $\tilde\delta(t) = t - \delta$, $t \in \mathbb{T}$ with $\tau > \delta > 0$;*

(H_2) *$a, b : \mathbb{T} \to \mathbb{R}$ are rd-continuous and $r : \mathbb{T} \to \mathbb{R} \setminus \{0\}$, $q : \mathbb{T} \to \mathbb{R}$ are continuous;*

(H_3) *$f : \mathbb{R} \to \mathbb{R}$ is continuous;*

(H_4') *$\int\limits_{t_1}^{\infty} \left(\frac{1}{|r(u)|} \int\limits_{u}^{\infty} |a(s)| \Delta s \right) \Delta u < +\infty, \ \int\limits_{t_1}^{\infty} \left(\frac{1}{|r(u)|} \int\limits_{u}^{\infty} |b(s)| \Delta s \right) \Delta u < +\infty$, for some $t_1 \in \mathbb{T}$;*

(H_5') *$\inf\limits_{t \in \mathbb{T}} q(t) = q_\star > 0$, $\sup\limits_{t \in \mathbb{T}} q(t) = q^\star < 1$.*

Then:

  i) *equation (1) possesses a bounded solution $x : \mathbb{T} \to \mathbb{R}$, which satisfies (1) for $t \geq t_1'$, if $t_1' := \min\{t_1 \in \mathbb{T}$ which fulfils $(H_4')\}$;*

  ii) *equation (1) possesses a bounded solution $x : \mathbb{T} \to \mathbb{R}$, which satisfies (1) for $t \geq t_1''$ for any $t_1'' > \inf\{t_1 \in \mathbb{T}$ which fulfills $(H_4')\}$, if $\min\{t_1 \in \mathbb{T}$ which fulfills $(H_4')\}$ does not exist.*

Proof At the beginning, we assume that $\min\{t_1 \in \mathbb{T} \text{ which fulfills } (H_4')\}$ exists and is equal to t_1'. Let $M > 0$. In an analogous way as in Theorem 2.1 we get that there exist $t_2^M \geq t_1'$ and $x : \mathbb{T} \to \mathbb{R}$ such that x is a fixed point of the operator $T : \overline{B}_{\mathrm{BC}(\mathbb{T})}(0, M) \to \overline{B}_{\mathrm{BC}(\mathbb{T})}(0, M)$ defined as follows:

$$
T(x)(t) = \begin{cases} -q(t)x(t - \tau) + \int\limits_t^\infty G_x(u)\Delta u, & \text{for } t \in [t_2^M, \infty)_{\mathbb{T}} \\ T(x)(t_2^M), & \text{for } t \in (-\infty, t_2^M)_{\mathbb{T}}, \end{cases}
$$

where

$$
G_x(u) = \frac{1}{r(u)} \left(\int\limits_u^\infty (a(s)f(x(s - \delta)) + b(s)) \, \Delta s \right), \ u \geq t_2^M.
$$

This means that x is a bounded solution to (1), which satisfied it for $t \geq t_2^M$ and $x|_{(-\infty, t_2^M]_{\mathbb{T}}} = x(t_2^M)$, $|x(t)| \leq M$, $t \in \mathbb{T}$. We claim that (H_1') implies $\rho(t + \tau) - \tau = \rho(t)$ and $\rho(t) + \tau - \delta \in \mathbb{T}$, $t \in \mathbb{T}$, where $\rho(\cdot)$ denotes the backward jump operator on $\mathbb{T}$. Now we divide the proof into two parts:

1. $\rho(t_2^M) < t_2^M$. Notice that $\rho(t_2^M) + \tau - \delta \geq t_2^M$. Putting $\rho(t_2^M + \tau)$ into the equation

$$
x(t - \tau) = \frac{1}{q(t)} \left(-x(t) + \int\limits_t^\infty \frac{1}{r(u)} \left(\int\limits_u^\infty (a(s)f(x(s - \delta)) + b(s)) \, \Delta s \right) \Delta u \right), \tag{6}
$$

we get that $\tilde{x} : \mathbb{T} \to \mathbb{R}$ defined by the formula $\tilde{x}|_{[t_2^M, +\infty)_{\mathbb{T}}} = x|_{[t_2^M, +\infty)_{\mathbb{T}}}$, $\tilde{x}|_{(-\infty, \rho(t_2^M)]_{\mathbb{T}}} = \tilde{x}(\rho(t_2^M))$, and

$$
\tilde{x}(\rho(t_2^M)) = \frac{1}{q(\rho(t_2^M + \tau))} \left(-x(\rho(t_2^M + \tau)) + \int\limits_{\rho(t_2^M + \tau)}^\infty G_x(u)\Delta u \right)
$$

satisfies (1) for $t \geq \rho(t_2^M)$, which means it is a solution to (1). Moreover, $|\tilde{x}(t)| \leq M$ for $t \geq t_2^M$ and from (6)

$$
|\tilde{x}(\rho(t_2^M))| \leq \frac{1}{q_\star} \left(M + \int\limits_{t_1'}^\infty G(u)\Delta u \right),
$$

where $Q := \max\limits_{|y| \leq M} |f(y)|$ and

$$
G(u) = \frac{1}{r(u)} \int\limits_u^\infty (a(s)Q + b(s)) \, \Delta s, \quad u \geq t_1'. \tag{7}
$$

2. $\rho(t_2^M) = t_2^M$. Let $t \in [t_2^M + \delta - \tau, t_2^M)_{\mathbb{T}}$. Hence, we have that $t + \tau - \delta \geq t_2^M$. Putting $t + \tau$ into equation (6) we get that $\tilde{x} : \mathbb{T} \to \mathbb{R}$ defined by the formula $\tilde{x}|_{[t_2^M, +\infty)_{\mathbb{T}}} = x|_{[t_2^M, +\infty)_{\mathbb{T}}}$, $\tilde{x}|_{(-\infty, t_2^M + \delta - \tau]_{\mathbb{T}}} = \tilde{x}(t_2^M + \delta - \tau)$ and

$$
\tilde{x}(t) = \frac{1}{q(t + \tau)} \left(-x(t + \tau) + \int\limits_{t+\tau}^\infty G_x(u)\Delta u \right), \ t \in [t_2^M - \delta + \tau, t_2^M)_{\mathbb{T}}
$$

satisfies (1) for $t \geq t_2^M - \delta + \tau$, which means $\tilde{x}$ is a solution to (1). Moreover, $|\tilde{x}(t)| \leq M$ for $t \geq t_2^M$ and from (6)

$$|\tilde{x}(t)| \leq \frac{1}{q_\star}\left(M + \int_{t_1'}^{\infty} G(u)\Delta u\right), \quad t \in [t_2^M - \delta + \tau, t_2^M)_\mathbb{T}.$$

Our next goal is to prove that there exists a solution to (1) that satisfies this equation on $t \geq t_1'$. Thanks to (H_0), without loss of generality, we can assume that there exists $k \in \mathbb{N}$ such that

$$[t_1, t_2^M]_\mathbb{T} = \{t_2^M, \rho(t_2^M), \rho^2(t_2^M), \ldots, \rho^k(t_2^M)\} \cup [t_1, \rho^k(t_2^M))_\mathbb{T}$$

with $t_2^M > \rho(t_2^M) > \rho^2(t_2^M) > \ldots > \rho^k(t_2^M)$ and the interval $[t_1, \rho^k(t_2^M))_\mathbb{T}$ does not include left-scattered points. Similarly to the case $\rho(t_2^M) < t_2^M$ there exists a solution to (1) $\hat{x} : \mathbb{T} \to \mathbb{R}$ that satisfies this equation on $[\rho^k(t_2^M), \infty)_\mathbb{T}$, $\hat{x}|_{(-\infty, \rho^k(t_2^M)]_\mathbb{T}} = \hat{x}(\rho^k(t_2^M))$ and

$$|\hat{x}(t)| \leq \frac{1}{q_\star}\left(M + k\int_{t_1'}^{\infty} G(u)\Delta u\right), \quad t \in \mathbb{T}.$$

Results from the case $\rho(t_2^M) = t_2^M$ prove the existence of a solution to (1) $\check{x} : \mathbb{T} \to \mathbb{R}$ that satisfies this equation on $[\rho^k(t_2^M) + \delta - \tau, \infty)_\mathbb{T}$, $\check{x}|_{(-\infty, \rho^k(t_2^M) + \delta - \tau]_\mathbb{T}} = \check{x}(\rho^k(t_2^M) + \delta - \tau)$ and

$$|\check{x}(t)| \leq \frac{1}{q_\star}\left(M + (k+1)\int_{t_1'}^{\infty} G(u)\Delta u\right), \quad t \in \mathbb{T}.$$

Since $\lim_{l \to \infty} \rho^k(t_2^M) + l(\delta - \tau) = -\infty$, there exists such $l_0 \in \mathbb{N}$ that $\rho^k(t_2^M) + l_0(\delta - \tau) \leq t_1' < \rho^k(t_2^M) + (l_0 - 1)(\delta - \tau)$. Similarly to the above we prove the existence of a solution to (1) $\bar{x} : \mathbb{T} \to \mathbb{R}$, which satisfies it on $[t_1', \infty)_\mathbb{T}$, $\bar{x}|_{(-\infty, t_1']_\mathbb{T}} = \bar{x}(t_1')$ with the estimation

$$|\bar{x}(t)| \leq \frac{1}{q_\star}\left(M + (k + l_0)\int_{t_1'}^{\infty} G(u)\Delta u\right), \quad t \in \mathbb{T}.$$

Case $ii)$ is analogous and is left to the reader. □

Remark 2.1 *Notice that* (H_1') *implies that* $\inf \mathbb{T} = -\infty$. *It is worth noting that besides* $\mathbb{R}$ *with any* $\delta, \tau \in \mathbb{R}$, $\delta > \tau$ *and* $\mathbb{Z}$ *with any* $\delta, \tau \in \mathbb{Z}$, $\delta > \tau$ *assumption* (H_1') *satisfies for example* $\mathbb{Z} \cup \bigcup_{k \in \mathbb{Z}}[\frac{1}{4} + k, \frac{3}{4} + k]$ *with* $\delta, \tau \in \mathbb{Z}$, $\delta > \tau$.

The last theorem presents the necessary condition for the existence of an asymptotically zero solution to (1).

Theorem 2.3 *Assume that:*

(H_1) $\tau : \mathbb{T} \to \mathbb{T}$ *is rd-continuous such that* $\tau(t) \leq t$, $t \in \mathbb{T}$ *and* $\lim\limits_{s \to \infty} \tau(s) = +\infty$, *and* $\delta : \mathbb{T} \to \mathbb{T}$ *is rd-continuous such that* $\delta(t) \leq t$, $t \in \mathbb{T}$ *and* $\lim\limits_{s \to \infty} \delta(s) = +\infty$;

(H_2') $a, b : \mathbb{T} \to (0, +\infty)$ *are rd-continuous;*

(H_3') $f : \mathbb{R} \to \mathbb{R}$ *is continuous with* $\min\limits_{|y| \leq \eta} f(y) =: d > 0$ *for some* $\eta > 0$;

(H_6) $q : \mathbb{T} \to \mathbb{R}$ *is bounded, continuous;*

(H_7) $r : \mathbb{T} \to (0, +\infty)$ *is continuous with* $\int\limits_{t_0'}^{\infty} \frac{\Delta u}{r(u)} < +\infty$ *for some* $t_0' \in \mathbb{T}$.

If (1) *possesses a solution* $x : \mathbb{T} \to \mathbb{R}$ *such that* $\lim\limits_{t \to \infty} x(t) = 0$, *then for some* $t_2 \in \mathbb{T}$

$$\int\limits_{t_2}^{\infty} \frac{1}{r(u)} \left(\int\limits_{t_2}^{u} a(s) \Delta s \right) \Delta u < +\infty \;\wedge\; \int\limits_{t_2}^{\infty} \frac{1}{r(u)} \left(\int\limits_{t_2}^{u} b(s) \Delta s \right) \Delta u < +\infty.$$

Proof From $\delta(t) \to \infty$ and $x(t) \to 0$ as $t \to \infty$ it is clear that there exists $t_1 \in \mathbb{T}$ such that $|x(\delta(t))| \leq \eta$ for $t \geq t_1$. Moreover, there exists $t_0 \in \mathbb{T}$, such that x satisfies (1) for $t \geq t_0$. Integrating (1) from t to $t_2 := \max\{t_0, t_0', t_1\}$, we get that

$$r(t) \left(x(t) + q(t) x(\tau(t)) \right)^{\Delta} - A_{t_2} = \int\limits_{t_2}^{t} a(s) f(x(\delta(s))) + b(s) \, \Delta s,$$

where $A_{t_2} := r(t_2) \left(x(t_2) + q(t_2) x(\tau(t_2)) \right)^{\Delta}$. Dividing the last equality by $r(t)$ and integrating above from t_2 to ∞, thanks to $\lim\limits_{t \to \infty} x(t) = 0$, (H_7), and boundedness of q, we see that

$$x(t_2) + q(t_2) x(\tau(t_2)) + A_{t_2} \int\limits_{t_2}^{\infty} \frac{\Delta u}{r(u)} = - \int\limits_{t_2}^{\infty} \frac{1}{r(u)} \left(\int\limits_{t_2}^{u} (a(s) f(x(\delta(s))) + b(s) \Delta s \right) \Delta u. \tag{8}$$

From (H_2') and (H_3'), we get that

$$\int\limits_{t_2}^{\infty} \frac{1}{r(u)} \left(\int\limits_{t_2}^{u} (a(s) f(x(\delta(s))) + b(s)) \Delta s \right) \Delta u \geq \int\limits_{t_2}^{\infty} \frac{1}{r(u)} \left(\int\limits_{t_2}^{u} (a(s) d + b(s)) \Delta s \right) \Delta u. \tag{9}$$

Hence, from (8), (9), and (H_2'), we see that

$$\int\limits_{t_2}^{\infty} \frac{1}{r(u)} \left(\int\limits_{t_2}^{u} a(s) \Delta s \right) \Delta u < +\infty \;\wedge\; \int\limits_{t_2}^{\infty} \frac{1}{r(u)} \left(\int\limits_{t_2}^{u} b(s) \Delta s \right) \Delta u < +\infty.$$

$\square$

3. Examples

Now we present two examples of equations that can be considered by the presented method.

Example 3.1 *Let us consider the time scale* $\mathbb{T}_p := \{p^n : n \in \mathbb{N}\}$ *with* $p > 1$ *and the second-order nonlinear neutral dynamic equation*

$$\left(t^3 \left(x(t) + \tfrac{1}{t} x\left(\tfrac{t}{p^2}\right)\right)^\Delta\right)^\Delta = -\tfrac{1}{p}\left|x\left(\tfrac{t}{p}\right)\right| + \tfrac{(p+1)^2}{t^2},\ t \ge t_0. \tag{10}$$

It is evident that assumptions (H_1), (H_2), *and* (H_3) *of Theorem 2.1 are satisfied. Moreover, for checking* (H_4) *and* (H_5), *notice that* $\sup_{t\in\mathbb{T}} q(t) = \sup_{t\in\mathbb{T}} \tfrac{1}{t} = \tfrac{1}{p} < 1$ *and*

$$\int_p^\infty \left(\tfrac{1}{u^3} \int_p^u \tfrac{1}{p} \Delta s\right) \Delta u = \tfrac{p^2+p-1}{p+1} < \infty,\ \int_p^\infty \left(\tfrac{1}{u^3} \int_p^u \tfrac{(p+1)^2}{s^2} \Delta s\right) \Delta u = \tfrac{p+1}{(p^2+p+1)} < \infty.$$

Hence, there exist uncountably many nonoscillatory bounded solutions to (10). *It is easy to see that for any* $c \ge 0$, $x_c(t) = c + \tfrac{1}{t^2}$, $t \in \mathbb{T}$ *satisfies* (10) *for* $t \in \mathbb{T}$, *which means that* x_c *is a bounded nonoscillatory solution to* (10). *Notice that results from [10] cannot be applied to* (10).

Example 3.2 *Consider the time scale* $\mathbb{T} = \mathbb{Z}$ *and the second-order nonlinear neutral difference equation*

$$\Delta \left(r_n \Delta \left(x_n + q_n x_{n-\tau}\right)\right) = a_n f(x_{n-\delta}) + b_n,\ n \ge \tau \tag{11}$$

with $\tau, \delta \in \mathbb{N}$, $\tau > \delta$, $\sup_{n\in\mathbb{Z}} q_n =: q^\star < 1$, $\inf_{n\in\mathbb{Z}} q_n =: q_\star > 0$, $r_n = \sqrt{n+1}$, $a_n = \tfrac{1}{4n^2-1}$, $b_n = 4^{-n}$, $n \in \mathbb{N} \cup \{0\}$, $r_n = a_n = b_n = 0$, $-n \in \mathbb{N}$, *and* $f : \mathbb{R} \to \mathbb{R}$ *a continuous function.* (11) *satisfies assumptions of Theorem 2.2, because*

$$\sum_{n=0}^\infty \tfrac{1}{\sqrt{n+1}} \sum_{k=n}^\infty \tfrac{1}{4k^2-1} = \sum_{n=0}^\infty \tfrac{1}{\sqrt{n+1}} \tfrac{1}{2(2n-1)} < \infty,\ \sum_{n=0}^\infty \tfrac{1}{\sqrt{n+1}} \sum_{k=n}^\infty 4^{-k} < \infty.$$

This means that there exists a bounded sequence $x : \mathbb{N} \cup \{0\} \to \mathbb{R}$ *that satisfied*

$$\Delta \left(\sqrt{n+1} \Delta \left(x_n + q_n x_{n-\tau}\right)\right) = \tfrac{1}{4n^2-1} f(x_{n-\delta}) + 4^{-n}$$

for $n \ge \tau$.

Acknowledgment

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