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Convolution properties for a family of analytic functions involving q-analogue of Ruscheweyh differential operator

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Abstract: The main object of the present paper is to investigate convolution properties for a new subfamily of analytic functions that are defined by q-analogue of Ruscheweyh differential operator. Several consequences of the main results are also given.

Key words: Analytic functions, subordinations, Ruscheweyh q-differential operator, circular domain

1. Introduction and definitions

The study of q-extension of calculus attracted the attention of prominent researchers due to its applications in various branches of mathematics and physics, as for example, in optimal control problems, areas of ordinary fractional calculus, quantum physics and in operator theory, for details see [6, 10]. Jackson [12, 13] was the first to give some application of q-calculus and introduced the q-analogue of derivative and integral. Recently, Srivastava and Bansal [25, pp. 62] used the q-analogue of derivative for the first time in Geometric function theory by introducing the family $\mathcal{S}_q^*$ of q-extension of starlike functions. The authors studied some useful geometric properties of this class, see also [24, pp. 347 et seq.]. Later on, the theory of q-extension of starlike functions was further nourished by Agrawal and Sahoo in [1] and they introduced the family $\mathcal{S}_q^*(\xi)$ with order ξ $(0 \le \xi < 1)$. After that, many mathematicians wrote wonderful articles and played important roles in developing this area of geometric function theory. Recently in 2014, Kanas and Răducanu [15] defined q-analogue of Ruscheweyh differential operator using the ideas of convolution and then studied some of its properties. For the recent extension of different operator in q-analogue, see [2, 7–9, 16]. The aim of the current paper is to discuss some useful convolution properties for a family of analytic functions associated with circular domain involving q-analogue of Ruscheweyh differential operator.

Let $\mathfrak{A}$ represent the family of functions f that are analytic in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and which have the following normalization

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{D}). \tag{1.1}$$

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For two functions f and g that are analytic in $\mathbb{D}$, we define the convolution of these functions by

$$f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k, \ \ (z \in \mathbb{D}),$$

where f has the form (1.1) and

$$g(z) = z + \sum_{k=2}^{\infty} b_k z^k, \ \ (z \in \mathbb{D}).$$

For $0 < q < 1$, the q-derivative of a function f is defined by

$$\partial_q f(z) = \frac{f(qz) - f(z)}{z(q-1)}, \ \ (z \neq 0, \ q \neq 1). \tag{1.2}$$

It can easily be seen that for $n \in \mathbb{N} := \{1, 2, 3, \ldots\}$ and $z \in \mathbb{D}$

$$\partial_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n, q]\, a_n z^{n-1}, \tag{1.3}$$

where

$$[n, q] = \frac{1 - q^n}{1 - q} = 1 + \sum_{l=1}^{n-1} q^l, \quad [0, q] = 0.$$

For any nonnegative integer n the q-number shift factorial is defined by

$$[n, q]! = \begin{cases} 1, \ n = 0, \\ [1, q]\,[2, q]\,[3, q] \cdots [n, q], \ n \in \mathbb{N}. \end{cases}$$

Also the q-generalized Pochhammer symbol for $x > 0$ is given by

$$[x, q]_n = \begin{cases} 1, \ n = 0, \\ [x, q][x + 1, q] \cdots [x + n - 1, q], \ n \in \mathbb{N}, \end{cases}$$

and for $x > 0$, let q-gamma function is defined as

$$\Gamma_q (x + 1) = [x, q]\, \Gamma_q (t) \ \text{ and } \Gamma_q (1) = 1.$$

We now define a function

$$\Phi (q, \mu + 1; z) = z + \sum_{n=2}^{\infty} \wedge_n \ z^n, \ (\mu > -1, \ z \in \mathbb{D}), \tag{1.4}$$

with

$$\wedge_n = \frac{[\mu + 1, q]_{n+1}}{[n + 1, q]!}.$$

Using $\Phi (q, \mu; z)$ and definition of q-derivative, Kanas and Răducanu [15] defined the differential operator $\mathcal{L}_q^{\mu} : \mathfrak{A} \to \mathfrak{A}$ by

$$\mathcal{L}_q^{\mu} f(z) = \Phi (q, \mu; z) * f(z) = z + \sum_{n=2}^{\infty} \wedge_n \ a_n z^n, \ (\mu > -1, \ z \in \mathbb{D}). \tag{1.5}$$

We note that $\mathcal{L}_q^0 f(z) = f(z)$, $\mathcal{L}_q^1 f(z) = z\partial_q f(z)$ and

$$\mathcal{L}_q^m f(z) = \frac{z\partial_q^m \left(z^{m-1} f(z)\right)}{[m,q]!}, \quad (m \in \mathbb{N}).$$

By making $q \to 1^-$, the q-differential operator defined in (1.5) reduces to the familiar differential operator introduced in [19], see also [18].

From (1.5), one can easily obtain the following identity

$$[\mu+1,q]\mathcal{L}_q^{\mu+1} f(z) = [\mu,q]\mathcal{L}_q^{\mu} f(z) + q^{\mu} z\partial_q \left(\mathcal{L}_q^{\mu} f(z)\right). \tag{1.6}$$

Motivated by [4, 5, 17, 20], we now define a subfamily $\mathcal{H}_q^b(\mu, A, B)$ of $\mathfrak{A}$ by using the operator $\mathcal{L}_q^{\mu}$ as follows;

Definition 1.1 *Let $-1 \le B < A \le 1$ and $0 < q < 1$. Then a function $f \in \mathfrak{A}$ is in the class $\mathcal{H}_q^b(\mu, A, B)$, if it satisfies*

$$1 + \frac{q+1}{b}\left(\frac{\mathcal{L}_q^{\mu+1} f(z)}{\mathcal{L}_q^{\mu} f(z)} - 1\right) \prec \frac{1+Az}{1+Bz}, \tag{1.7}$$

where the notation "$\prec$" denotes the familiar subordinations.

Equivalently, a function $f \in \mathfrak{A}$ is in the class $\mathcal{H}_q^b(\mu, A, B)$, if and only if

$$\left|\frac{(q+1)\left(\frac{\mathcal{L}_q^{\mu+1} f(z)}{\mathcal{L}_q^{\mu} f(z)} - 1\right)}{b(A-B) + (q+1)B\left(1 - \frac{\mathcal{L}_q^{\mu+1} f(z)}{\mathcal{L}_q^{\mu} f(z)}\right)}\right| < 1. \tag{1.8}$$

It is noticed that, by giving particular values of the parameters in $\mathcal{H}_q^b(\mu, A, B)$, we get various familiar subfamilies of univalent functions, for example;

(i). Taking $\mu = 0$ and $b = \gamma(q+1)$ with $\gamma \in \mathbb{C} \setminus \{0\}$ in $\mathcal{H}_q^b(\mu, A, B)$, we obtain the class $\mathcal{S}_q^*(\gamma, A, B)$ and it is a special form of the one studied in [20]. Further for $\gamma = 1$, we have the class $\mathcal{S}_q^*(1, A, B) \cong \mathcal{S}_q^*(A, B)$ investigated in [21] and the family $\mathcal{S}_q^*(1, 1, -1) \cong \mathcal{S}_q^*$ was studied in [11]. Also we see that $\mathcal{H}_q^{q+1}(0, 1-2\xi, -1) \cong \mathcal{S}_q^*(\xi)$, introduced by Agrawal and Sahoo [1].

(ii). Making $\mu = 0$ and $b = \gamma q$ with $\gamma \in \mathbb{C} \setminus \{0\}$ in $\mathcal{H}_q^b(\mu, A, B)$, we have the class $\mathcal{K}_q(\gamma, A, B)$ investigated in [20] and further putting $\gamma = 1$, $B = -1$ and $A = (1-2\xi)$ with $0 \le \xi < 1$, we get the class $\mathcal{K}_q(1, 1-2\xi, -1) \cong \mathcal{K}_q(\xi)$ studied in [21].

(iii). Also $\lim\limits_{q\to 1^-} \mathcal{H}_q^2(0, A, B) \cong \mathcal{S}^*(A, B)$ and $\lim\limits_{q\to 1^-} \mathcal{H}_q^1(1, A, B) \cong \mathcal{K}(A, B)$ are the familiar families of Janwoski starlike and Janowski convex functions respectively which was introduced and studied by Janowski [14].

To avoid repetition, we shall assume, unless otherwise stated, that

$$0 \le \theta < 2\pi,\ 0 < q < 1,\ -1 \le B < A \le 1,\ \mu > -1 \text{ and } b \in \mathbb{C} \setminus \{0\}.$$

2. Convolution Properties and Their Consequences

Theorem 2.1 *Let $f \in \mathfrak{A}$ be given by (1.1). Then the function f is in the class $\mathcal{H}_q^b(\mu, A, B)$, if and only if*

$$\frac{1}{z}\left[\mathcal{L}_q^\mu f(z) * \frac{Mz - Lqz^2}{(1-z)(1-qz)}\right] \neq 0, \quad (z \in \mathbb{D}), \tag{2.1}$$

for all

$$\begin{aligned} L &= L_\theta = \left(\frac{e^{-i\theta}+B}{A-B}\frac{q+1}{b}+1\right), \\ M &= M_\theta = \frac{e^{-i\theta}+B}{B-A}\frac{q+1}{b}\left(\frac{q^\mu-1}{[\mu+1,q]}+\frac{[\mu,q]}{q^\mu}\right)+1, \end{aligned} \tag{2.2}$$

and also for $L = M = 1$.

Proof Since the function $f \in \mathcal{H}_q^b(\mu, A, B)$ is analytic in $\mathbb{D}$, it implies that $\mathcal{L}_q^\mu f(z) \neq 0$ for all $z \in \mathbb{D}^* = \mathbb{D}\backslash\{0\}$; that is $\frac{1}{z}\mathcal{L}_q^\mu f(z) \neq 0$ for $z \in \mathbb{D}$ and this is equivalent to (2.1) for $L = M = 1$. From (1.7), according to the definition of the subordination, there exists an analytic function w with the property that $w(0) = 0$ and $|w(z)| < 1$ such that

$$1 + \frac{q+1}{b}\left(\frac{\mathcal{L}_q^{\mu+1} f(z)}{\mathcal{L}_q^\mu f(z)} - 1\right) = \frac{1+Aw(z)}{1+Bw(z)} \quad (z \in \mathbb{D}),$$

which is equivalent for $z \in \mathbb{D},\ 0 \leq \theta < 2\pi$

$$1 + \frac{q+1}{b}\left(\frac{\mathcal{L}_q^{\mu+1} f(z)}{\mathcal{L}_q^\mu f(z)} - 1\right) \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}}. \tag{2.3}$$

Using the identity (1.6), we can rewrite (2.3) as

$$1 + \frac{q+1}{b}\left(\frac{q^\mu}{[\mu+1,q]}\frac{z\partial_q \mathcal{L}_q^\mu f(z)}{\mathcal{L}_q^\mu f(z)} + \frac{[\mu,q]}{[\mu+1,q]} - 1\right) \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}},$$

and further written in more simplified form

$$\left[Q\mathcal{L}_q^\mu f(z) + Rz\partial_q \mathcal{L}_q^\mu f(z)\right]\left(1+Be^{i\theta}\right) \neq \mathcal{L}_q^\mu f(z)\left(1+Ae^{i\theta}\right), \tag{2.4}$$

where

$$Q = \left(1 + \frac{[\mu,q](q+1)}{[\mu+1,q]b} - \frac{(q+1)}{b}\right) \text{ and } R = \frac{q^\mu(q+1)}{[\mu+1,q]b}. \tag{2.5}$$

Now using the following convolution properties in (2.4)

$$\mathcal{L}_q^\mu f(z) * \frac{z}{1-z} = \mathcal{L}_q^\mu f(z) \quad \text{and} \quad \mathcal{L}_q^\mu f(z) * \frac{z}{(1-z)(1-qz)} = z\partial_q \mathcal{L}_q^\mu f(z),$$

and then simple computation gives

$$\mathcal{L}_q^{\mu} f\left(z\right) * \left[\left\{\frac{Qz}{1-z} + \frac{Rz}{(1-z)(1-qz)}\right\}\left(1+Be^{i\theta}\right) - \frac{z}{1-z}\left(1+Ae^{i\theta}\right)\right] \neq 0,$$

or equivalently

$$\frac{1}{z}\left[\mathcal{L}_q^{\mu} f\left(z\right) * \left\{\frac{Mz - Lqz^2}{(1-z)(1-qz)}\right\}\right] \neq 0,$$

which is the required necessary condition.

Conversely, Assume that the condition (2.1) hold for $L_\theta = M_\theta = 1$, it follows that $\frac{1}{z}\mathcal{L}_q^{\mu} f\left(z\right) \neq 0$ for all $z \in \mathbb{D}$. Thus, the function $h\left(z\right) = 1 + \frac{(q+1)}{b}\left(\frac{\mathcal{L}_q^{\mu+1} f(z)}{\mathcal{L}_q^{\mu} f(z)} - 1\right)$ is analytic in $\mathbb{D}$ and $h\left(0\right) = 1$. Since we have shown that (2.4) and (2.1) are equivalent, we have

$$1 + \frac{(q+1)}{b}\left(\frac{\mathcal{L}_q^{\mu+1} f\left(z\right)}{\mathcal{L}_q^{\mu} f\left(z\right)} - 1\right) \neq \frac{1+Ae^{i\theta}}{1+Be^{i\theta}}. \quad \left(z \in \mathbb{D}\right). \tag{2.6}$$

Suppose that

$$H\left(z\right) = \frac{1+Az}{1+Bz}, \quad z \in \mathbb{D}.$$

Now from relation (2.6) it is clear that $H\left(\partial\mathbb{D}\right) \cap h\left(\mathbb{D}\right) = \phi$. Therefore, the simply connected domain $h\left(\mathbb{D}\right)$ is contained in a connected component of $\mathbb{C}\backslash H\left(\partial\mathbb{D}\right)$. The univalence of the function h together with the fact $H\left(0\right) = h\left(0\right) = 1$ shows that $h \prec H$ which shows that $f \in \mathcal{H}_q^b\left(\mu, A, B\right)$. □

Substituting different values of the parameters μ, b, A and B in Theorem 2.1, we obtain the following familiar corollaries studied earlier in [21].

Corollary 2.2 *Let the function $f \in \mathfrak{A}$ be of the form (1.1). Then the function f is in the family $\mathcal{H}_q^{q+1}\left(0, 1-2\xi, -1\right) \cong \mathcal{S}_q^*\left(\xi\right)$, if and only if*

$$\frac{1}{z}\left[f(z) * \frac{z - L_1 qz^2}{(1-z)(1-qz)}\right] \neq 0, \quad \left(z \in \mathbb{D}\right),$$

for all $L_1 = \left(e^{-i\theta} + 1 - 2\xi\right)/2\left(1-\xi\right)$ and also for $L_1 = 1$.

Corollary 2.3 *Let the function $f \in \mathfrak{A}$ be of the form (1.1). Then the function $f \in \mathcal{H}_q^q\left(1, 1-2\xi, -1\right) \cong \mathcal{K}_q\left(\xi\right)$, f and only if*

$$\frac{1}{z}\left[f(z) * \frac{z + \left(1 - (q+1)L_1\right)qz^2}{(1-z)(1-qz)(1-q^2z)}\right] \neq 0, \quad \left(z \in \mathbb{D}\right),$$

for all $L_1 = \left(e^{-i\theta} + 1 - 2\xi\right)/2\left(1-\xi\right)$ and also for $L_1 = 1$.

Furthermore, by making $q \to 1^-$ in Corollaries 2.2 and 2.3, we obtain the results for the familiar sets $\mathcal{S}^*\left(\xi\right)$ and $\mathcal{K}\left(\xi\right)$ of starlike and convex functions of order ξ respectively which improves the convolution results proved in [22].

Theorem 2.4 *Let $f \in \mathfrak{A}$ be of the form* (1.1)*. Then a necessary and sufficient condition for $f \in \mathcal{H}_q^b(\mu, A, B)$ is that*

$$1 - \sum_{n=2}^{\infty} \wedge_n \left([n,q]\left(L/M - 1\right) - L/M\right) a_n z^{n-1} \neq 0, \quad (z \in \mathbb{D}), \tag{2.7}$$

where L and M are defined in (2.2)*.*

Proof In Theorem 2.1, we prove that $f \in \mathcal{H}_q^b(\mu, A, B)$, if and only if

$$\frac{1}{z}\left[\mathcal{L}_q^\mu f(z) * \left\{\frac{Mz - Lqz^2}{(1-z)(1-qz)}\right\}\right] \neq 0,$$

for all L, M defined in (2.2) and also for $L = M = 1$. We can rewrite it as

$$\frac{1}{z}\left[\mathcal{L}_q^\mu f(z) * \frac{Mz}{(1-z)(1-qz)} - \mathcal{L}_q^\mu f(z) * \frac{Lqz^2}{(1-z)(1-qz)}\right] \neq 0,$$

equivalently

$$\frac{1}{z}\left[Mz\partial_q \mathcal{L}_q^\mu f(z) - L\left\{z\partial_q \mathcal{L}_q^\mu f(z) - \mathcal{L}_q^\mu f(z)\right\}\right] \neq 0.$$

Now putting series expansion (1.5) instead of $\mathcal{L}_q^\mu f(z)$, we obtain

$$\frac{1}{z}\left[Mz - \sum_{n=2}^{\infty} \wedge_n \left([n,q](L-M) - L\right) a_n z^{n-1}\right] \neq 0,$$

and this completes the proof. □

Taking different particular values of the parameters μ, b, A and B in the last Theorem, we get the following results studied in [21].

Corollary 2.5 *Let $f \in \mathfrak{A}$ be of the form* (1.1)*. Then a necessary and sufficient condition for $f \in \mathcal{S}_q^*(\xi)$ is that*

$$1 - \sum_{n=2}^{\infty} \frac{[n,q]\left(e^{-i\theta} - 1\right) - e^{-i\theta} - 1 + 2\xi}{2(1-\xi)} a_n z^{n-1} \neq 0, \quad (z \in \mathbb{D}).$$

Corollary 2.6 *Let $f \in \mathfrak{A}$ be of the form* (1.1)*. Then a necessary and sufficient condition for the function $f \in \mathcal{K}_q(\xi)$ is*

$$1 - \sum_{n=2}^{\infty} [n,q] \frac{[n,q]\left(e^{-i\theta} - 1\right) - e^{-i\theta} - 1 + 2\xi}{2(1-\xi)} a_n z^{n-1} \neq 0, \quad (z \in \mathbb{D}).$$

Furthermore, by making $q \to 1^-$ in Corollary 2.5, we obtain the results for the familiar sets $\mathcal{S}^*(\xi)$ and $\mathcal{K}(\xi)$ which improves the results proved in [3].

Theorem 2.7 *If $f \in \mathfrak{A}$ is of the form* (1.1) *and satisfy the inequality*

$$\sum_{n=2}^{n} \left\{\frac{[2,q]\left([n,q](1+\nu) - 1\right)(1-B) + (A-B)}{(A-B)\Re b - [2,q](1+B)\nu}\right\} \wedge_n |a_n| \leq 1, \tag{2.8}$$

with $\nu = \frac{q^{\mu}-1}{[\mu+1,q]} + \frac{[\mu,q]}{q^{\mu}}$, *then* $f \in \mathcal{H}_q^b(\mu, A, B)$.

Proof To prove this result, we need to show that (2.7). For this, consider

$$\left|1 - \sum_{n=2}^{\infty} \wedge_n ([n,q](L/M-1) - L/M) a_n z^{n-1}\right|$$

$$\begin{aligned} &> 1 - \sum_{n=2}^{\infty} \left|\wedge_n ([n,q](L/M-1) - L/M)\right| |a_n|, \\ &= 1 - \sum_{n=2}^{\infty} \wedge_n \left|([n,q](L/M-1) - L/M)\right| |a_n|, \\ &> 1 - \sum_{n=2}^{n} \left\{ \frac{[2,q]([n,q](1+\nu)-1)(1-B)+(A-B)}{(A-B)\Re b - [2,q](1+B)\nu} \right\} \wedge_n |a_n| > 0, \end{aligned}$$

where we have used (2.8). Thus, the result follows by using Theorem 2.4. □

Varying the parameters μ, b, A and B in the last Theorem, we get the following known results discussed earlier in [21].

Corollary 2.8 *Let* $f \in \mathfrak{A}$ *be given by* (1.1) *and satisfy the inequality*

$$\sum_{n=2}^{\infty} ([n,q](1-B) - 1 + A) |a_n| \leq A - B.$$

Then the function $f \in \mathcal{S}_q^*[A, B]$.

Corollary 2.9 *Let* $f \in \mathfrak{A}$ *be given by* (1.1) *and satisfy the inequality*

$$\sum_{n=2}^{\infty} [n,q]([n,q](1-B) - 1 + A) |a_n| \leq A - B.$$

Then the function $f \in \mathcal{K}_q[A, B]$.

By choosing $q \to 1^-$ in the last two corollaries, we get the known results proved by Ahuja [3] and furthermore for $A = 1 - \alpha$ and $B = -1$, we obtain the results for the families $\mathcal{S}^*(\xi)$ and $\mathcal{K}(\xi)$ which was proved by Silverman [23].

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Competing Interests

The authors declare that they have no competing interests.

Authors Contributions

All authors jointly worked on the results and they read and approved the final manuscript.

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