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Some ergodic properties of multipliers on commutative Banach algebras

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Abstract: A commutative semisimple regular Banach algebra A with the Gelfand space Σ_A is called a Ditkin algebra if each point of $\Sigma_A \cup \{\infty\}$ is a set of synthesis for A. Generalizing the Choquet–Deny theorem, it is shown that if T is a multiplier of a Ditkin algebra A, then $\{\varphi \in A^* : T^*\varphi = \varphi\}$ is finite dimensional if and only if $\mathrm{card}\,\mathcal{F}_T$ is finite, where $\mathcal{F}_T = \left\{\gamma \in \Sigma_A : \widehat{T}(\gamma) = 1\right\}$ and $\widehat{T}$ is the Helgason–Wang representation of T.

Key words: Commutative Banach algebra, multiplier, Choquet–Deny theorem

1. Introduction

This note was motivated by the classical result of Choquet and Deny [2] on ergodic properties of measures on locally compact abelian groups.

We begin with some basic notations and definitions. For a commutative Banach algebra A, by Σ_A, we will denote the Gelfand space of A equipped with the w^*-topology and by $a \to \widehat{a}$, where $\widehat{a}(\gamma) = \gamma(a)$ $(\gamma \in \Sigma_A)$, the Gelfand transform of $a \in A$. A linear operator $T : A \to A$ is called a multiplier of A if

$$(Ta)b = a(Tb) \ \ (= T(ab)), \ \ \forall a, b \in A.$$

When A is semisimple, the set $M(A)$ of all multipliers of A is a commutative, closed, and unital subalgebra of $B(A)$, the algebra of all bounded linear operators on A. Unless otherwise stated, we always assume that A is a commutative semisimple Banach algebra.

For an arbitrary $a \in A$, the multiplication operator L_a given by $L_a b = ab$ $(b \in A)$ is a multiplier of A. The algebra A embeds into $M(A)$ via the mapping $a \mapsto L_a$ and therefore the Gelfand space of $M(A)$ may be represented as the disjoint union of Σ_A and $\mathrm{hull}(A)$, where Σ_A is canonically embedded in $\Sigma_{M(A)}$ and $\mathrm{hull}(A)$ denotes the hull of A in $\Sigma_{M(A)}$.

For each $T \in M(A)$, there is a uniquely determined bounded continuous function $\widehat{T}$ on Σ_A such that

$$\sup_{\gamma \in \Sigma_A} \left|\widehat{T}(\gamma)\right| \leq \|T\|$$

and

$$\widehat{(Ta)}(\gamma) = \widehat{T}(\gamma)\,\widehat{a}(\gamma), \ \ \forall a \in A, \ \forall \gamma \in \Sigma_A.$$

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In fact, $\widehat{T}$ is the restriction to Σ_A of the Gelfand transform of T on $\Sigma_{M(A)}$. The function $\widehat{T}$ is often called the Helgason–Wang representation of T. Standard references to multipliers are the books [1, 5, 7].

2. Ditkin algebras

Throughout this paper, G will denote a locally compact abelian group with the Haar measure. By $\widehat{G}$, we will denote the dual group of G. As usual, $L^1(G)$ and $M(G)$ will denote the group algebra and the convolution measure algebra of G, respectively. By the Wendel–Helson theorem [5, Theorem 0.1.1], an operator T on $L^1(G)$ is a multiplier of $L^1(G)$ if and only if there exists a measure $\mu \in M(G)$ such that $T = T_\mu$, where $T_\mu f = \mu * f$, $f \in L^1(G)$. Moreover, the map $\mu \mapsto T_\mu$ is an isometric isomorphism.

Let $\widehat{f}$ and $\widehat{\mu}$ denote the Fourier and the Fourier–Stieltjes transform of $f \in L^1(G)$ and $\mu \in M(G)$, respectively. The classical Choquet–Deny theorem [2] characterizes a certain ergodic property of measures on G as follows. Given $\mu \in M(G)$, the following two conditions are equivalent:

(i) For any $\varphi \in L^\infty(G)$, the identity $\mu * \varphi = \varphi$ implies that φ is constant.

(ii) $\widehat{\mu}(\chi) \neq 1$, for all $\chi \in \widehat{G} \setminus \{0\}$.

In [10], Ramsey and Weit give a different proof of the Choquet–Deny theorem. Granirer [3, Theorem 3] obtained an extension of the Choquet–Deny theorem for the Herz algebras $A_p(G)$ $(1 < p < \infty)$. In [4, Theorem 3.6], more general Choquet–Deny type results are established for some class of commutative Banach algebras (for the related results, see also [8, 9]).

Recall that a commutative Banach algebra A is said to be regular if, given a closed subset S of Σ_A and $\gamma \in \Sigma_A \setminus S$, there exists an $a \in A$ such that $\widehat{a}(S) = \{0\}$ and $\widehat{a}(\gamma) \neq 0$. A regular Banach algebra A is said to be Tauberian if $\overline{A_{00}} = A$, where

$$A_{00} := \{a \in A : \mathrm{supp}\widehat{a} \text{ is compact}\}.$$

The Tauberian condition implies that every proper closed ideal of A is contained in a maximal modular ideal.

Let A be a regular semisimple Banach algebra. Given a closed set S in Σ_A, there are two distinguished closed ideals of A with hull equal to S; namely $J_S := \overline{J_S^o}$ is the smallest closed ideal

$$J_S^o := \{a \in A_{00} : \mathrm{supp}\widehat{a} \cap S = \emptyset\}$$

and

$$I_S := \{a \in A : \widehat{a}(\gamma) = 0, \quad \forall \gamma \in S\}$$

is the largest closed ideal whose hulls are S. The set S is a set of synthesis for A if $J_S = I_S$ (for instance, see [6, Sect. 8.3]). Thus, S is a set of synthesis for A if and only if I_S is the only closed ideal of A whose hull is S. It is a famous theorem of Malliavin that, for each noncompact locally compact abelian group G, there exists a set of nonsynthesis for $L^1(G)$.

We say that a regular semisimple Banach algebra A is a w-Ditkin algebra if each point of $\Sigma_A \cup \{\infty\}$ is a set of synthesis for A. Since $J_{\{\infty\}}^o = A_{00}$ and $I_{\{\infty\}} = A$, the algebra A is a w-Ditkin algebra if $J_{\{\gamma\}} = I_{\{\gamma\}}$, for all $\gamma \in \Sigma_A$ and $\overline{A_{00}} = A$.

Recall that a weight function ω is a continuous function on G such that

$$\omega(g) \geq 1 \text{ and } \omega(g+s) \leq \omega(g)\,\omega(s), \ \ \forall g, s \in G.$$

For a weight function ω on G, by $L^1_\omega(G)$ we will denote the Banach space of the functions $f \in L^1(G)$ with the norm

$$\|f\|_{1,\omega} = \int_G |f(g)|\,\omega(g)\,dg < \infty.$$

The space $L^1_\omega(G)$ with convolution product and the norm $\|\cdot\|_{1,\omega}$ is a commutative semisimple Banach algebra with a bounded approximate identity and is called Beurling algebra. The dual space of $L^1_\omega(G)$, denoted by $L^\infty_\omega(G)$, is the space of all measurable functions φ on G such that

$$\|\varphi\|_{\omega,\infty} := \operatorname{ess\,sup}_{g\in G} \frac{|\varphi(g)|}{\omega(g)} < \infty.$$

Let $M_\omega(G)$ denote the Banach algebra (with respect to the convolution product) of all complex regular Borel measures on G with the norm

$$\|\mu\|_{1,\omega} = \int_G \omega(g)\,d|\mu|(g) < \infty.$$

There is a version of the Wendel–Helson theorem for Beurling algebras. This result says that an operator T on $L^1_\omega(G)$ is a multiplier of $L^1_\omega(G)$ if and only if there exists a measure $\mu \in M_\omega(G)$ for which $Tf = \mu * f$, $f \in L^1_\omega(G)$ [7, 4.1.7].

We mention the following result [11, Ch. 6, §3.2].

Theorem 2.1 *Let ω be a weight function on G satisfying the following conditions for each $g \in G$:*

(i) $\omega(g^n) = O(|n|^{\alpha_g})$ $(|n| \to \infty)$, *for some* $\alpha_g > 0$;

(ii) $\liminf\limits_{|n|\to\infty} \frac{\omega(g^n)}{|n|} = 0$.

Then:

(a) *The Gelfand space of $L^1_\omega(G)$ is the dual group of G.*

(b) *The Gelfand transform of $f \in L^1_\omega(G)$ is just the Fourier transform of f.*

(c) *$L^1_\omega(G)$ is a w-Ditkin algebra.*

Let A be a commutative Banach algebra. For $\varphi \in A^*$ and $a \in A$, the functional $\varphi \cdot a$ on A is defined by

$$\langle \varphi \cdot a, b\rangle = \langle \varphi, ab\rangle,\ b \in A.$$

If $T \in M(A)$, then as $T(ab) = a(Tb)$ $(a, b \in A)$, we have

$$T^*(\varphi \cdot a) = (T^*\varphi) \cdot a,\ \forall a \in A,\ \forall \varphi \in A^*. \tag{2.1}$$

Further, note that for an arbitrary $\varphi \in A^*$,

$$I_\varphi := \{a \in A : \varphi \cdot a = 0\}$$

is a closed ideal of A. If the algebra A has an approximate identity, then $\varphi \in I_\varphi^\perp$. The w^*-spectrum of $\varphi \in A^*$, denoted by $\sigma_*(\varphi)$, is the set

$$\sigma_*(\varphi) = \overline{\{\varphi \cdot a : a \in A\}}^{w^*} \cap \Sigma_A.$$

We will need the following well-known results (for instance, see [4]).

Lemma 2.2 *If A is a regular semisimple Banach algebra, then the following assertions hold for every $\varphi \in A^*$ and $a \in A$:*

(a) $\sigma_*(\varphi) = \mathrm{hull}(I_\varphi)$.

(b) $\sigma_*(\varphi \cdot a) \subseteq \sigma_*(\varphi) \cap \mathrm{supp}\widehat{a}$.

(c) *If A is Tauberian with an approximate identity, then $\sigma_*(\varphi) \neq \emptyset$, whenever $\varphi \neq 0$.*

We have the following.

Lemma 2.3 *If A is a Tauberian Banach algebra, then*

$$\sigma_*(\varphi) \cap \{\gamma \in \Sigma_A : \widehat{a}(\gamma) \neq 0\} \subseteq \sigma_*(\varphi \cdot a),$$

for all $\varphi \in A^*$ and $a \in A$.

Proof Let $\varphi \in A^*$ and $a \in A$ be given. Let $\gamma \in \Sigma_A$ be such that $\gamma \in \sigma_*(\varphi)$ and $\widehat{a}(\gamma) \neq 0$. Assume that $\gamma \notin \sigma_*(\varphi \cdot a)$. Then there exists $b \in A$ such that $\widehat{b}(\gamma) \neq 0$ and $\widehat{b}$ vanishes in a neighborhood of $\sigma_*(\varphi \cdot a)$. Since A is Tauberian, there exists a sequence $\{b_n\}$ in A_{00} such that $\|b_n - b\| \to 0$. It follows that $\widehat{b_n}(\gamma) \neq 0$, for some n. If $c := bb_n$, then we have $\widehat{c}(\gamma) \neq 0$, $c \in A_{00}$, and $\widehat{c}$ vanishes in a neighborhood of $\sigma_*(\varphi \cdot a)$. Consequently, c belongs to the smallest ideal of A whose hull is $\sigma_*(\varphi \cdot a)$. By Lemma 2.2 (a), $c \in I_{\varphi \cdot a}$ and therefore $\varphi \cdot (ac) = 0$. It follows that $\widehat{a}\widehat{c}$ vanishes on $\sigma_*(\varphi)$. Since $\gamma \in \sigma_*(\varphi)$ and $\widehat{c}(\gamma) \neq 0$, we have $\widehat{a}(\gamma) = 0$. This contradicts $\widehat{a}(\gamma) \neq 0$. □

Notice that if $T \in M(A)$, then

$$F_T := \overline{(I - T)A}$$

is a closed ideal of A associated with T and $\mathrm{hull}(F_T) = \mathcal{F}_T$, where

$$\mathcal{F}_T = \left\{\gamma \in \Sigma_A : \widehat{T}(\gamma) = 1\right\}.$$

The main result of this note is the following:

Theorem 2.4 *Let A be a w-Ditkin algebra with an approximate identity (not necessarily bounded) and $T \in M(A)$. Then the subspace $\{\varphi \in A^* : T^*\varphi = \varphi\}$ is finite dimensional if and only if $\mathrm{card}\mathcal{F}_T$ is finite. In this case,*

$$\dim\{\varphi \in A^* : T^*\varphi = \varphi\} = \mathrm{card}\mathcal{F}_T$$

and

$$\{\varphi \in A^* : T^*\varphi = \varphi\} = \mathrm{span}\mathcal{F}_T.$$

Proof Assume that the subspace $\{\varphi \in A^* : T^*\varphi = \varphi\}$ is finite dimensional. It follows from the identity

$$T^*\gamma = \widehat{T}(\gamma)\gamma \quad (\gamma \in \Sigma_A)$$

that

$$\mathcal{F}_T \subseteq \{\varphi \in A^* : T^*\varphi = \varphi\}.$$

Since Σ_A is a linearly independent subset of A^*, we have

$$\dim\{\varphi \in A^* : T^*\varphi = \varphi\} \geq \mathrm{card}\mathcal{F}_T.$$

Now, assume that $\mathrm{card}\mathcal{F}_T$ is finite, say $\mathcal{F}_T = \{\gamma_1, ..., \gamma_n\}$. Clearly,

$$\mathrm{span}\mathcal{F}_T \subseteq \{\varphi \in A^* : T^*\varphi = \varphi\}.$$

Let $\varphi \in A^*$ be such that $T^*\varphi = \varphi$ and $\gamma \in \sigma_*(\varphi)$. Then

$$\gamma = w^* - \lim_{\lambda} (\varphi \cdot a_\lambda),$$

for some net $\{a_\lambda\}$ in A. By (2.1), we can write

$$T^*\gamma = w^* - \lim_{\lambda} [(T^*\varphi) \cdot a_\lambda] = w^* - \lim_{\lambda} (\varphi \cdot a_\lambda) = \gamma.$$

It follows that $\widehat{T}(\gamma) = 1$ and therefore $\gamma \in \mathcal{F}_T$. We have $\sigma_*(\varphi) \subseteq \{\gamma_1, ..., \gamma_n\}$. Let us show that $\varphi = c_1\gamma_1 + ... + c_n\gamma_n$, for some $c_1, ..., c_n \in \mathbb{C}$. We may assume that $\sigma_*(\varphi) = \{\gamma_1, ..., \gamma_n\}$. Let $U_1, ..., U_n$ be the disjoint neighborhoods of $\gamma_1, ..., \gamma_n$, respectively. Let V_i be a compact neighborhood of γ_i such that $\overline{V_i} \subset U_i$. Then there exist elements $a_1, ..., a_n$ in A such that $\widehat{a_i} = 1$ on $\overline{V_i}$ and $\widehat{a_i} = 0$ outside U_i $(i = 1, ..., n)$. Let $a := a_1 + ... + a_n$. Since $\widehat{a} = 1$ in a neighborhood of $\sigma_*(\varphi)$, the Gelfand transform of $ab - b$ vanishes in a neighborhood of $\sigma_*(\varphi)$, for every $b \in A_{00}$. Consequently, $ab - b$ belongs to the smallest ideal of A whose hull is $\sigma_*(\varphi)$ and therefore $ab - b \in I_\varphi$. Hence,

$$(\varphi \cdot a) \cdot b = \varphi \cdot b, \ \ \forall b \in A_{00}.$$

Since A_{00} is dense in A, we have

$$(\varphi \cdot a) \cdot b = \varphi \cdot b, \ \ \forall b \in A.$$

If $\{e_\lambda\}$ is an approximate identity for A, then from the identities $(\varphi \cdot a) \cdot e_\lambda = \varphi \cdot e_\lambda$, we obtain that $\varphi \cdot a = \varphi$. Thus, we have $\varphi = \varphi_1 + ... + \varphi_n$, where $\varphi_i = \varphi \cdot a_i$ $(i = 1, ..., n)$. By Lemmas 2.2 and 2.3, we can write

$$\{\gamma_i\} \subseteq \sigma_*(\varphi \cdot a_i) \subseteq \sigma_*(\varphi) \cap \mathrm{supp}\widehat{a_i} = \{\gamma_i\}.$$

Consequently, $\sigma_*(\varphi_i) = \{\gamma_i\}$, so that $\mathrm{hull}(I_{\varphi_i}) = \{\gamma_i\}$. Since $\{\gamma_i\}$ is a set of synthesis for A, we have $I_{\varphi_i} = I_{\{\gamma_i\}}$. It follows that

$$\varphi_i \in I_{\varphi_i}^{\perp} = I_{\{\gamma_i\}}^{\perp} = \mathbb{C}\gamma_i.$$

Hence, $\varphi_i = c_i\gamma_i$, for some $c_i \in \mathbb{C}$, and therefore $\varphi = c_1\gamma_1 + ... + c_n\gamma_n$. It follows that

$$\dim\{\varphi \in A^* : T^*\varphi = \varphi\} \leq \mathrm{card}\mathcal{F}_T.$$

Thus, we have

$$\dim\{\varphi \in A^* : T^*\varphi = \varphi\} = \mathrm{card}\mathcal{F}_T.$$

This completes the proof. □

Let ω be a weight function on G. For an arbitrary $\mu \in M_\omega(G)$, we put

$$F_\mu = \overline{\{(f - \mu * f) : f \in L^1_\omega(G)\}}$$

and

$$\mathcal{F}_\mu = \left\{\chi \in \widehat{G} : \widehat{\mu}(\chi) = 1\right\}.$$

The following result is an immediate consequence of Theorem 2.4.

Corollary 2.5 *Let ω be a weight function on G satisfying the hypotheses of Theorem 2.1. If $\mu \in M_\omega(G)$, then the subspace*

$$\{\varphi \in L^\infty_\omega(G) : \mu * \varphi = \varphi\}$$

is finite dimensional if and only if $\mathrm{card}\mathcal{F}_\mu$ *is finite. In this case, we have*

$$\dim\{\varphi \in L^\infty_\omega(G) : \mu * \varphi = \varphi\} = \mathrm{card}\mathcal{F}_\mu$$

and

$$\{\varphi \in L^\infty_\omega(G) : \mu * \varphi = \varphi\} = \mathrm{span}\mathcal{F}_\mu.$$

The following example shows that without the $w-$-Ditkin algebra condition, Theorem 2.4 does not hold in general.

Example 2.6 *Let $A = L^1_\omega(\mathbb{R})$ be the Beurling algebra with weight $\omega(t) = 1 + |t|$ $(t \in \mathbb{R})$. Then,*

$$I_{\{0\}} = \left\{f \in A : \widehat{f}(0) = 0\right\}$$

and

$$J_{\{0\}} = \left\{f \in A : \widehat{f}(0) = \widehat{f}'(0) = 0\right\}.$$

Define a multiplier T on A by $Tf = h * f$, where $h(t) = \frac{1}{2\sqrt{\pi}}e^{-\frac{t^2}{4}}$. Then $\widehat{T} = \widehat{h}$, and as $\widehat{h}(\lambda) = e^{-\lambda^2}$, we have $\mathcal{F}_T = \{0\}$. Hence, $\mathrm{card}\mathcal{F}_T = 1$. *If $f \in (I - T)A$, then $f = k - h * k$ for some $k \in A$ and*

$$\widehat{f}(\lambda) = \widehat{k}(\lambda)\left(1 - e^{-\lambda^2}\right).$$

Notice that $\widehat{f}(0) = \widehat{f}'(0) = 0$ and therefore $F_T \subseteq J_{\{0\}}$. Since $J_{\{0\}}$ is the smallest closed ideal of A with hull equal to $\{0\}$, we obtain that $F_T = J_{\{0\}}$. We thus have

$$\dim\{\varphi \in A^* : T^*\varphi = \varphi\} = \dim F_T^\perp = \dim J_{\{0\}}^\perp = 2.$$

Recall that a regular semisimple Banach algebra A is said to satisfy Ditkin's condition [6, Definition 8.5.1] at $\gamma \in \Sigma_A \cup \{\infty\}$ if for every $a \in A$ with $\widehat{a}(\gamma) = 0$, there exists a sequence $\{a_n\}$ in A such that each $\widehat{a_n}$ vanishes in a neighborhood U_n of γ and $\|aa_n - a\| \to 0$. We say that A is an s-Ditkin algebra if it satisfies Ditkin's condition at each point of $\Sigma_A \cup \{\infty\}$. For example, if $\omega(g) = (1 + |g|)^\alpha$ $(0 \leq \alpha < 1)$, then $L^1_\omega(\mathbb{R}^n)$ is an s-Ditkin algebra [11, Ch. 6, §3.3], where for $g = (x_1, ..., x_n)$, we have $|g| = \left(x_1^2 + ... + x_n^2\right)^{\frac{1}{2}}$. Clearly, every s-Ditkin algebra is a w-Ditkin algebra.

A locally compact Hausdorff space Ω is said to be scattered if it contains no nonempty compact perfect subset. As usual, ∂S will denote the topological boundary of $S \subset \Omega$.

The following result is another extension of the Choquet–Deny theorem.

Theorem 2.7 *Let A be an s-Ditkin algebra and let S be a closed subset of Σ_A such that ∂S is scattered. The following conditions are equivalent for $T \in M(A)$:*

(a) $\{\varphi \in A^* : T^*\varphi = \varphi\} \subseteq \overline{\mathrm{span} S}^{w^*}$.

(b) $\widehat{T}(\gamma) \neq 1, \ \forall \gamma \in \Sigma_A \diagdown S$.

Moreover, if $\partial \mathcal{F}_T$ is scattered, then

$$\{\varphi \in A^* : T^*\varphi = \varphi\} = \overline{\mathrm{span} \mathcal{F}_T}^{w^*}.$$

Proof (a)$\Rightarrow$(b) Since

$$\{\varphi \in A^* : T^*\varphi = \varphi\} = F_T^{\perp},$$

we have $F_T^{\perp} \subseteq \overline{\mathrm{span} S}^{w^*}$, and therefore,

$$I_S =^{\perp} \left(\overline{\mathrm{span} S}^{w^*}\right) \subseteq F_T.$$

It follows that

$$\left\{\gamma \in \Sigma_A : \widehat{T}(\gamma) = 1\right\} = \mathrm{hull}(F_T) \subseteq \mathrm{hull}(I_S) = S.$$

Hence, $\widehat{T}(\gamma) \neq 1, \ \forall \gamma \in \Sigma_A \diagdown S$.

(b)$\Rightarrow$(a) Since

$$\mathcal{F}_T = \mathrm{hull}(F_T) \subseteq S$$

and S is a set of synthesis for A [6, Corollary 8.5.1], we can write

$$I_S = J_S \subseteq J_{\mathcal{F}_T} \subseteq F_T.$$

This implies

$$\{\varphi \in A^* : T^*\varphi = \varphi\} = F_T^{\perp} \subseteq I_S^{\perp} = \overline{\mathrm{span} S}^{w^*}.$$

If $S = \mathcal{F}_T$, then as $\widehat{T}(\gamma) \neq 1, \ \forall \gamma \in \Sigma_A \diagdown \mathcal{F}_T$, by (a),

$$\{\varphi \in A^* : T^*\varphi = \varphi\} \subseteq \overline{\mathrm{span} \mathcal{F}_T}^{w^*}.$$

On the other hand, since

$$T^*\gamma = \widehat{T}(\gamma)\gamma, \quad \forall \gamma \in \Sigma_A,$$

we have

$$\overline{\mathrm{span} \mathcal{F}_T}^{w^*} \subseteq \{\varphi \in A^* : T^*\varphi = \varphi\}$$

and so

$$\{\varphi \in A^* : T^*\varphi = \varphi\} = \overline{\mathrm{span}\mathcal{F}_T}^{w^*}.$$

□

Recall that a linear subspace of $L^1(G)$ is said to be a Segal algebra and denoted by $S(G)$ if it satisfies the following conditions:

a) $S(G)$ is a translation invariant dense subalgebra of $L^1(G)$;

b) For an arbitrary $f \in S(G)$ and $g \in G$, $\|f_g\|_S = \|f\|_S$, where $f_g(s) := f(g+s)$ and $\|\cdot\|_S$ is the norm of $S(G)$;

c) For each $f \in S(G)$, the mapping $g \mapsto f_g$ is continuous from G into $S(G)$.

About Segal algebras, ample information can be found in Reiter's book [11]. The following examples show that the class of Segal algebras is sufficiently large.

1) The algebra $L^1(G) \cap L^p(G)$ $(1 \leq p < \infty)$, equipped with the norm $\|f\| = \|f\|_1 + \|f\|_p$, is a Segal algebra.

2) The algebra $L^1(G) \cap C_0(G)$, equipped with the norm $\|f\| = \|f\|_1 + \|f\|_\infty$, is a Segal algebra, where $C_0(G)$ is the space of all complex valued continuous functions on G vanishing at infinity.

A Segal algebra $S(G)$ is a commutative semisimple regular Banach algebra with respect to convolution. The Gelfand space of $S(G)$ is $\widehat{G}$ and the Gelfand transform of $f \in S(G)$ is just the Fourier transform of f. Moreover, $S(G)$ is an s-Ditkin algebra [13]. Moreover, a Segal algebra $S(G)$ has an approximate identity (not bounded in $S(G)$-norm unless $S(G) = L^1(G)$).

Let $A(G)$ be the Fourier algebra of G. We know that $A(G)$ is isometrically isomorphic to the algebra $L^1\left(\widehat{G}\right)$ via the Fourier transform. The elements of $A(G)^*$ are called pseudomeasures. If T is a multiplier of $S(G)$, then there exists a unique pseudomeasure σ such that $Tf = \sigma * f$, $f \in S(G)$ [12]. It follows that $\widehat{T} = \widehat{\sigma}$, where $\widehat{\sigma}$ is the Fourier transform of σ, which is defined by

$$\langle \widehat{\sigma}, f \rangle = \langle \sigma, \widehat{f} \rangle, \quad f \in L^1\left(\widehat{G}\right).$$

Consequently, Theorems 2.4 and 2.7 can be applied to the Segal algebras.

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