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# On isotropic projective Ricci curvature of C-reducible Finsler metrics

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**Abstract:** Projective Ricci curvature is a projective invariant quantity in Finsler geometry which is introduced by Z. Shen. In this paper, we study special projective Ricci curvature of **C**-reducible Finsler metrics. The necessary and sufficient conditions of these metrics, which cause these metrics to be weak or isotropic projective Ricci curvature, are found and it is proved that **C**-reducible Douglas metric of isotropic **PRic**-curvature must be PRic flat. The same theorem for **C**-reducible metrics of scalar flag curvature is also investigated.

**Key words:** Isotropic **PRic**-curvature, **C**-reducible Finsler metrics, Douglas metric, scalar flag curvature

## 1. Introduction

There are some well-known projective invariants of Finsler metrics namely Douglas curvature, Weyl curvature [2], generalized Douglas-Weyl curvature [3], and another projective invariant by Akbar-Zadeh [1] (For another special projective invariant, see [15, 19]). Recently, Z. Shen [17] defined the concept of projective Ricci curvature **PRic** for a Finsler metric $F$ as

$$\mathbf{PRic} := \mathbf{Ric} + (n-1)\{\bar{\mathbf{S}}_{|m}y^m + \bar{\mathbf{S}}^2\},$$

where **Ric** and **S** denote the Ricci curvature and $S$-curvature and "$|$" is the horizontal covariant derivative with respect to the Berwald connection (or the Chern connection), and

$$\bar{\mathbf{S}} := \frac{1}{n+1}\mathbf{S}.$$

A Finsler metric is called projective Ricci flat if $\mathbf{PRic} = 0$. It is remarkable that the $S$-curvature is a non-Riemannian quantity and plays an important role in Finsler geometry, which was introduced by Shen [16]. In [8], Cheng-Shen-Ma rewrote the projective Ricci curvature as

$$\mathbf{PRic} = \mathbf{Ric} + \frac{n-1}{n+1}\mathbf{S}_{|m}y^m + \frac{n-1}{(n+1)^2}\mathbf{S}^2. \tag{1.1}$$

Moreover, they completely classified projective Ricci flat Randers metrics. In [22], Zhu and Zhang studied the projective Ricci curvature and characterized projective Ricci flat spherically symmetric Finsler metrics. Recently, Rezaei et al. defined the concept of weak, isotropic and constant **PRic**-curvature*.

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**Definition 1.1** *Let $F$ be a Finsler metric on n-dimensional manifold $M$ and $\boldsymbol{PRic}$ denote the projective Ricci curvature of $F$.*

- *$F$ is of weak $\boldsymbol{PRic}$-curvature if*
$$\boldsymbol{PRic} = (n-1)[\frac{3\theta}{F} + \kappa]F^2, \tag{1.2}$$
*where $\theta = \theta_i y^i$ is a $1-$form and $\kappa = \kappa(x)$ is scalar function on $M$;*
- *$F$ is of isotropic $\boldsymbol{PRic}$-curvature if $\theta = 0$ i.e. $\boldsymbol{PRic} = (n-1)\kappa(x)F^2$;*
- *$F$ is of constant $\boldsymbol{PRic}$-curvature if $\boldsymbol{PRic} = (n-1)cF^2$, where $c$ is a real constant;*
- *$F$ is called $\boldsymbol{PRic}$ flat if $\boldsymbol{PRic} = 0$.*

$(\alpha, \beta)$-metrics are a rich and important class of Finsler metrics. The Randers, Kropina, and Matsumoto metrics are special $(\alpha, \beta)$-metrics. The second author and others classified Matsumoto metric of weak projective Ricci curvature and they showed that projective Ricci flat Matsumoto metrics with constant length one-forms reduces to Ricci flat metric[†].

There is a class of Finsler metrics which is called **C**-reducible Finsler metrics. These spaces first were introduced by Matsumoto [12] and were classified by special form of Cartan torsion. In [14] Matsumoto and Hojo proved that a Finsler space is **C**-reducible if and only if the space is either a Randers or a Kropina space. In this paper, we study special projective Ricci curvature of Randers and Kropina metrics and we prove:

**Theorem 1.2** ***C**-reducible Douglas metric of isotropic $\boldsymbol{PRic}$-curvature on a manifold $M$ with dimension $n > 2$ must be projective Ricci flat.*

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry. The flag curvature of a Finsler metric $F$ is a function $K = K(P, y)$ of a two-dimensional plane called "flag" $P \subset T_xM$ and a "flagpole" $y \in P \setminus \{0\}$. A Finsler metric $F$ is said to be of scalar flag curvature if $K = K(x, y)$ is independent of $P$ containing $y \in T_xM$. This quantity tells us how curved the space is. Finsler metric of scalar flag curvature has vanishing weyl curvature. X. Cheng showed that F is of scalar flag curvature and of vanishing S-curvature if and only if the flag curvature K = 0 and F is a Berwald metric. In this case, F is a locally Minkowski metric [5]. In this paper, we prove the following theorem:

**Theorem 1.3** *Let $F$ be a **C**-reducible Finsler metric of isotropic $\boldsymbol{PRic}$-curvature on a manifold $M$ with dimension $n > 2$. If $F$ is of scalar flag curvature, then it is projective Ricci flat.*

## 2. Preliminaries

Let $M$ be an n-dimensional $C^\infty$ manifold. Denote by $T_xM$ the tangent space at $x \in M$, and by $TM = \cup_{x \in M} T_xM$ the tangent bundle of $M$. Each element of $TM$ has the form $(x, y)$, where $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \backslash \{0\}$. The natural projection $\pi : TM \to M$ is given by $\pi(x, y) = x$. The pull-back tangent bundle $\pi^*TM$ is a vector bundle over $TM_0$ whose fiber $\pi^*_vTM$ at $v \in TM_0$ is just $T_xM$, where $\pi(v) = x$. Then
$$\pi^*TM = \{(x, y, v) | y \in T_xM_0, v \in T_xM\}.$$

[†]Rezaei B, Tayebi A, Gabrani M. On projective Ricci curvature of Finsler metrics. (submitted)

A *Finsler metric* on a manifold $M$ is a function $F : TM \to [0, \infty)$ which has the following properties:
(i) $F$ is $C^\infty$ on $TM_0$;
(ii) $F(x, \lambda y) = \lambda F(x, y) \quad \lambda > 0$;
(iii) For any tangent vector $y \in T_xM$, the vertical Hessian of $F^2/2$ given by

$$g_{ij}(x, y) = \left[\frac{1}{2}F^2\right]_{y^i y^j},$$

is positive definite. The non-Riemannian quantity $C = C_{ijk}dx^i \otimes dx^j \otimes dx^k$ is a tensor on $TM$ which is defined by fundamental tensor

$$C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k} = \frac{1}{4}\frac{\partial^3 F^2}{\partial y^i \partial y^j \partial y^k},$$

and called the Cartan tensor. A Finsler space of dimension $n > 2$ is called **C**-reducible if the Cartan tensor $C_{ijk}$ satisfies

$$C_{ijk} = \frac{1}{n+1}(h_{ij}C_k + h_{jk}C_i + h_{ki}C_j),$$

where $h_{ij} = F\frac{\partial^2 F}{\partial y^i \partial y^j}$ is called angular metric tensor and $C_i = C_{ijk}g^{jk}$.

For a Finsler metric $F = F(x, y)$, its geodesics are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and are given by

$$G^i = \frac{1}{4}g^{il}\left\{[F^2]_{x^k y^l}y^k - [F^2]_{x^l}\right\},$$

where $y \in T_xM$ and $(g^{ij}) := (g_{ij})^{-1}$.

Berwald curvature $B = B^{\ i}_{jkl}\frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^l$ is defined by spray geodesic coefficients as $B^{\ i}_{jkl} = \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}$. The Finsler space is called Berwald space if $B = 0$.

The tensor $D = D^i_{\ jkl}\frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k \otimes dx^l$, where

$$D^i_{\ jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m}y^i),$$

is called Douglas curvature tensor. Finsler metric called Douglas if $D = 0$. It is easy to see that Berwald spaces are subspace of Douglas spaces.

The Riemann curvature $R_y = R^i_{\ k}\frac{\partial}{\partial x^i} \otimes dx^k$ of $F$ is defined by

$$R^i_{\ k} = 2\frac{\partial G^i}{\partial x^k} - \frac{\partial^2 G^i}{\partial x^j \partial y^k}y^j + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k}.$$

When $F(x, y) = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric, $R^i_{\ k} = R^i_{\ jkl}(x)y^j y^l$, where $R^i_{\ jkl}(x)$ denotes the coefficients of the usual Riemannian curvature tensor. Thus, the quantity $R_y$ in Finsler geometry is still called the Riemann curvature [20]. The Ricci curvature **Ric** is defined by $\mathbf{Ric} := R^i{}_i$. By definition, the Ricci curvature is a positively homogeneous function of degree two in $y \in TM$.

The flag curvature is a natural extension of the sectional curvature of Riemannian metrics. Let $(x, y) \in T_xM$ and $u$ be an arbitrary vector in $T_xM$ such that $P = span\{y, u\} \subset T_xM$, Then

$$K(P, y) := \frac{g_y(u, R_y(u))}{g_y(y, y)g_y(u, u) - g_y(y, u)^2},$$

is called flag curvature. $F$ is of scalar flag curvature if $K(P, y)$ is independent of $P \subset T_xM$, It is known that F is of scalar flag curvature if and only if in a standard local coordinate system

$$R^i{}_k = K(F^2\delta^i{}_k - FF_{y^k}y^i).$$

For a Finsler metric F, the Busemann–Hausdorff volume form $dV_{BH} := \sigma_{BH}(x)\omega^1 \wedge \cdots \wedge \omega^n$, is defined by

$$\sigma_{BH} := \frac{Vol(B^n(1))}{Vol\Big\{(y^i) \in R^n \mid F(x, y^i\frac{\partial}{\partial x^i}\mid_x)\Big\}}.$$

Here Vo$l\{.\}$ denotes the Euclidean volume function and $B^n(1)$ denotes the unit ball on $R^n$. When $F(x, y) = \sqrt{g_{ij}(x)y^iy^j}$ is a Riemannian metric, then $\sigma_{BH}(x) = \sqrt{det(g_{ij})}$. There is a notion of distortion $\tau = \tau(x, y)$ on $TM$ associated with the Busemann–Hausdorff volume form $dV_{BH} := \sigma_{BH}(x)\omega^1 \wedge \cdots \wedge \omega^n$, i.e.

$$\tau(x, y) := \ln\left[\frac{\sqrt{det(g_{ij}(x, y))}}{\sigma_{BH}(x)}\right].$$

The $S-$curvature is defined by

$$\mathbf{S}(x, y) := \frac{d}{dt}[\tau(c(t), \dot{c}(t))] \mid_{t=0},$$

where $c(t)$ is the geodesic with $c(0) = x$ and $\dot{c}(0) = y$ [16]. From the definition, we see that the $S$-curvature measures the rate of change of the distortion on $(T_xM, F_x)$ in the direction $y \in T_xM$. For a Finsler metric $F$, the $S$-curvature is given by following:

$$\mathbf{S} = \frac{\partial G^m}{\partial y^m} - y^m\frac{\partial}{\partial x^m}\Big[\ln\sigma_{BH}\Big]. \tag{2.1}$$

The class of $(\alpha, \beta)$-metrics forms a special and important class of Finsler metrics which can be expressed in the form $F = \alpha\phi(s), s = \beta/\alpha$, where $\alpha := \alpha(x, y) = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta := \beta(y) = b_i(x)y^i$ is a 1-form on $M$, and $\phi(s)$ is a $C^\infty$ positive function on some open interval. In particular, when $\phi(s) = 1+s$, the Finsler metrics $F = \alpha + \beta$ is called Randers metrics, which were introduced and studied by Randers. If $\phi(s) = 1/s$, the Finsler metric $F = \alpha^2/\beta$ is called a Kropina metric. Kropina metrics were first introduced by Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and were investigated by Kropina [11].

For generic $(\alpha,\beta)$-metric usually use the following notations

$$r_{ij} := \frac{1}{2}(b_{i;j} + b_{j;i}), \qquad s_{ij} := \frac{1}{2}(b_{i;j} - b_{j;i}),$$

where ";" denotes the covariant derivative with respect to the Levi-Civita connection of $\alpha$. Further, put

$$r^i{}_j := a^{im} r_{mj},\ s^i{}_j := a^{im} s_{mj},\ r_j := b^m r_{mj},\ s_j := b^m s_{mj},\ r := r_{ij} b^i b^j = b^j r_j,$$

$$q_{ij} := r_{im} s^m_{\ j},\ t_{ij} := s_{im} s^m_{\ j},\ q_j := b^i q_{ij} = r_m s^m_{\ j},\ t_j := b^i t_{ij} = s_m s^m_{\ j},\ s^i{}_j s^j{}_i := t^m_{\ m},$$

where $(a^{ij}) := (a_{ij})^{-1}$ and $b^i := a^{ij} b_j$. Let us define:

$$r_{i0} := r_{ij} y^j, \quad s_{i0} := s_{ij} y^j, \quad r_{00} := r_{ij} y^i y^j, \quad r_0 := r_i y^i, \quad s_0 := s_i y^i.$$

## 3. Weak projective Ricci curvature of C-reducible Finsler metrics

Now we study special **PRic**-curvature of Randers metrics. Projective Ricci curvature of Randers metrics is as follows

$$\mathbf{PRic} = {}^{\alpha}\mathbf{Ric} + 2\alpha s^m_{\ 0;m} - 2t_{00} - \alpha^2 t^m_{\ m} + (n-1)\left\{2\alpha(\rho_m s^m_{\ 0}) - \rho_{0;0} + \rho_0^2\right\},$$

where $\rho_m = -\frac{r_m + s_m}{1-b^2}$ and $\rho_0 = \rho_m y^m$ (see [8]). Let Randers metrics be of weak isotropic **PRic**-curvature, then by (1.2) we have

$$\begin{aligned} 0 &= {}^{\alpha}\mathbf{Ric} - (n-1)[\frac{3\theta}{F} + \kappa]F^2 + 2\alpha s^m_{\ 0;m} - 2t_{00} \\ &- \alpha^2 t^m_{\ m} + (n-1)\left\{2\alpha(\rho_m s^m_{\ 0}) - \rho_{0;0} + \rho_0^2\right\}. \end{aligned}$$

We can sort equation mentioned above by $\alpha$ as follows

$$A_2\alpha^2 + A_1\alpha + A_0 = 0, \tag{3.1}$$

where

$$\begin{aligned} A_2 &= -(n-1)\kappa - t^m{}_m, \\ A_1 &= (n-1)[-3\theta - 2\kappa\beta + 2\rho_m s^m_{\ 0}] + 2s^m_{\ 0;m}, \\ A_0 &= {}^{\alpha}\mathbf{Ric} + (n-1)[-3\theta\beta - \kappa\beta^2 - \rho_{0;0} + \rho_0^2] - 2t_{00}. \end{aligned}$$

From 3.1 we obtain two fundamental equations:

$$A_1 = 0, \tag{3.2}$$

$$A_2\alpha^2 + A_0 = 0. \tag{3.3}$$

From (3.2) and (3.3) we can get equations that characterize Randers metric of weak projective Ricci curvature.

**Theorem 3.1** *Let* $F = \alpha + \beta$ *be the non-Riemannian Randers metric on manifold* $M$. *Then* $F$ *is a weak projective Ricci curvature metric if and only if* $\alpha$ *and* $\beta$ *satisfies..*

$$\begin{aligned} s^m{}_{0;m} &= (n-1)\Big\{\frac{3}{2}\theta + \kappa\beta - \rho_m s^m{}_0\Big\}, \\ {}^\alpha\boldsymbol{Ric} &= (n-1)\Big\{\kappa\alpha^2 + 3\theta\beta + \kappa\beta^2 + \rho_{0;0} - \rho_0^2\Big\} + t^m{}_m\alpha^2 + 2t_{00}. \end{aligned}$$

If $F = \alpha + \beta$ is of isotropic **PRic**$-$curvature then $\theta = 0$ and these classifing equations are changed as follows

$$s^m_{0;m} = (n-1)\Big\{\kappa\beta - \rho_m s^m_0)\Big\}, \tag{3.4}$$

$${}^\alpha\mathbf{Ric} = (n-1)\Big\{\kappa(\alpha^2+\beta^2) + \rho_{0;0} - \rho_0^2\Big\} + t^m{}_m\alpha^2 + 2t_{00}. \tag{3.5}$$

**PRic** flat Randers metric had been studied by Cheng et al. [8], later Cheng and Rezaei wrote the modification to this paper and corrected the results [7]. It is easy to get the same classifing theorem in [7] by putting $\theta = 0$ and $\kappa = 0$ In theorem 3.1.

**Example 3.2** *For a constant number* $a \in \boldsymbol{R}^n$, *let us define the Randers metirc* $F = \alpha + \beta$ *by*

$$\begin{aligned} \alpha &:= \frac{\sqrt{(1-|a|^2|x|^4)|y|^2 + (|x|^2 < a, y > -2 < a, x >< x, y >)^2}}{1-|a|^2|x|^4}, \\ \beta &:= \frac{|x|^2 < a, y > -2 < a, x >< x, y >}{1-|a|^2|x|^4}. \end{aligned}$$

*This Randers metric satisfies following equations*

$$\begin{aligned} \boldsymbol{S} &= (n+1)cF, \\ \boldsymbol{Ric} &= (n-1)(3c_0F + \delta F^2), \end{aligned}$$

*where*

$$c :=< a, x >, \quad c_0 := c_{x^m}y^m, \quad \delta := 3 < a, x >^2 -2|a|^2|x|^2.$$

*For more details see [10]. Then by* (1.1) *we can see*

$$\boldsymbol{PRic} = (n-1)[\frac{4c_0}{F} + c^2 + \delta]F^2.$$

*Therefore,* $F$ *is of weak projective Ricci curvature with* $\theta = \frac{4c_0}{3}$ *and* $\kappa = c^2 + \delta$.

Cheng et al. compute the **PRic**-curvature of Kropina metrics in [6], but some terms of this curvature were missing. We added these terms and stated this theorem completely as:

**Theorem 3.3** *Let $F = \alpha^2/\beta$ be a Kropina metric on an $n$-dimensional manifold $M$. Then the projective Ricci curvature of $F$ is given by*

$$\begin{aligned}\boldsymbol{PRic} &= {}^{\alpha}\boldsymbol{Ric} + (n-2)\left[\frac{1}{b^2}r_{0;0} + \frac{1}{b^2}s_{0;0} - \frac{1}{b^4}(r_0+s_0)^2\right] + (n-1)\frac{F}{b^2}t_0 \\ &+ \frac{2}{b^2}q_{00} - \frac{nF}{b^4}s_0r - \frac{n}{b^4}rr_{00} + (n-3)\frac{F}{b^2}q_0 - Fs^m{}_{0;m} - \frac{F^2}{4}t^m{}_m \\ &+ \frac{F}{b^2}b^m s_{0;m} + \frac{1}{b^2}b^m r_{00;m} + \frac{1}{b^2}(Fs_0 + r_{00})r^m{}_m - \frac{F^2}{2b^2}s^m s_m. \end{aligned} \tag{3.6}$$

Now assume the Kropina metric is of weak **PRic**-curvature, by (1.2) and (3.6) we can obtain

$$\begin{aligned} 0 &= {}^{\alpha}\mathbf{Ric} - (n-1)[\frac{3\theta}{F} + \kappa]F^2 + (n-2)\left[\frac{1}{b^2}r_{0;0} + \frac{1}{b^2}s_{0;0} - \frac{1}{b^4}(r_0+s_0)^2\right] \\ &+ \frac{2}{b^2}q_{00} + (n-1)\frac{F}{b^2}t_0 - \frac{nF}{b^4}s_0r - \frac{n}{b^4}rr_{00} + (n-3)\frac{F}{b^2}q_0 - Fs^m{}_{0;m} \\ &+ \frac{F}{b^2}b^m s_{0;m} + \frac{1}{b^2}b^m r_{00;m} + \frac{1}{b^2}(Fs_0 + r_{00})r^m{}_m - \frac{F^2}{2b^2}s^m s_m - \frac{F^2}{4}t^m{}_m. \end{aligned}$$

Multiplying $4b^4\beta^2$ on both sides of this equation to remove denominators yields

$$\begin{aligned} 0 &= 4b^4\beta^{2\alpha}\mathbf{Ric} + 4\beta^2(n-2)\left[b^2r_{0;0} + b^2s_{0;0} - (r_0+s_0)^2\right] - b^4\alpha^4t^m{}_m \\ &+ 4b^2\beta(n-1)\alpha^2t_0 - 4n\beta\alpha^2s_0r - 4n\beta^2rr_{00} + 4(n-3)b^2\beta\alpha^2q_0 \\ &- 4b^4\beta\alpha^2s^m{}_{0;m} + 4b^2\beta\alpha^2b^ms_{0;m} + 4b^2\beta^2b^mr_{00;m} + 4b^2\beta^2r_{00}r^m{}_m \\ &+ 4b^2\beta\alpha^2s_0r^m{}_m - 2b^2\alpha^4s^ms_m + 8\beta^2b^2q_{00} - 4b^4(n-1)\kappa(x)\alpha^4 \\ &- 12(n-1)b^4\theta\beta. \end{aligned} \tag{3.7}$$

Equation (3.7) is equivalent to the following equation

$$\Xi_4\alpha^4 + \Xi_2\alpha^2 + \Xi_0 = 0, \tag{3.8}$$

where

$$\begin{aligned} \Xi_4 &= -2b^2s^ms_m - b^4t^m{}_m - 4b^4(n-1)\kappa(x), \\ \Xi_2 &= 4\beta\{b^2(n-1)t_0 - ns_0r + (n-3)b^2q_0 \\ &\quad -b^4s^m{}_{0;m} + b^2b^ms_{0;m} + b^2s_0r^m{}_m - 3(n-1)b^4\theta\}, \end{aligned} \tag{3.9}$$

$$\begin{aligned} \Xi_0 &= 4\beta^2\{b^{4\alpha}\mathbf{Ric} + (n-2)\left[b^2r_{0;0} + b^2s_{0;0} - (r_0+s_0)^2\right] \\ &\quad -nrr_{00} + 2b^2q_{00} + b^2b^mr_{00;m} + b^2r_{00}r^m{}_m\}. \end{aligned} \tag{3.10}$$

By factoring of $\alpha^2$ we can rewrite equation (3.8) as

$$(\Xi_4\alpha^2 + \Xi_2)\alpha^2 + \Xi_0 = 0. \tag{3.11}$$

Since $\alpha^2$ and $\beta^2$ are relatively prime polynomials in $y$, then by (3.10) and (3.11), we can conclude that there is scalar function $\mu = \mu(x)$ in such a way that

$$\begin{aligned}\mu(x)\alpha^2 &= {}^{\alpha}\mathbf{Ric}b^4 + (n-2)\left[b^2 r_{0;0} + b^2 s_{0;0} - (r_0+s_0)^2\right] \\ &\quad -nrr_{00} + 2b^2 q_{00} + b^2 b^m r_{00;m} + b^2 r_{00} r^m_{\ m}. \end{aligned} \tag{3.12}$$

Substitute (3.12) into (3.7) and simplify this equation, it yields

$$\begin{aligned} 0 &= \alpha^2\{-2b^2 s^m s_m - b^4 t^m_{\ m} - 4b^4(n-1)\kappa(x)\} \\ &+ 4\beta\{b^2(n-1)t_0 - ns_0 r + (n-3)b^2 q_0 - b^4 s^m_{\ 0;m} \\ &+ b^2 b^m s_{0;m} + b^2 s_0 r^m_{\ m} - 3(n-1)b^4\theta + \mu(x)\beta\}. \end{aligned}$$

By the same way, $\alpha^2$ and $\beta$ are relatively prime polynomials in $y$ then equation mentioned above is equivalent

$$0 = -2b^2 s^m s_m - b^4 t^m_{\ m} - 4b^4(n-1)\kappa(x), \tag{3.13}$$

$$\begin{aligned} 0 &= b^2(n-1)t_0 - ns_0 r + (n-3)b^2 q_0 - b^4 s^m_{\ 0;m} \\ &\quad + b^2 b^m s_{0;m} + b^2 s_0 r^m_{\ m} - 3(n-1)b^4\theta + \mu(x)\beta. \end{aligned} \tag{3.14}$$

By (3.13) we have

$$\kappa(x) = \frac{-1}{4b^2(n-1)}(2s^m s_m + b^2 t^m_{\ m}). \tag{3.15}$$

Differentiating both sides of (3.14) with respect to $y^i$ yields

$$\begin{aligned} 0 &= b^2(n-1)t_i + (n-3)b^2 q_i - b^4 s^m_{\ i;m} + b^2 b^m s_{i;m} \\ &\quad - ns_i r + b^2 s_i r^m_{\ m} - 3(n-1)b^4\theta_i + \mu(x) b_i. \end{aligned} \tag{3.16}$$

Remark that for contracting this equation with $b^i$ its neccesary to know

$$\begin{aligned} b^i t_i &= b^i s_m s^m_{\ i} = -s_m s^m. \\ b^i q_i &= b^i r_m s^m_{\ i} = -r_m s^m. \\ b^i s^m_{\ i;m} &= (b^i s^m_{\ i})_{;m} - s^m_{\ i} b^i_{;m} = -s^m_{;m} - q^m_m - t^m_m. \\ b^m b^i s_{i;m} &= b^m[(s_i b^i)_{;m} - s_i b^i_{;m}] = s_m(s^m - r^m). \end{aligned} \tag{3.17}$$

Contraction (3.16) with $b^i$ implies that

$$\begin{aligned} 0 &= -b^2(n-1)s_m s^m - (n-3)b^2 r_m s^m + b^4(s^m_{\ ;m} + q^m_{\ m} + t^m_{\ m}) \\ &\quad + b^2 s_m(s^m - r^m) - 3(n-1)b^4\theta_i b^i + \mu(x) b^2. \end{aligned}$$

From this equation, it is easy to get

$$\mu(x) = (n-2)s_m s^m + (n-2)r_m s^m - b^2(s^m_{\ ;m} + q^m_{\ m} + t^m_{\ m}) + 3(n-1)b^2\theta_i b^i. \tag{3.18}$$

By replacing (3.15) into (3.18) we can obtain

$$\mu(x) = -b^2(s^m_{\ ;m} + q^m_{\ m} - (n-1)[3\theta_i b^i + 4\kappa]) + ns^m s_m + (n-2)s^m r_m. \tag{3.19}$$

By the above calculation, we get classifying equation for weak **PRic**-curvature Kropina metric.

**Theorem 3.4** *Let $F = \alpha^2/\beta$ be the non-Riemannian Kropina metric on manifold $M$. Then $F$ is a weak projective Ricci curvature metric if and only if $\alpha$ and $\beta$ satisfies*

$$\begin{aligned} 0 &= b^2\Big\{(n-1)[t_0 - 3b^2\theta] + (n-3)q_0 - b^2 s^m_{\ 0;m} \\ &\quad + b^m s_{0;m} + s_0 r^m_{\ m}\Big\} - ns_0 r + \mu(x)\beta, \\ {}^{\alpha}\boldsymbol{Ric} &= \frac{1}{b^4}\Big\{\alpha^2\mu(x) - (n-2)\left[b^2 r_{0;0} + b^2 s_{0;0} - (r_0+s_0)^2\right] \\ &\quad + r_{00}(nr - b^2 r^m_{\ m}) - 2b^2 q_{00} - b^2 b^m r_{00;m}\Big\}. \end{aligned}$$

*where*

$$\mu(x) = -b^2(s^m_{\ ;m} + q^m_m - (n-1)[3\theta_m b^m + 4\kappa]) + ns^m s_m + (n-2)s^m r_m,$$

*and $\theta = \theta_i y^i$ is a $1-$form and $\kappa = \kappa(x)$ is scalar function on $M$.*

By replacing $\theta = 0$ into equations above, we can charactrize isotropic **PRic**-curvature of Kropina metrics,

**Corollary 3.5** *Let $F = \alpha^2/\beta$ be the non-Riemannian Kropina metric on manifold $M$. Then $F$ is of isotropic projective Ricci curvature metric if and only if $\alpha$ and $\beta$ satisfies*

$$\begin{aligned} 0 = &\quad b^2(n-1)t_0 - ns_0 r + (n-3)b^2 q_0 - b^4 s^m_{\ 0;m} \\ &\quad + b^2 b^m s_{0;m} + b^2 s_0 r^m_{\ m} + \mu(x)\beta, \end{aligned} \tag{3.20}$$

$$\begin{aligned} {}^{\alpha}\boldsymbol{Ric} &= \frac{1}{b^4}\Big\{\alpha^2\mu(x) - (n-2)\left[b^2 r_{0;0} + b^2 s_{0;0} - (r_0+s_0)^2\right] \\ &\quad + nrr_{00} - 2b^2 q_{00} - b^2 b^m r_{00;m} - b^2 r_{00} r^m_{\ m}\Big\}, \end{aligned} \tag{3.21}$$

*where*

$$\mu(x) = -b^2(s^m_{\ ;m} + q^m_{\ m} - 4(n-1)\kappa) + ns^m s_m + (n-2)s^m r_m. \tag{3.22}$$

In [6] Cheng gave a formula for the **PRic** flat Kropina metric. However, we find some errors in his formula. Let $F$ be the Kropina metric with $PRic = 0$ then (3.15) must be as follows

$$t^m_{\ m} = -\frac{2}{b^2} s^m s_m. \tag{3.23}$$

By replacing quantity above into (3.18), we can get Corollary 3.6

**Corollary 3.6** *Let $F = \alpha^2/\beta$ be the non-Riemannian Kropina metric on manifold $M$. Then $F$ is projective Ricci flat $\boldsymbol{PRic} = 0$ if and only if $\alpha$ and $\beta$ satisfy 3.20 and 3.21, where*

$$\mu(x) = -b^2(s^m_{\ ;m} + q^m_m) + ns^m s_m + (n-2)s^m r_m.$$

## 4. Proof of theorems

**Proof of theorem 1.2**: Suppose that **C**-reducible Finsler metric has vanishing Douglas curvature.

It is shown that Randers metric is Douglas metric if and only if $s_{ij} = 0$ [9]. By this assumption, (3.4) must be as follows:

$$\kappa\beta = 0.$$

This means $\kappa = 0$ and **PRic**-curvature must be flat. Moreover, by (3.5) Ricci curvature of $\alpha$ is

$${}^{\alpha}\mathbf{Ric} \quad = \quad (n-1)\Big\{(\frac{-r_0}{1-b^2})_{;0} - (\frac{r_0}{1-b^2})^2\Big\}.$$

In [13], Matsumoto shows that Kropina metric is a Douglas metric if and only if

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i).$$

It is easy to see that $s_0 = s^m{}_0 = 0$. Replacing these vanishing quantities in corollary (3.5), by (3.20) we get $\mu(x) = 0$. According to the definition of this scalar function (3.22), we see that $\kappa = 0$ i.e. isotropic **PRic**-curvature of Douglas Kropina metric must be **PRic** flat, and by (3.21) Ricci curvature of $\alpha$ must be as follows

$${}^{\alpha}\mathbf{Ric} \quad = \quad \frac{1}{b^4}\Big\{-(n-2)\left[b^2 r_{0;0} - r_0^2\right] + nrr_{00} - b^2 b^m r_{00;m} - b^2 r_{00} r^m_{\ m}\Big\}.$$

Berwald spaces are subspaces of Douglas space, then we can conclude that **C**-reducible Berwald metric of isotropic **PRic**-curvature must be projective Ricci flat.

**Corollary 4.1** *Randers Berwald metric of isotropic* ***PRic****-curvature must be projective Ricci flat and* ${}^{\alpha}$***Ric*** *flat.*

Let Randers metric be Berwaldian, then $\beta$ is parallel with respect to $\alpha$ i.e. $r_{ij} = s_{ij} = 0$. by (3.4) we see that $\kappa = 0$ and $\mathbf{PRic} = 0$ , and by (3.5) we can get ${}^{\alpha}\mathbf{Ric} = 0$.

**Corollary 4.2** *Kropina Berwald metric of isotropic* ***PRic****-curvature must be projective Ricci flat and Ricci curvature of* $\alpha$ *satisfies*

$${}^{\alpha}\boldsymbol{Ric} = \frac{-(n-2)}{b^4}\Big\{c_0 b^2\beta + c^2(b^2\alpha^2 + \beta^2)\Big\}. \tag{4.1}$$

It was proved that Kropina metric is a Berwald metric if and only if (see [21])

$$s_{ij} = \frac{1}{b^2}(b_i s_j - b_j s_i), \quad r_{ij} = c(x)a_{ij}.$$

By simple computation we see that

$$s_0 = 0,\ s^m_{\ 0} = 0,\ q_0 = r_m s^m_{\ 0} = 0,\ q_{00} = 0,$$

$$r^m_{\ m} = nc(x),\ r_{00} = c(x)\alpha^2,\ r = c(x)b^2,$$

$$r_0 = c(x)\beta, \; r_{0;0} = c_0\beta + c^2(x)\alpha^2.$$

We put the above values in corollary (3.5) and get $\kappa = \mu(x) = 0$. Kropina Berwald metric of isotropic **PRic** curvature reduces to **PRic** flat. By replacing above quantities in (3.21), we can get (4.1).

**Proof of theorem 1.3**: Shen and Yildirim classified Randers metrics of scalar flag curvature [18] and proved this theorem

**Theorem 4.3** *Let $F = \alpha + \beta$ be a Randers metric on an n-dimensional manifold $M$. $F$ is of scalar flag curvature $K = \sigma(x,y)$, if and only if the Riemann curvature of $\alpha$ and the covariant derivatives of $\beta$ satisfy the following equations:*

$$\begin{aligned} {}^{\alpha}\boldsymbol{R}^i{}_k &= \kappa(\alpha^2\delta^i{}_k - y_k y^i) + \alpha^2 t^i{}_k + t_{00}\delta^i{}_k - t_{k0}y^i - t^i{}_0 - 3s^i{}_0 s_{k0}, \\ s_{ij;k} &= \frac{1}{n-1}(a_{ik}s^m{}_{j;m} - a_{jk}s^m{}_{i;m}). \end{aligned} \tag{4.2}$$

Equation (4.2) is equivalent to the following relation (see Lemma 5.4.4 in [9])

$$\alpha^2 s_{ij;0} = s_{i0;0}y_j - s_{j0;0}y_i. \tag{4.3}$$

Multiplying both sides of equality mentioned above by $a^{im}$ and using equation (3.4), we obtain this relation

$$0 = \alpha^2(\kappa b_j y^m - \rho_0 s^m_j) + \rho_0 s^m_0 y_j - \rho_0 s_{j0}y^m.$$

Contracting above equation by $b^j b_m$ yields $\kappa b^2\alpha^2\beta = 0$, $\kappa$ must be vanished.

Scalar flag curvature Kropina metrics were studied by Ceyhan and Civi [4] and they showed that Kropina metric $F = \alpha^2/\beta$ with dimension $n > 2$ is of scalar flag curvature if the following conditions hold

$$r_{00} = c\alpha^2, \qquad (c = \frac{n-1}{n-2}), \tag{4.4}$$

$$s_0 = 0, \qquad s^r{}_j s_{ri} = 0. \tag{4.5}$$

By using (4.4) and (4.5), we can get

$$\begin{aligned} &r = cb^2, \quad r_0 = c\beta, \quad r^m_m = nc, \quad r_{0;0} = c^2\alpha^2, \\ &s_i = 0, \quad t_0 = 0, \quad r_{00;m} = 0, \quad q^m_m = q_{00} = q_0 = 0. \end{aligned} \tag{4.6}$$

Replace above quantities into (3.20) and (3.22), then we get

$$0 = 4(n-1)\kappa\beta - b^2 s^m_{0;m}.$$

This equation is equivalent to

$$0 = 4(n-1)\kappa - b^i s^m_{i;m}.$$

By (3.17) and (4.6), we see that $b^i s^m_{i;m} = 0$, then by equation mentioned above, $\kappa$ must be zero and Ricci curvature of $\alpha$ has the following form

$${}^{\alpha}Ric = \frac{n-1}{b^4(n-2)}\{\beta^2 - b^2\alpha^2\}.$$

This completes the proof of theorem 1.3.

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