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Perfect numerical semigroups

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Abstract: A numerical semigroup is perfect if it does not have isolated gaps. In this paper we will order the perfect numerical semigroups with a fixed multiplicity. This ordering allows us to give an algorithm procedure to obtain them. We also study the perfect monoid, which is a subset of $\mathbb{N}$ that can be expressed as an intersection of perfect numerical semigroups, and we present the perfect monoid generated by a subset of $\mathbb{N}$. We give an algorithm to calculate it. We study the perfect closure of a numerical semigroup, as well as the perfect numerical semigroup with maximal embedding dimension, in particular Arf and saturated numerical semigroups.

Key words: Arf semigroup, embedding dimension, Frobenius number, genus, multiplicity, numerical semigroup, saturated semigroup

1. Introduction

Let $\mathbb{Z} = \{0, 1, -1, 2, -2, \dots\}$ be the set of integer numbers and $\mathbb{N} = \{x \in \mathbb{Z} \mid x \geq 0\}$.

A submonoid of $(\mathbb{N}, +)$ is a subset M of $\mathbb{N}$ that is closed by the sum and $0 \in M$. A numerical semigroup is a submonoid S of $(\mathbb{N}, +)$ such that $\mathbb{N} \backslash S = \{x \in \mathbb{N} \mid x \notin S\}$ is finite.

If S is a numerical semigroup, the set of elements in $\mathrm{G}(S) = \mathbb{N} \backslash S$ is known as the set of gaps of S. Its cardinality is the genus of S and it will be denoted by $\mathrm{g}(S)$. We will say that a gap h of S is isolated if $\{h-1, h+1\} \subseteq S$. Numerical semigroups without isolated gaps are called perfect numerical semigroups.

Let M be a submonoid of $(\mathbb{N}, +)$ such that $M \neq \{0\}$. The multiplicity of M, denoted by $\mathrm{m}(M)$, is the smallest positive integer that belongs to M.

If m is a positive integer, we denote by $\mathscr{P}_m$ the set of all perfect numerical semigroups with multiplicity m; that is:

$$\mathscr{P}_m = \{S \mid S \text{ is a perfect numerical semigroup and } \mathrm{m}(S) = \mathrm{m}\}.$$

It is clear that $\mathscr{P}_1 = \{\mathbb{N}\}$ and $\mathscr{P}_2 = \emptyset$ (note that if S is a numerical semigroup with multiplicity 2 then 1 is an isolated gap of S).

In Section 2 we will see that if m is an integer number greater than or equal to 3, we can order the elements of $\mathscr{P}_m$ making a tree with the root the perfect numerical semigroup $\{0, m, \rightarrow\}$ (the symbol $\rightarrow$ means that every integer greater than m belongs to the set). We characterize the children of an arbitrary vertex and this will provide us an algorithmic procedure that allows us to recurrently build the elements of $\mathscr{P}_m$.

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In Section 3 we will see that the finite intersection of a perfect numerical semigroup is again a perfect numerical semigroup. In general, this is not true for infinite intersections (although it is always a submonoid of $(\mathbb{N},+)$).

We say that a submonoid of $(\mathbb{N},+)$ is perfect if it can be expressed as an intersection of perfect numerical semigroups.

The intersection of perfect monoids is a perfect monoid. Therefore, we can speak about the least (with respect to inclusion order) perfect monoid containing a set X of nonnegative integers. Such monoid will be denoted by $\mathscr{P}(X)$ and we will give an algorithm to calculate it.

In Section 4 we will speak about the perfect closure of a numerical semigroup, S, i.e. the smallest perfect numerical semigroup containing S. Our main aim in this section will be the following: given a perfect numerical semigroup T, we want to calculate and to study the set of all numerical semigroups whose perfect closure is T.

If A is a subset nonempty of $\mathbb{N}$, we denote by $\langle A\rangle$ the submonoid of $(\mathbb{N},+)$ generated by A; that is, $\langle A\rangle = \{\lambda_1 a_1 + \cdots + \lambda_n a_n \mid n \in \mathbb{N}\setminus\{0\}, \{a_1,\ldots,a_n\} \subseteq A \text{ and } \{\lambda_1,\ldots,\lambda_n\} \subseteq \mathbb{N}\}$. It is well known (see, for example, [10, Lema 2.1]) that $\langle A\rangle$ is a numerical semigroup if and only if $\gcd(A)=1$.

If M is a submonoid of $(\mathbb{N},+)$ and $M=\langle A\rangle$, then we say that A is a system of generators of M. Moreover, if $M\neq\langle B\rangle$ for all $B\subsetneq A$, then we will say that A is a minimal system of generators of M. In [10, Corollary 2.8] it is shown that every submonoid of $(\mathbb{N},+)$ has a unique minimal system of generators, which in addition is finite. We denote by $\mathrm{msg}(M)$ the minimal system of generators of M. The cardinality of $\mathrm{msg}(M)$ is called the embedding dimension of M and will be denoted by $\mathrm{e}(M)$. From Propositions 2.2 and 2.10 of [10], we deduce that $\mathrm{e}(M)\leq \mathrm{m}(M)$.

A numerical semigroup S has maximal embedding dimension if $\mathrm{e}(S)=\mathrm{m}(S)$. In Section 5 we will study the perfect numerical semigroups with maximal embedding dimension. In particular, we are interested in saturated numerical semigroups and in those having the Arf property.

2. The tree of perfect numerical semigroups

The following result can be easily deduced from the definition of perfect numerical semigroups and it will be used many times throughout this work.

Proposition 1 *Let S be a numerical semigroup. The following conditions are equivalent:*

1) *S is a perfect numerical semigroup.*

2) *If $\{s, s+2\}\subseteq S$, then $s+1\in S$.*

If S is a numerical semigroup, then the greatest integer number that does not belong to S is called the Frobenius number of S and it will be denoted by $\mathrm{F}(S)$.

In the rest of this section we will assume that m is an integer number greater than or equal to 3.

Lemma 2 *If $S\in\mathscr{P}_m$, then $\{\mathrm{F}(S), \mathrm{F}(S)-1\}\subseteq \mathrm{G}(S)$.*

Proof As $m\geq 3$, then $\mathrm{F}(S)\geq 2$. Moreover, $\mathrm{F}(S)\in\mathrm{G}(S)$ and $\mathrm{F}(S)+1\in S$. Therefore, applying Proposition 1, we have $\mathrm{F}(S)-1\in\mathrm{G}(S)$. □

Lemma 3 *If* $S \in \mathscr{P}_m$ *and* $S \neq \{0, m, \rightarrow\}$, *then* $S \cup \{\mathrm{F}(S)\} \in \mathscr{P}_m$ *or* $S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1\} \in \mathscr{P}_m$.

Proof As $S \neq \{0, m, \rightarrow\}$, then $\mathrm{F}(S) > m$ and clearly $S \cup \{\mathrm{F}(S)\}$ and $S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1\}$ are numerical semigroups. If $\mathrm{F}(S)-2 \notin S$ then $S \cup \{\mathrm{F}(S)\} \in \mathscr{P}_m$. If $\mathrm{F}(S)-2 \in S$ it is clear that $S \cup \{\mathrm{F}(S), \mathrm{F}(S)-1\} \in \mathscr{P}_m$. □

If $S \in \mathscr{P}_m$, we denote $\theta(S) = \begin{cases} \{\mathrm{F}(S)\} & \text{if } \mathrm{F}(S)-2 \notin S, \\ \{\mathrm{F}(S), \mathrm{F}(S)-1\} & \text{otherwise.} \end{cases}$

Given an element $S \in \mathscr{P}_m$, we define recurrently the following sequence of elements of $\mathscr{P}_m$:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \theta(S_n) & \text{if } S_n \neq \{0, m, \rightarrow\}, \\ S_n & \text{otherwise.} \end{cases}$

The following proposition is easily deduced from the above results. If X is a set, its cardinality is denoted by $\sharp X$.

Proposition 4 *If* $S \in \mathscr{P}_m$, *then there exist* $S_0, S_1, \ldots, S_l \in \mathscr{P}_m$ *such that* $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_l = \{0, m, \rightarrow\}$ *and* $\sharp(S_{i+1} \setminus S_i) \in \{1, 2\}$ *for all* $i \in \{0, 1, \ldots, l-1\}$.

We will refer to the chain of the above proposition,

$$S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_l = \{0, m, \rightarrow\},$$

as the chain associated to S; the number l will be denoted by $\mathrm{l}(S)$ and it is called the length of the chain associated to S.

Example 5 *Let* $S = \{0, 9, 14, 18, 21, \rightarrow\} \in \mathscr{P}_9$. *Then the associated chain to* S *is* $S = S_0 \subsetneq S_1 = \{0, 9, 14, 18, \rightarrow\} \subsetneq S_2 = \{0, 9, 14, 17, \rightarrow\} \subsetneq S_3 = \{0, 9, 14, \rightarrow\} \subsetneq S_4 = \{0, 9, 13, \rightarrow\} \subsetneq S_5 = \{0, 9, 12, \rightarrow\} \subsetneq S_6 = \{0, 9, \rightarrow\}$. *Therefore,* $\mathrm{l}(S) = 6$.

If S is a numerical semigroup, then the set of its small elements is $\mathrm{N}(S) = \{s \in S \mid s < \mathrm{F}(S)\}$. Its cardinality is denoted by $\mathrm{n}(S)$. Note that $\mathrm{g}(S) + \mathrm{n}(S) = \mathrm{F}(S) + 1$.

If $\{a, b\} \subseteq \mathbb{N}$, then $]a, b[= \{x \in \mathbb{N} \mid a < x < b\}$. We remark that if $]a, b[\neq \emptyset$, then $\sharp(]a, b[) = b - a - 1$.

Theorem 6 *If* $S \in \mathscr{P}_m$, *then* $\mathrm{l}(S) = \mathrm{g}(S) - \mathrm{n}(S) - \mathrm{m}(S) + 2$.

Proof Let $S = \{s_0 = 0, s_1 = \mathrm{m}(S), s_2, \ldots, s_{\mathrm{n}(S)} = \mathrm{F}(S) + 1, \rightarrow\}$. Then it is clear that $\mathrm{G}(S) =]0, s_1[\cup]s_1, s_2[\cup \cdots \cup]s_{\mathrm{n}(S)-1}, s_{\mathrm{n}(S)}[$. Hence, the first links of the chain associated to S are of the form $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k = S \cup \{s_{\mathrm{n}(S)-1}, \rightarrow\} \subsetneq \ldots$ and $k = \sharp(]s_{\mathrm{n}(S)-1}, s_{\mathrm{n}(S)}[) - 1 = s_{\mathrm{n}(S)} - s_{\mathrm{n}(S)-1} - 2$. The following links of the chain associated to S would have the form $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_k \subsetneq S_{k+1} \subsetneq \cdots \subsetneq S_{k+r} = S \cup \{s_{\mathrm{n}(S)-2}, \rightarrow\} \subsetneq \ldots$ where $r = s_{\mathrm{n}(S)-1} - s_{\mathrm{n}(S)-2} - 2$. Applying this idea recursively we obtain that $\mathrm{l}(S) = s_2 - s_1 - 2 + s_3 - s_2 - 2 + \cdots + s_{\mathrm{n}(S)} - s_{\mathrm{n}(S)-1} - 2 = s_{\mathrm{n}(S)} - s_1 - 2(\mathrm{n}(S) - 1) = \mathrm{F}(S) + 1 - \mathrm{m}(S) - 2(\mathrm{n}(S) - 1)$. Applying now that $\mathrm{g}(S) + \mathrm{n}(S) = \mathrm{F}(S) + 1$, we obtain that $\mathrm{l}(S) = \mathrm{g}(S) - \mathrm{n}(S) - \mathrm{m}(S) + 2$. □

Let S be a numerical semigroup. If $s \in \mathrm{N}(S)$, then it is clear that $\mathrm{F}(S) - s \in \mathrm{G}(S)$, so $\mathrm{n}(S) \leq \mathrm{g}(S)$. In [10] it is shown that numerical semigroups that achieve the previous bound are called symmetric numerical semigroups.

The following result is a consequence of Theorem 6.

Corollary 7 *If S is a perfect numerical semigroup such that $S \neq \mathbb{N}$, then* $\mathrm{n}(S) \leq \mathrm{g}(S) - \mathrm{m}(S) + 2$.

A graph G is a pair (V, E) where V is a nonempty set and E is a subset of $\{(u, v) \in V \times V \mid u \neq v\}$. The elements of V and E are called vertices and edges, respectively. A path (of length n) connecting the vertices x and y of G is a sequence of different edges of the form $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n)$ such that $v_0 = x$ and $v_n = y$. A graph G is a tree if there exists a vertex r (known as the root of G) such that for any other vertex x of G there exists a unique path connecting x and r. If there exists a path connecting the vertices x and y, then we will say that x is a descendant of y. In particular, if (x, y) is an edge of the tree, we say that x is a child of y. We define the graph $G(\mathscr{P}_m)$ as follows: $\mathscr{P}_m$ is its set of vertices and $(S, T) \in \mathscr{P}_m \times \mathscr{P}_m$ is an edge if $T = S \cup \theta(S)$.

The following result is an easy consequence of Proposition 4.

Proposition 8 *$G(\mathscr{P}_m)$ is a tree with root $\{0, m, \rightarrow\}$.*

A tree can be recurrently built starting from its root and, adding to each vertex already built, its children. Now we want to characterize the children of an arbitrary vertex from $G(\mathscr{P}_m)$. The following result is well known and it appears in [10].

Lemma 9 *Let S be a numerical semigroup and $x \in S$. Then $S \setminus \{x\}$ is a numerical semigroup if and only if $x \in \mathrm{msg}(S)$.*

The proof of the following result is trivial.

Lemma 10 *Let $T \in \mathscr{P}_m$. We have:*

1. *If $\mathrm{F}(T) + 1 \in \mathrm{msg}(T) \setminus \{m\}$, then $T \setminus \{\mathrm{F}(T) + 1\} \in \mathscr{P}_m$.*
2. *If $\{x, x + 1\} \subseteq \mathrm{msg}(T)$ and $x > \mathrm{F}(T) + 1$, then $T \setminus \{x, x + 1\} \in \mathscr{P}_m$.*

Theorem 11 *Let S and T be two elements of $\mathscr{P}_m$. Then S is a child of T in the tree $G(\mathscr{P}_m)$ if and only if one of the following conditions is verified:*

1. *$\mathrm{F}(T) + 1 \in \mathrm{msg}(T) \setminus \{m\}$ and $S = T \setminus \{\mathrm{F}(T) + 1\}$.*
2. *$S = T \setminus \{x, x + 1\}$ with $\{x, x + 1\} \subseteq \mathrm{msg}(T)$ and $x > \mathrm{F}(T) + 1$.*

Proof *Necessity.* If S is a child of T then $T = S \cup \theta(S)$. We will distinguish two cases:

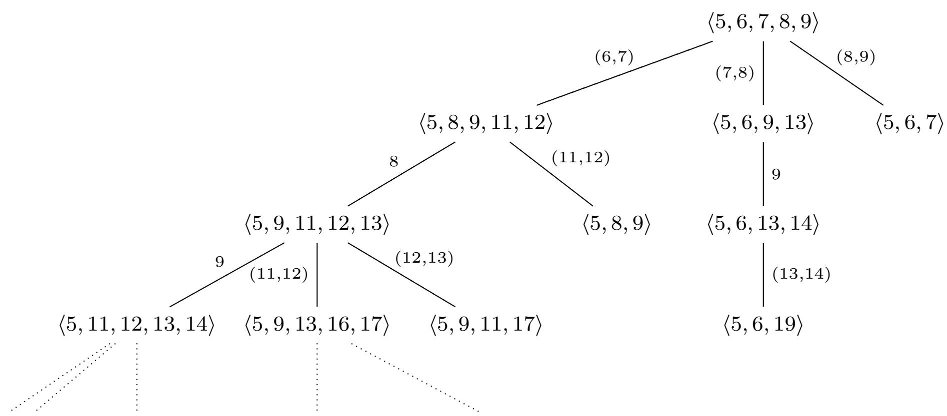
1. If $\theta(S) = \{\mathrm{F}(S)\}$, then $T = S \cup \{\mathrm{F}(S)\}$. By Lemma 2 we know that $\mathrm{F}(S) - 1 \notin S$ and therefore $\mathrm{F}(T) = \mathrm{F}(S) - 1$, so $S = T \setminus \{\mathrm{F}(T) + 1\}$. Applying Lemma 9 we have that $\mathrm{F}(T) + 1 \in \mathrm{msg}(T) \setminus \{m\}$.
2. If $\theta(S) = \{\mathrm{F}(S), \mathrm{F}(S) - 1\}$, then $T = S \cup \{\mathrm{F}(S), \mathrm{F}(S) - 1\}$ and $\mathrm{F}(S) - 2 \in S$. Hence, we easily deduce that $S = T \setminus \{\mathrm{F}(S) - 1, \mathrm{F}(S)\}$ with $\{\mathrm{F}(S) - 1, \mathrm{F}(S)\} \subseteq \mathrm{msg}(T)$ and $\mathrm{F}(S) - 1 > \mathrm{F}(T) + 1$.

Sufficiency. This is an immediate consequence of Lemma 10.

□

The previous theorem provides us an algorithmic process that allows to recursively build the tree $G(\mathscr{P}_m)$, as the following example shows.

Example 12 *We are going to build* $G(\mathscr{P}_5)$. *For this purpose we have to take the root as the starting point* $\langle 5,6,7,8,9\rangle$ *and we will connect each vertex already built with its children (which are perfectly determined by Theorem 11). The character that appears on the edge means the minimal generators that we have removed from the parent vertex to obtain the child vertex.*



We finish this section observing that, for every $k \in \mathbb{N}\backslash\{0\}$, we have that $S(k) = \langle m\rangle \cup \{km+1, \rightarrow\} \in \mathscr{P}_m$. Hence, $\mathscr{P}_m$ has infinite cardinality. Therefore, we cannot calculate all elements of $\mathscr{P}_m$. However, the cardinality of the set of elements of $\mathscr{P}_m$ with a fixed Frobenius number or a fixed genus is finite. As we go forward through the branches of the tree $G(\mathscr{P}_m)$, the numerical semigroups that we find have an increasing genus and an increasing Frobenius number. The reader will not have difficulty in designing an algorithm that allows us to determine all the elements of $\mathscr{P}_m$ with a fixed Frobenius number and/or genus.

3. Perfect monoids

It is well known and easy to prove that if S and T are numerical semigroups, then $S \cap T$ is also a numerical semigroup. Using Proposition 1, we easily deduce that the finite intersection of perfect numerical semigroups is again a perfect numerical semigroup. This result does not hold for infinite intersections, as is shown in the following example.

Example 13 *It is clear that* $\{0, k, \rightarrow\}$ *is a perfect numerical semigroup for all* $k \in \mathbb{N} \setminus \{0,1,2\}$. *Also, it is clear that* $\bigcap_{k\in\mathbb{N}\setminus\{0,1,2\}}\{0,k,\rightarrow\} = \{0\}$.

Thus, the infinite intersection of perfect numerical semigroups, in general, is not a numerical semigroup.

The intersection (finite or infinite) of numerical semigroups is always a submonoid of $(\mathbb{N}, +)$. We say that a submonoid M of $(\mathbb{N}, +)$ is a perfect submonoid if it can be written as an intersection of perfect numerical semigroups. Then we have that the intersection of perfect monoids is also a perfect monoid.

If $X \subseteq \mathbb{N}$, then the perfect monoid generated by X, which we will denote by $\mathscr{P}(X)$, is the intersection of all perfect monoids containing X. As a consequence of the above results, we have that $\mathscr{P}(X)$ is the least (with respect to inclusion) perfect monoid containing X. Also, it is easy to deduce that $\mathscr{P}(X)$ is the intersection of all perfect numerical semigroups containing X.

If $M = \mathscr{P}(X)$, then we will say that X is a *$\mathscr{P}$-system of generators* of M. Moreover, if $M \neq \mathscr{P}(Y)$ for all $Y \subsetneq X$, then we will say that X is a *minimal $\mathscr{P}$-system of generators* of M.

The following result can be easily deduced from the above comments.

Lemma 14 *Let X and Y be two subsets of $\mathbb{N}$ and M a perfect monoid. Then:*

1. *If $X \subseteq Y$ then $\mathscr{P}(X) \subseteq \mathscr{P}(Y)$.*
2. *$\mathscr{P}(X) = \mathscr{P}(\langle X \rangle)$.*
3. *$\mathscr{P}(M) = M$.*

The following result tells us that the set formed by all perfect monoids is $\{\mathscr{P}(X) \mid X \text{ is a finite subset of } \mathbb{N}\}$. Therefore, giving a perfect monoid is the same as giving a finite subset of $\mathbb{N}$.

Proposition 15 *Every perfect monoid has a $\mathscr{P}$-system of generators, which in addition is finite.*

Proof Let M be a perfect monoid. Then M is a submonoid of $(\mathbb{N}, +)$ and by [10, Corollary 2.8] there exists X, a finite subset of $\mathbb{N}$ such that $M = \langle X \rangle$. Applying Lemma 14, we have $M = \mathscr{P}(M) = \mathscr{P}(\langle X \rangle) = \mathscr{P}(X)$. Hence, X is a $\mathscr{P}$-system of finite generators of M. □

Now our purpose is to show an algorithmic method that allows us to calculate $\mathscr{P}(X)$ from X.

Theorem 16 *Let M be a submonoid of $(\mathbb{N}, +)$ such that $M \neq \{0\}$. Then the following conditions are equivalent:*

1) M is a perfect monoid.

2) If $\{x, x+2\} \subseteq M$ then $x+1 \in M$.

Proof *1) implies 2).* If M is a perfect monoid then there exists a family $\{S_i \mid i \in I\}$ of perfect numerical semigroups such that $M = \bigcap_{i \in I} S_i$.

As $\{x, x+2\} \subseteq M$, then $\{x, x+2\} \subseteq S_i$ for all $i \in I$. Applying Proposition 1, we have that $x+1 \in S_i$ for all $i \in I$. Therefore, $x+1 \in M$.

2) implies 1). For every $x \in M$ we denote $S_x = M \cup \{x, \rightarrow\}$. It is clear that S_x is a perfect numerical semigroup and $M = \bigcap_{x \in M} S_x$, so M is a perfect monoid. □

If M is a submonoid of $(\mathbb{N}, +)$, then the perfect closure of M is $\overline{M} = M \cup \{x \in \mathbb{N} \mid \{x-1, x+1\} \subseteq M\}$.

Corollary 17 *If M is a submonoid of $(\mathbb{N}, +)$, then $\overline{M} = \mathscr{P}(M)$.*

Proof First we show that $\overline{M}$ is a submonoid of $(\mathbb{N}, +)$. This can be easily deduced from the two following points:

- If $s \in \mathbb{N} \setminus M$, $\{s-1, s+1\} \subseteq M$ and $x \in M$, then $\{x+s-1, x+s+1\} \subseteq M$ and so $s+x \in \overline{M}$.
- If $\{s-1, s+1\} \subseteq M$ and $\{s'-1, s'+1\} \subseteq M$, then $s'+1+s-1 \in M$ and so $s+s' \in \overline{M}$.

Applying Theorem 16 we have that $\overline{M}$ is a perfect monoid. As $M \subseteq \overline{M}$ then $\mathscr{P}(M) \subseteq \overline{M}$. To conclude the proof we will see $\overline{M} \subseteq \mathscr{P}(M)$. Indeed, let $x \in \overline{M}$. If $x \in M$, then $x \in \mathscr{P}(M)$. If $x \notin M$, then

$\{x-1, x+1\} \subseteq M$, so $\{x-1, x+1\} \subseteq \mathscr{P}(M)$. Applying that $\mathscr{P}(M)$ is a perfect monoid and Theorem 16, we have $x \in \mathscr{P}(M)$. □

As a consequence of these results, we have the following result.

Corollary 18 *If* $X \subseteq \mathbb{N}$, *then* $\mathscr{P}(X) = \overline{\langle X \rangle}$.

The previous corollary allows us an algorithmic method to compute the least perfect monoid containing a finite set of nonnegative integers. Next we illustrate this method with an example.

Example 19 *We are going to build* $\mathscr{P}(\{5,8\})$. *As* $\langle 5,8\rangle = \{0,5,8,10,13,15,16,$ $18,20,21,23,24,25,26,28,\rightarrow\}$, *then applying Corollary 18, we have that* $\mathscr{P}(\{5,8\}) = \overline{\langle 5,8\rangle} = \{0,5,8,9,10,13,\rightarrow\} = \langle 5,8,9\rangle$.

Now we aim to show the condition that must verify X so that $\mathscr{P}(X)$ is a numerical semigroup.

Proposition 20 *Let* M *be a submonoid of* $(\mathbb{N},+)$ *such that* $M \neq \{0\}$ *and* $d = \gcd(M)$.

1. *If* $d \geq 3$ *then* $\overline{M} = M$.
2. $\overline{M}$ *is a numerical semigroup if and only if* $d \in \{1,2\}$.

Proof

1. If $\{x, x+2\} \subseteq M$ then $d \leq \gcd\{x, x+2\} \leq 2$, which is absurd. Therefore, applying Theorem 16 we have that M is a perfect monoid and so $M = \overline{M}$.
2. If $\overline{M}$ is a numerical semigroup and $d \neq 1$, then $\overline{M} \neq M$. Therefore, there exists $\{x, x+2\} \subseteq M$. Therefore, $d = 2$.

   Conversely, if $d = 1$, then M is a numerical semigroup and so $\overline{M}$ is also a numerical semigroup. If $d = 2$, then there exists $\{x, x+2\} \subseteq M$ and hence $\{x, x+1, x+2\} \subseteq \overline{M}$. Therefore, $\gcd(\overline{M}) = 1$ and consequently $\overline{M}$ is a numerical semigroup.

□

An immediate consequence of the above proposition is the following result.

Corollary 21 *Let* $X \subseteq \mathbb{N}$. *Then* $\mathscr{P}(X)$ *is a perfect numerical semigroup if and only if* $\gcd(X) \in \{1,2\}$.

Note that a perfect numerical semigroup does not have to have a unique minimal $\mathscr{P}$-system of generators because, for example, $\mathscr{P}(\{3,4\}) = \mathscr{P}(\{3,5\}) = \{0,3,\rightarrow\}$, and so $\{3,4\}$ and $\{3,5\}$ are minimal $\mathscr{P}$-systems of generators of $\{0,3,\rightarrow\}$.

Lemma 22 *If* X *is a* $\mathscr{P}$*-system of generators of* T, *then* $Y = \{x \in X \mid x \leq \max(\mathrm{msg}(T)) + 1\}$ *is also a* $\mathscr{P}$*-system of generators of* T.

Proof We will see that $\overline{\langle Y\rangle} = T$. For this it suffices to see that $\mathrm{msg}(T) \subseteq \overline{\langle Y\rangle}$, but this is clear because $\mathrm{msg}(T) \subseteq \overline{\langle X\rangle}$. □

Note that, as a consequence of the above lemma, we have that if X is a minimal $\mathscr{P}$-system of generators of a perfect numerical semigroup T, then X is a subset of $\{1, 2, \ldots, \max(\mathrm{msg}(T) + 1)\}$. Therefore, we have the following result.

Proposition 23 *A perfect numerical semigroup has a finite number of minimal $\mathscr{P}$-systems of generators.*

4. The perfect closure of a numerical semigroup

If S is a numerical semigroup, then by Corollary 17, we know that $\overline{S} = S \cup \{x \in \mathbb{N} \mid \{x-1, x+1\} \subseteq S\}$ is a perfect numerical semigroup, which we will call the perfect closure of S.

Let $\mathscr{L}$ be the set of all numerical semigroups. We define the following equivalence relation $\mathcal{R}$ on $\mathscr{L}$: $S\mathcal{R}T$ if $\overline{S} = \overline{T}$. For every $S \in \mathscr{L}$, we denote by $[S]$ its class. That is, $[S] = \{S' \in \mathscr{L} \mid S\mathcal{R}S'\}$. The reader will not have difficulty to prove the following result.

Proposition 24 *The set $\{[T] \mid T$ is a perfect numerical semigroup$\}$ is a partition of $\mathscr{L}$. Moreover, if S and T are perfect numerical semigroup where S is not equal to T, then $[S] \cap [T] = \emptyset$.*

If S is a numerical semigroup that is not perfect, then we denote by $\mathrm{h}(S)$ the maximum isolated gap of S. If S is a perfect numerical semigroup then $\mathrm{h}(S) = -1$.

Proposition 25 *If S is a numerical semigroup that is not perfect, then $S \cup \{\mathrm{h}(S)\}$ is a numerical semigroup.*

Proof As $\mathrm{h}(S)$ is an isolated gap of S, then $\{\mathrm{h}(S)-1, \mathrm{h}(S)+1\} \subseteq S$. Therefore, $\mathrm{h}(S) + \mathrm{h}(S) = \mathrm{h}(S) - 1 + \mathrm{h}(S) + 1 \in S$. To conclude the proof, we will see that if $s \in S \setminus \{0\}$ then $\mathrm{h}(S) + s \in S$. In fact, $\{\mathrm{h}(S)-1, \mathrm{h}(S)+1\} \subseteq S$ and so $\{\mathrm{h}(S)+s-1, \mathrm{h}(S)+s+1\} \subseteq S$. Applying that $\mathrm{h}(S)$ is the maximum isolated gap of S, we deduce that $\mathrm{h}(S) + s \in S$. □

If S is a numerical semigroup, then we denote by $\mathrm{a}(S)$ the number of isolated gaps of S.

If S is a numerical semigroup, we define recursively the following sequence of numerical semigroups:

- $S_0 = S$,
- $S_{n+1} = \begin{cases} S_n \cup \{\mathrm{h}(S_n)\} & \text{if } \mathrm{h}(S_n) \neq -1, \\ S_n & \text{otherwise.} \end{cases}$

The proof of the following result is trivial.

Proposition 26 *If S is a numerical semigroup, then $S = S_0 \subsetneq S_1 \subsetneq \cdots \subsetneq S_{\mathrm{a}(S)}$. Moreover, $S_{\mathrm{a}(S)}$ is a perfect numerical semigroup and $\sharp(S_{i+1} \setminus S_i) = 1$ for all $i \in \{0, \ldots, \mathrm{a}(S) - 1\}$.*

If T is a perfect numerical semigroup, then we define the graph $G([T])$ of the following form: $[T]$ is its set of vertex and $(S, S') \in [T] \times [T]$ is an edge if $S \cup \{\mathrm{h}(S)\} = S'$.

The following result is an immediate consequence of Proposition 26.

Corollary 27 *If T is a perfect numerical semigroup, then $G([T])$ is a tree with root T.*

In the following result we will show how the children of an arbitrary vertex of tree $G([T])$ are.

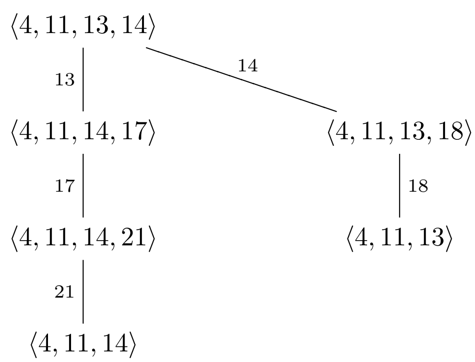
Proposition 28 *Let T be a perfect numerical semigroup and $S \in [T]$. Then the set formed by the children of S in the tree $G([T])$ is $\{S \setminus \{x\} \mid x \in \mathrm{msg}(S),\ x > \mathrm{h}(S)$ and $\{x-1, x+1\} \subseteq S\}$.*

Proof If S' is a child of S, then $S' \cup \{\mathrm{h}(S')\} = S$. Therefore, $S' = S \setminus \{\mathrm{h}(S')\}$ with $\mathrm{h}(S') \in \mathrm{msg}(S)$, $\mathrm{h}(S') > \mathrm{h}(S)$, and $\{\mathrm{h}(S')-1, \mathrm{h}(S')+1\} \subseteq S' \subseteq S$.

Conversely, if $x \in \mathrm{msg}(S)$, $x > \mathrm{h}(S)$ and $\{x-1, x+1\} \subseteq S$, then it is clear that $S \setminus \{x\} \in [T]$ and $\mathrm{h}(S \setminus \{x\}) = x$. Thus, $(S \setminus \{x\}) \cup \mathrm{h}(S \setminus \{x\}) = S$ and consequently $S \setminus \{x\}$ is a child of S. □

The above proposition will enable us to recursively construct the tree $G([T])$, starting from the root T and adding their children to each vertex already built.

Example 29 *It is clear that $T = \langle 4, 11, 13, 14\rangle = \{0, 4, 8, 11, \rightarrow\}$ is a perfect numerical semigroup. We are going to build the tree $G([T])$ using Proposition 28. The number that appears in the following figure over every edge marks the minimal generator that we have removed from the parent vertex to obtain the corresponding child.*



We remark that $\sharp[\langle 4, 11, 13, 14\rangle] = 6$ and so the set formed by all numerical semigroups that have $\langle 4, 11, 13, 14\rangle$ as perfect closure is finite. This is not true in general as the following example shows.

Example 30 *It is clear that if n is an integer such that $n \geq 6$, then $S(n) = \langle 4, 6\rangle \cup \{n, \rightarrow\}$ is a numerical semigroup and $\overline{S(n)} = \langle 4, 5, 6, 7\rangle$. Therefore, $[\langle 4, 5, 6, 7\rangle]$ is a set with infinite elements.*

Another example of this type is provided by the family of numerical semigroups $\{T(n) = \langle 8, 12, 14\rangle \cup \{n, \rightarrow\} \mid n \in \mathbb{N} \text{ and } n \geq 21\}$. It is clear that $\overline{T(n)} = \langle 8, 12, 13, 14, 15\rangle$ for all $n \in \{21, \rightarrow\}$. Therefore, $[\langle 8, 12, 13, 14, 15\rangle]$ is also a set with infinite cardinality.

Let T be a perfect numerical semigroup. Now we propose to study which conditions must fulfill T so that $\sharp[T]$ is infinite.

Lemma 31 *Let T be a perfect numerical semigroup, X a $\mathscr{P}$-system of generators of T, and S a numerical semigroup. If $X \subseteq S \subseteq T$, then $\overline{S} = T$.*

Proof By Lemma 14, we know that $\mathscr{P}(X) \subseteq \mathscr{P}(S) \subseteq \mathscr{P}(T) = T$. As $\mathscr{P}(X) = T$, then by Corollary 18 we have $\overline{S} = \mathscr{P}(S) = T$. □

Recall that by Corollary 21, we know that $\mathscr{P}(X)$ is a perfect numerical semigroup if and only if $\gcd(X) \in \{1, 2\}$.

Lemma 32 *Let T be a perfect numerical semigroup, $X \subseteq \mathbb{N}$, such that $\gcd(X) = 2$ and $\mathscr{P}(X) = T$. Then $[T]$ has infinite cardinality.*

Proof It is clear that $S(n) = \langle X \rangle \cup \{n, \rightarrow\}$ is a numerical semigroup for every $n \in \{\mathrm{F}(T) + 1, \rightarrow\}$. Applying Lemma 31, we have that $\overline{S(n)} = T$. Therefore, $[T]$ has infinite cardinality. □

Recall that the minimal $\mathscr{P}$-system of generators of a perfect numerical semigroup is not unique.

Lemma 33 *Let T be a perfect numerical semigroup and S a numerical semigroup. Then $\overline{S} = T$ if and only if there exists a minimal $\mathscr{P}$-system of generators X of T such that $X \subseteq S \subseteq T$.*

Proof *Necessity.* As $\overline{S} = T$, then $\mathscr{P}(S) = T$ and so S is a $\mathscr{P}$-system of generators of T. Let X be a subset of S minimal with the condition of $\mathscr{P}(X) = T$. Then X is a minimal $\mathscr{P}$-system of generators of T and $X \subseteq S \subseteq T$.

Sufficiency. It is a consequence of Lemma 31. □

Recall that by Proposition 23, we know that a perfect numerical semigroup has only a finite number of minimal $\mathscr{P}$-systems of generators.

Theorem 34 *Let T be a perfect numerical semigroup. Then $[T]$ is a set with infinite cardinality if and only if T has a minimal $\mathscr{P}$-system of generators whose elements have as greatest common divisor two.*

Proof *Necessity.* Let $X_1, X_2, \ldots, X_n$ be all the minimal $\mathscr{P}$-systems of generators of T. If $\gcd(X_1) = \gcd(X_2) = \cdots = \gcd(X_n) = 1$, then $\langle X_1 \rangle, \langle X_2 \rangle, \ldots, \langle X_n \rangle$ are numerical semigroups. Therefore, $\{S \mid S \text{ is numerical semigroup and } X_i \subseteq S \subseteq T\}$ is finite for all $i \in \{1, \ldots, n\}$. Applying Lemma 33, we deduce that $[T]$ is a finite set.

Sufficiency. It is an immediate consequence of Lemma 32. □

5. Perfect numerical semigroups with maximal embedding dimension

Recall that a numerical semigroup S has maximal embedding dimension (MED-semigroup) if $\mathrm{e}(S) = \mathrm{m}(S)$. The following result is deduced from [2, Proposition I.2.9].

Lemma 35 *Let S be a numerical semigroup. Then S is a* MED*-semigroup if and only if $\{s - \mathrm{m}(S) \mid s \in S \backslash \{0\}\}$ is a numerical semigroup.*

If S is a numerical semigroup we will denote $\Delta(S) = \{s - \mathrm{m}(S) \mid s \in S \setminus \{0\}\}$.

Proposition 36 *Let S be a* MED*-semigroup such that $\mathrm{m}(S) \geq 3$. Then S is a perfect numerical semigroup if and only if $\Delta(S)$ is a perfect numerical semigroup.*

Proof *Necessity.* By Lemma 35, we know that $\Delta(S)$ is a numerical semigroup. To prove that $\Delta(S)$ is perfect, we will use Proposition 1. In fact, if $\{x, x+2\} \subseteq \Delta(S)$, then there exists $\{s_1, s_2\} \subseteq S$ such that $x = s_1 - \mathrm{m}(S)$ and $x + 2 = s_2 - \mathrm{m}(S)$. Thus, $s_2 = s_1 + 2$ and consequently $\{s_1, s_1 + 2\} \subseteq S$. Applying that S is perfect, we have that $s_1 + 1 \in S$. Hence, $x + 1 = s_1 + 1 - \mathrm{m}(S) \in \Delta(S)$.
Sufficiency. To prove that S is perfect, we will use Proposition 1. If $\{s, s+2\} \subseteq S$, then applying that $\mathrm{m}(S) \geq 3$, we deduce that $S \neq 0$. Therefore, $\{s, s+2\} \subseteq S \setminus \{0\}$ and consequently $\{s - \mathrm{m}(S), s + 2 - \mathrm{m}(S)\} \subseteq \Delta(S)$. As $\Delta(S)$ is perfect, we have $s + 1 - \mathrm{m}(S) \in \Delta(S)$. Hence, $s + 1 \in S$. □

The following result is an immediate consequence of Lemma 35.

Lemma 37 *Let S be a numerical semigroup and $x \in S \setminus \{0\}$. Then $(\{x\} + S) \cup \{0\}$ is a* MED*-semigroup with multiplicity x. Moreover, every* MED*-semigroup has this form.*

A PMED-semigroup is a perfect MED-semigroup.

Theorem 38 *Let S be a perfect numerical semigroup and $x \in S$ such that $x \geq 3$. Then $S(x) = (\{x\} + S) \cup \{0\}$ is a* PMED*-semigroup. Moreover, every* PMED*-semigroup different from $\mathbb{N}$ has this form.*

Proof By Lemma 37, we know that $S(x)$ is a MED-semigroup with multiplicity x. Applying now Proposition 36, we have that $S(x)$ is a perfect numerical semigroup. Thus, $S(x)$ is a PMED-semigroup.

Let T be a PMED-semigroup such that $T \neq \mathbb{N}$. Then $\mathrm{m}(T) \geq 3$, and by Proposition 36, we have that $\Delta(T)$ is a perfect numerical semigroup. Therefore, $T = (\{\mathrm{m}(T)\} + \Delta(T)) \cup \{0\}$ where $\mathrm{m}(T) \in \Delta(T)$, $\mathrm{m}(T) \geq 3$, and $\Delta(T)$ is a perfect numerical semigroup. □

The following result can be deduced from [7, Proposition 9].

Proposition 39 *Let S be a numerical semigroup, $x \in S \setminus \{0, 1\}$, and $S(x) = (\{x\} + S) \cup \{0\}$. Then $\mathrm{F}(S(x)) = \mathrm{F}(S) + x$ and $\mathrm{g}(S(x)) = \mathrm{g}(S) + x - 1$.*

Theorem 38 and Proposition 39 allow us to build PMED-semigroups with a fixed Frobenius number and fixed genus, starting from a perfect numerical semigroup as shown in the following example.

Example 40 *Let $S = \langle 5, 8, 9 \rangle = \{0, 5, 8, 9, 10, 13, \rightarrow\}$. It is clear that S is a perfect numerical semigroup, $\mathrm{F}(S) = 12$, and $\mathrm{g}(S) = 8$. As $9 \in S$, then applying Theorem 38 and Proposition 39, we have that $S(9) = (\{9\} + S) \cup \{0\} = \{0, 9, 14, 17, 18, 19, 22, \rightarrow\}$ is a PMED-semigroup with Frobenius number $12 + 9 = 21$ and genus $8 + 9 - 1 = 16$.*

Inspired by [1], Lipman introduced and encouraged in [5] the study of Arf rings. The characterization given in this article of Arf rings via their value semigroups leads to the notion of Arf numerical semigroup. We say that a numerical semigroup is Arf if it verifies the following condition: if $\{x, y, z\} \subseteq S$ and $x \geq y \geq z$ then $x + y - z \in S$. It is well known (see, for example, [10]) that every Arf numerical semigroup is a MED-semigroup. We say that a sequence of integer numbers $(x_1, \ldots, x_n)$ is an Arf sequence if it verifies the following conditions:

1. $x_n \geq \cdots \geq x_1 \geq 2$,

2. $x_{i+1} \in \{x_i, x_i + x_{i-1}, \ldots, x_i + \cdots + x_1, \rightarrow\}$ for all $i \in \{1, \ldots, n-1\}$.

The following result is [4, Proposition 1].

Proposition 41 *Let S be a proper and nonempty subset of $\mathbb{N}$. Then S is an Arf numerical semigroup if and only if there exists an Arf sequence $(x_1, \ldots, x_n)$ such that $S = \{0, x_n, x_n + x_{n-1}, \ldots, x_n + x_{n-1} + \cdots + x_1, \rightarrow\}$.*

A PARF-semigroup is an Arf numerical semigroup that is also a perfect numerical semigroup. An immediate consequence of Proposition 41 is the following result.

Corollary 42 *Let $(x_1, \ldots, x_n)$ be an Arf sequence such that $x_1 \geq 3$. Then $S = \{0, x_n, x_n + x_{n-1}, \ldots, x_n + x_{n-1} + \cdots + x_1, \rightarrow\}$ is a* PARF*-semigroup. Moreover, every* PARF*-semigroup has this form.*

In [4] a certain algorithm appears to calculate all Arf sequences whose associated numerical semigroup has a fixed Frobenius number or genus. As for the algorithm, if we require it to hold that $x_1 \geq 3$, then we will obtain another algorithm that allows us to compute all PARF-semigroup with a fixed Frobenius number or genus.

The so-called saturated numerical semigroups are an especially interesting kind of Arf numerical semigroups (see [10]). The concept of the saturated ring was introduced in three different ways by Zariski [12], Pham-Teissier [6], and Campillo [3]. As for the Arf property, the characterization of saturated rings in terms of their value semigroups leads to the concept of the saturated numerical semigroup.

A numerical semigroup S is saturated if the following condition holds: if $\{s, s_1, \ldots, s_r\} \subseteq S$ where $s_i \leq s$ for all $i \in \{1, \ldots, r\}$, $\{z_1, \ldots, z_r\} \subseteq \mathbb{Z}$ and $z_1 s_1 + \cdots + z_r s_r \geq 0$, then $s + z_1 s_1 + \cdots + z_r s_r \in S$.

We say that a sequence of positive integers $(n_1, n_2, \ldots, n_p)$ is saturated if it verifies the following conditions:

1. $n_1 < n_2 \cdots < n_p$,
2. If $d_i = \gcd\{n_1, \ldots, n_i\}$ for all $i \in \{1, \ldots, p\}$, then $d_1 > d_2 > \cdots > d_p = 1$.

As a consequence of the results of [11], we have the following proposition.

Proposition 43 *Let $(n_1, \ldots, n_p)$ be a saturated sequence. For every $j \in \{1, \ldots, p-1\}$, let $k_j = \max\{k \in \mathbb{N} \mid n_j + kd_j < n_{j+1}\}$. Then $S(n_1, \ldots, n_p) = \{0, n_1, n_1 + d_1, \ldots, n_1 + k_1 d_1, n_2, n_2 + d_2, \ldots, n_2 + k_2 d_2, \ldots, n_{p-1}, n_{p-1} + d_{p-1}, \ldots, n_{p-1} + k_{p-1} d_{p-1}, n_p, \rightarrow\}$ is a saturated numerical semigroup. Moreover, every saturated numerical semigroup has this form.*

A PSAT-semigroup is a saturated and perfect numerical semigroup.

Given two integers a and b with $b \neq 0$, we denote by $a \bmod b$ the remainder of the division of a by b. The following result has an easy proof.

Corollary 44 *Let $(n_1, \ldots, n_p)$ be a saturated sequence. Then $S(n_1, \ldots, n_p)$ is a* PSAT*-semigroup if and only if $d_{p-1} \geq 3$ and $n_p \bmod d_{p-1} \neq 2$.*

We illustrate the above result with an example.

Example 45 *It is clear that* $(27,36,48,55)$ *is a saturated sequence,* $d_1 = 27$*,* $d_2 = 9$*,* $d_3 = 3$*, and* $d_4 = 1$*. By Proposition 43, we know that* $S = \{0,27,36,45,48,51,54,55,\rightarrow\}$ *is a saturated numerical semigroup. Moreover, as* $d_3 = 3 \geq 3$ *and* $55 \, mod 3 = 1 \neq 2$*, then applying Corollary 44, we have that* S *is a PSAT-semigroup.*

In [8] and [9] some algorithms appear to calculate all saturated numerical semigroups with a fixed Frobenius number or genus, respectively. From these algorithms and Corollary 44, it is easy to obtain algorithms that allow us to build all PSAT-semigroups with a fixed Frobenius number or genus.

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