

Nonnegative integer solutions of the equation $F_n - F_m = 5^a$

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Abstract: In this study, we solve the Diophantine equation in the title in nonnegative integers m, n , and a . The solutions are given by

$$F_1 - F_0 = F_2 - F_0 = F_3 - F_2 = F_3 - F_1 = F_4 - F_3 = 5^0$$

and

$$F_5 - F_0 = F_6 - F_4 = F_7 - F_6 = 5.$$

Then we give a conjecture that says that if $a \geq 2$ and $p > 7$ is prime, then the equation $F_n - F_m = p^a$ has no solutions in nonnegative integers m, n .

Key words: Diophantine equation, Fibonacci numbers, linear forms in logarithms

1. Introduction

The Fibonacci sequence (F_n) and Lucas sequence (L_n) are defined as $F_0 = 0$, $F_1 = 1$, $L_0 = 2$, $L_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. The terms of the Fibonacci and Lucas sequences are called Fibonacci and Lucas numbers, respectively. The Fibonacci and Lucas numbers for negative indices are defined by $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$ for $n \geq 1$. For a brief history of Fibonacci and Lucas sequences, one can consult [8]. The Fibonacci and Lucas sequences have many interesting properties and they are the most studied among the second-order recurrence sequences. In the last decade, some exponential Diophantine equations containing the terms of second-order linear recursive sequences were studied by mathematicians. In 2014, the Diophantine equation $L_n + L_m = 2^a$ was tackled in [4] by Bravo and Luca. Two years later, the same authors solved the Diophantine equation $F_n + F_m = 2^a$ in [5]. Meanwhile, the equation $F_n + F_m + F_l = 2^a$ was solved by Bravo and Bravo in [3]. Lastly, in [12], the authors dealt with the Diophantine equation $U_n + U_m = wp_1^{z_1} p_2^{z_2} \cdots p_s^{z_s}$ and they solved this equation in the case that $w = 1$, p_1, p_2, \dots, p_{46} are all prime numbers less than 200 and (U_n) is the Fibonacci sequence or the Lucas sequence. Similar equations were tackled in [2] and [7]. In [2], the equations $U_n = 2^a + 3^b$ and $U_n = 2^a + 3^b + 5^c$ were solved when (U_n) is one of the sequences (F_n) , (L_n) , (P_n) , or (Q_n) , where (P_n) and (Q_n) are the Pell and Pell–Lucas sequences, respectively. In [7], it was shown that if $F_n + F_m + F_r = 2^a + 2^b$, then $\max\{n, m, r, a, b\} \leq 16$. It was also shown that if $F_n + F_m = 2^a + 2^b + 2^c$, then $\max\{n, m, a, b, c\} \leq 18$. Luca and Patel, in [10], found that the Diophantine equation $F_n - F_m = y^p$ in integers (n, m, y, p) with $p \geq 2$ has solution either $\max\{|n|, |m|\} \leq 36$ or $y = 0$ and $|n| = |m|$ if $n \equiv m \pmod{2}$. However, it is still an open problem for the case $n \not\equiv m \pmod{2}$. In [13], the authors solved the equation

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$F_n - F_m = 2^a$. They proved that all nonnegative integer solutions of the Diophantine equation $F_n - F_m = 2^a$ are given by

$$(n, m, a) \in \{(1, 0, 0), (2, 0, 0), (3, 0, 1), (6, 0, 3), (3, 1, 0), (4, 1, 1), (5, 1, 2), (3, 2, 0)\}$$

and

$$(n, m, a) \in \{(4, 3, 0), (4, 2, 1), (5, 2, 2), (9, 3, 5), (5, 4, 1), (7, 5, 3), (8, 5, 4), (8, 7, 3)\}.$$

Furthermore, in an unpublished work, the authors proved that all solutions of the equation

$$F_n - F_m = 3^a \tag{1}$$

are given by

$$(n, m, a) \in \{(1, 0, 0), (2, 0, 0), (4, 0, 1), (3, 1, 0), (3, 2, 0), (4, 3, 0), (5, 3, 1), (6, 5, 1), (11, 6, 4)\}.$$

Motivated by the above studies, in this study, we consider the Diophantine equation

$$F_n - F_m = 5^a \tag{2}$$

in nonnegative integers m, n , and a . Our work is a continuation of the previous studies on this subject. We prove our main result following the approach and the method presented in [5]. In Section 2, we introduce necessary lemmas and theorems. Then, in Section 3, we prove our main theorem.

2. Auxiliary results

In order to solve Diophantine equations of the form (2), we use Baker’s theory of lower bounds for a nonzero linear form in logarithms of algebraic numbers. Since such bounds are very important in effectively solving Diophantine equations, we start by recalling some basic notions from algebraic number theory.

Let η be an algebraic number of degree d with the minimal polynomial

$$a_0x^d + a_1x^{d-1} + \dots + a_d = a_0 \prod_{i=1}^d (x - \eta^{(i)}) \in \mathbb{Z}[x],$$

where the a_i ’s are relatively prime integers with $a_0 > 0$ and the $\eta^{(i)}$ ’s are conjugates of η . Then

$$h(\eta) = \frac{1}{d} \left(\log a_0 + \sum_{i=1}^d \log \left(\max \{ |\eta^{(i)}|, 1 \} \right) \right) \tag{3}$$

is called the logarithmic height of η . If $\eta = a/b$ is a rational number with $\gcd(a, b) = 1$ and $b > 1$, then $h(\eta) = \log(\max\{|a|, b\})$.

The following properties of logarithmic height are found in many works stated in the references:

$$h(\eta \pm \gamma) \leq h(\eta) + h(\gamma) + \log 2, \tag{4}$$

$$h(\eta\gamma^{\pm 1}) \leq h(\eta) + h(\gamma), \tag{5}$$

$$h(\eta^k) = |k|h(\eta). \tag{6}$$

The following theorem can be deduced from Corollary 2.3 of Matveev [11], which provides a large upper bound for the subscript n in equation (2) (also see Theorem 9.4 in [6]).

Theorem 1 *Assume that $\gamma_1, \gamma_2, \dots, \gamma_t$ are positive real algebraic numbers in a real algebraic number field \mathbb{K} of degree D ; b_1, b_2, \dots, b_t are rational integers; and*

$$\Lambda := \gamma_1^{b_1} \dots \gamma_t^{b_t} - 1$$

is not zero. Then

$$|\Lambda| > \exp(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1 + \log D)(1 + \log B)A_1 A_2 \dots A_t),$$

where

$$B \geq \max\{|b_1|, \dots, |b_t|\},$$

and

$$A_i \geq \max\{Dh(\gamma_i), |\log \gamma_i|, 0.16\}$$

for all $i = 1, \dots, t$.

The following lemma was proved by Dujella and Pethő [9], which is a variation of a lemma of Baker and Davenport [1]. This lemma will be used to reduce the upper bound for the subscript n in equation (2). For a real number x , $\|x\|$ denotes the distance from x to the nearest integer. That is, $\|x\| = \min\{|x - n| : n \in \mathbb{Z}\}$.

Lemma 2 *Let M be a positive integer, let p/q be a convergent of the continued fraction of the irrational number γ such that $q > 6M$, and let A, B, μ be some real numbers with $A > 0$ and $B > 1$. Let $\epsilon := \|\mu q\| - M\|\gamma q\|$. If $\epsilon > 0$, then there exists no solution to the inequality*

$$0 < |u\gamma - v + \mu| < AB^{-w},$$

in positive integers u, v , and w with

$$u \leq M \text{ and } w \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

It is well known that

$$F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}} \text{ and } L_n = \alpha^n + \beta^n, \tag{7}$$

where $\alpha = (1 + \sqrt{5})/2$ is the golden section and $\beta = \bar{\alpha}$, which are the roots of the characteristic equation $x^2 - x - 1 = 0$. The relation between the Fibonacci number and Lucas number is given by

$$F_{n+1} + F_{n-1} = L_n, \tag{8}$$

and an induction method shows that

$$\alpha^{n-2} \leq F_n \leq \alpha^{n-1} \tag{9}$$

for $n \geq 1$. It is clear that $1 < \alpha < 2$ and $-1 < \beta < 0$.

The following theorem and lemma are given in [6] and [10], respectively.

Theorem 3 *The only perfect powers in the Fibonacci sequence are $F_0 = 0$, $F_1 = F_2 = 1$, $F_6 = 8$, and $F_{12} = 144$. The only perfect powers in the Lucas sequence are $L_1 = 1$ and $L_3 = 4$.*

Lemma 4 *Assume that $n \equiv m \pmod{2}$. Then*

$$F_n - F_m = \begin{cases} F_{(n-m)/2}L_{(n+m)/2} & \text{if } n \equiv m \pmod{4}, \\ F_{(n+m)/2}L_{(n-m)/2} & \text{if } n \equiv m + 2 \pmod{4}. \end{cases}$$

3. Main theorem

Theorem 5 *The only solutions of Diophantine equation (2) in nonnegative integers $m < n$, and a , are given by*

$$(n, m, a) \in \{(1, 0, 0), (2, 0, 0), (5, 0, 1), (3, 1, 0), (3, 2, 0), (4, 3, 0), (6, 4, 1), (7, 6, 1)\},$$

namely,

$$F_1 - F_0 = F_2 - F_0 = F_3 - F_2 = F_3 - F_1 = F_4 - F_3 = 5^0$$

and

$$F_5 - F_0 = F_6 - F_4 = F_7 - F_6 = 5.$$

Proof Assume that equation (2) holds. Let $n - m = 1$. Then we get $F_{m-1} = 5^a$. By Theorem 3, we have $(n, m, a) \in (1, 0, 0), (3, 2, 0), (4, 3, 0), (7, 6, 1)$. Similarly, in the case $n - m = 2$, we have $(n, m, a) \in \{(2, 0, 0), (3, 1, 0), (6, 4, 1)\}$. If $m = 0$, from Theorem 3, we get the solutions $F_1 - F_0 = F_2 - F_0 = 1$ and $F_5 - F_0 = 5$. With the help of the Mathematica program, we obtain the other solutions in Theorem 5 for $1 \leq m < n \leq 200$. This takes a little time. From now on, assume that $n > 200, m \geq 1$, and $n - m \geq 3$. Using the identity (9), we get the inequality

$$5^a = F_n - F_m < F_n < \alpha^n < 5^n,$$

which shows that $a < n$.

Rearranging equation (2) as $\frac{\alpha^n}{\sqrt{5}} - 5^a = F_m + \frac{\beta^n}{\sqrt{5}}$ and taking absolute values, we obtain

$$\left| \frac{\alpha^n}{\sqrt{5}} - 5^a \right| = \left| F_m + \frac{\beta^n}{\sqrt{5}} \right| \leq F_m + \frac{|\beta|^n}{\sqrt{5}} < \alpha^m + \frac{1}{2}$$

by identity (9). If we divide both sides of the above inequality by $\frac{\alpha^n}{\sqrt{5}}$, we get

$$\left| 1 - 5^a \alpha^{-n} \sqrt{5} \right| < \sqrt{5} \alpha^{m-n} + \frac{\sqrt{5}}{2} \alpha^{-n} = \sqrt{5} \alpha^{m-n} \left(1 + \frac{\alpha^{-m}}{2} \right) < \frac{4}{\alpha^{n-m}}. \tag{10}$$

Now we apply Theorem 1 with $\gamma_1 := 5, \gamma_2 := \alpha, \gamma_3 := \sqrt{5}$ and $b_1 := a, b_2 := -n, b_3 := 1$. Note that the numbers γ_1, γ_2 , and γ_3 are positive real numbers and elements of the field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, so $D = 2$. We show that $\Lambda_1 := 5^a \alpha^{-n} \sqrt{5} - 1$ is nonzero. Assume that $\Lambda_1 = 0$. Then we get $\alpha^{2n} = 5^{2a+1}$, which is impossible since α^{2n} is irrational. Moreover, since $h(\gamma_1) = \log 5 = 1.60943\dots$, $h(\gamma_2) = \frac{\log \alpha}{2} = \frac{0.4812\dots}{2}$, and $h(\gamma_3) = \log \sqrt{5} = 0.8047\dots$ by identity (3), we can take $A_1 := 3.22, A_2 := 0.5$, and $A_3 := 1.7$. Since $a < n$, it

follows that $B := \max\{|b_1|, |b_2|, |b_3|\} = \max\{|a|, |-n|, 1\} = n$. Thus, taking into account inequality (10) and using Theorem 1, we obtain

$$\frac{4}{\alpha^{n-m}} > |\Lambda_1| > \exp(-1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2(1 + \log 2)(1 + \log n)) \quad (3.22) \quad (0.5) \quad (1.7).$$

From the last inequality, a quick computation with Mathematica gives us the inequality

$$(n - m) \log \alpha - \log 4 < (1 + \log n) \cdot 2.65419 \cdot 10^{12}. \tag{11}$$

Now we apply Theorem 1 again. Rearranging equation (2) as

$$\frac{\alpha^n}{\sqrt{5}} - \frac{\alpha^m}{\sqrt{5}} - 5^a = \frac{\beta^n}{\sqrt{5}} - \frac{\beta^m}{\sqrt{5}}$$

and taking absolute values, we obtain

$$\left| \frac{\alpha^n(1 - \alpha^{m-n})}{\sqrt{5}} - 5^a \right| = \left| \frac{\beta^n}{\sqrt{5}} - \frac{\beta^m}{\sqrt{5}} \right| \leq \frac{|\beta|^n + |\beta|^m}{\sqrt{5}} < \frac{1}{3},$$

where we used the fact that $|\beta|^n + |\beta|^m < 2/3$ for $n > 200$. Dividing both sides of the above inequality by $\frac{\alpha^n(1 - \alpha^{m-n})}{\sqrt{5}}$, we get

$$\left| 1 - 5^a \alpha^{-n} \sqrt{5} (1 - \alpha^{m-n})^{-1} \right| < \frac{\sqrt{5} \alpha^{-n} (1 - \alpha^{m-n})^{-1}}{3}. \tag{12}$$

Since

$$\alpha^{m-n} = \frac{1}{\alpha^{n-m}} < \frac{1}{\alpha} < \frac{2}{3},$$

it is seen that

$$1 - \alpha^{m-n} > \frac{1}{3}$$

and therefore

$$\frac{1}{1 - \alpha^{m-n}} < 3.$$

Then from (12), it follows that

$$\left| 1 - 5^a \alpha^{-n} \sqrt{5} (1 - \alpha^{m-n})^{-1} \right| < \frac{3}{\alpha^n}. \tag{13}$$

Thus, taking $\gamma_1 := 5, \gamma_2 := \alpha, \gamma_3 := \sqrt{5}(1 - \alpha^{m-n})^{-1}$ and $b_1 := a, b_2 := -n, b_3 := 1$, we can apply Theorem 1. The numbers γ_1, γ_2 , and γ_3 are positive real numbers and elements of the field $\mathbb{K} = \mathbb{Q}(\sqrt{5})$, so $D = 2$. We claim that $\Lambda_2 := 5^a \alpha^{-n} \sqrt{5} (1 - \alpha^{m-n})^{-1} - 1$ is nonzero. Because if $\Lambda_2 = 0$, then we get

$$\frac{\alpha^n}{\sqrt{5}} - \frac{\alpha^m}{\sqrt{5}} = 5^a = F_n - F_m = \frac{\alpha^n}{\sqrt{5}} - \frac{\alpha^m}{\sqrt{5}} + \frac{\beta^m}{\sqrt{5}} - \frac{\beta^n}{\sqrt{5}},$$

which implies that $\beta^m = \beta^n$. However, this is impossible since $n > m$. Since $h(\gamma_1) = \log 5 = 1.60943\dots$, and $h(\gamma_2) = \frac{\log \alpha}{2} = \frac{0.4812\dots}{2}$ by (3), we can take $A_1 := 3.22$ and $A_2 := 0.5$. On the other hand, using (4), (5), and

(6), we get $h(\gamma_3) \leq \log 2\sqrt{5} + (n - m)\frac{\log \alpha}{2}$. A simple computation shows that $|\log \gamma_3| < \log 5 + (n - m)\log \alpha$, so we can take $A_3 := \log 20 + (n - m)\log \alpha$. Also, since $a < n$, it follows that $B := \max\{|a|, | -n|, 1\} = n$. Thus, taking into account inequality (13) and using Theorem 1, we obtain

$$\frac{3}{\alpha^n} > |\Lambda_2| > \exp(-C)(1 + \log n)(\log 20 + (n - m)\log \alpha),$$

or

$$n \log \alpha - \log 3 < C(1 + \log n)(\log 20 + (n - m)\log \alpha), \tag{14}$$

where $C = 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 2^2 \cdot (1 + \log 2)(3.22)(0.5)$. Substituting inequality (11) into the last inequality, we get

$$n \log \alpha - \log 3 < C(1 + \log n)(\log 20 + 2.65419 \cdot 10^{12}(1 + \log n) + \log 4). \tag{15}$$

With the help of Mathematica, it is seen that $n < 3.85 \cdot 10^{28}$.

Let

$$z_1 := a \log 5 - n \log \alpha + \log \sqrt{5}.$$

Then

$$|1 - e^{z_1}| < \frac{4}{\alpha^{n-m}}$$

by (10). The inequality

$$\frac{\alpha^n}{\sqrt{5}} = F_n + \frac{\beta^n}{\sqrt{5}} > F_n - 1 \geq F_n - F_m = 5^a$$

implies that $5^a \sqrt{5} \alpha^{-n} < 1$. Therefore, we get $z_1 < 0$. Since $\frac{4}{\alpha^{n-m}} < 0.95$ for $n - m \geq 3$, it follows that $e^{|z_1|} < 20$. Hence, since $x < e^x - 1$ for $x > 0$, we get

$$0 < |z_1| < e^{|z_1|} - 1 = e^{|z_1|} |1 - e^{z_1}| < \frac{80}{\alpha^{n-m}},$$

or

$$0 < |a \log 5 - n \log \alpha + \log \sqrt{5}| < \frac{80}{\alpha^{n-m}}.$$

Dividing this inequality by $\log \alpha$, we get

$$0 < \left| a \left(\frac{\log 5}{\log \alpha} \right) - n + \frac{\log \sqrt{5}}{\log \alpha} \right| < \frac{80}{\log \alpha \cdot \alpha^{n-m}}. \tag{16}$$

From (16), it follows that

$$0 < \left| a \left(\frac{\log 5}{\log \alpha} \right) - n + \frac{\log \sqrt{5}}{\log \alpha} \right| < 166.3 \cdot \alpha^{-(n-m)}. \tag{17}$$

Now we can apply Lemma 2. Put

$$\gamma := \frac{\log 5}{\log \alpha} \notin \mathbb{Q}, \mu := \frac{\log \sqrt{5}}{\log \alpha}, A := 166.3, B := \alpha, \text{ and } w := n - m.$$

Let $M := 3.85 \cdot 10^{28}$. Then the denominator of the 60th convergent of γ exceeds $6M$. Furthermore,

$$\epsilon := \|\mu q_{60}\| - M\|\gamma q_{60}\| \leq 0.496.$$

Thus, inequality (17) has no solutions for

$$n - m \geq \frac{\log(Aq_{60}/\epsilon)}{\log B} \geq 158.15.$$

A computer search with Mathematica yields $n - m \geq 158.15$, so $n - m \leq 158$. Substituting this upper bound for $n - m$ into (15), we obtain $n < 9.7 \cdot 10^{15}$.

Now we apply Lemma 2 to reduce the upper bound on n a little bit. Let

$$z_2 := a \log 5 - n \log \alpha + \log \left(\sqrt{5}(1 - \alpha^{m-n})^{-1} \right).$$

In this case,

$$|1 - e^{z_2}| < \frac{3}{\alpha^n}$$

by (13). If $z_2 > 0$, then $0 < z_2 < e^{z_2} - 1 < \frac{3}{\alpha^n} < \frac{1}{2}$. If $z_2 < 0$, then $|1 - e^{z_2}| = 1 - e^{z_2} < \frac{3}{\alpha^n} < \frac{1}{2}$. From this, we get $e^{|z_2|} < 2$ and so

$$0 < |z_2| < e^{|z_2|} - 1 = e^{|z_2|} |1 - e^{z_2}| < \frac{6}{\alpha^n}.$$

Therefore, it is true that

$$0 < |z_2| < \frac{6}{\alpha^n}.$$

That is,

$$0 < \left| a \log 5 - n \log \alpha + \log \left(\sqrt{5}(1 - \alpha^{m-n})^{-1} \right) \right| < \frac{6}{\alpha^n}.$$

Dividing both sides of the above inequality by $\log \alpha$, we get

$$0 < \left| a \left(\frac{\log 5}{\log \alpha} \right) - n + \frac{\log \left(\sqrt{5}(1 - \alpha^{m-n})^{-1} \right)}{\log \alpha} \right| < 13 \cdot \alpha^{-n}. \tag{18}$$

Let $\gamma := \frac{\log 5}{\log \alpha}$ and $M := 9.7 \cdot 10^{15}$. Then the denominator of the 32nd convergent of γ exceeds $6M$. Also, taking

$$\mu := \frac{\log \left(\sqrt{5}(1 - \alpha^{m-n})^{-1} \right)}{\log \alpha}$$

with $n - m \in [3, 158]$ and $n - m \neq 4, 56, 66$, a quick computation with Mathematica gives us the inequality

$$\epsilon := \epsilon(\mu) = \|\mu q_{32}\| - M\|\gamma q_{32}\| \leq 0.493272.$$

Let $A := 13, B := \alpha$, and $w := n$. Using Lemma 2, we can say that inequality (18) has no solution for

$$w = n \geq \frac{\log(Aq_{32}/\epsilon)}{\log B} \geq 91.3088$$

with $n - m \neq 4, 56, 66$. That is, there are no solutions for $n \geq 91.3088$ and therefore we get $n \leq 91$. This contradicts our assumption that $n > 200$. Now consider the cases $n - m = 4, 56, 66$ to complete the proof. If $n - m = 4$, then we have the equation $5^a = F_{m+4} - F_m = L_{m+2}$, which is impossible. If $n - m = 56$, then we have the equation $5^a = F_n - F_m = F_{28}L_{m+28}$ by Lemma 4. This is impossible. Lastly, assume that $n - m = 66$. Then we have the equation $5^a = F_n - F_m = F_{m+33}L_{33}$ by Lemma 4, which is impossible. This completes the proof. \square

3.1. Concluding remarks

In [10], it was shown that if $n \equiv m \pmod{2}$, then all solutions of the equation

$$F_n - F_m = y^p, p \geq 2, y \geq 1 \tag{19}$$

satisfy $\max\{n, m\} \leq 36$. After that, the authors conjectured that all solutions of equation (19) are

$$\begin{aligned} F_1 - F_0 &= 1, F_2 - F_0 = 1, F_3 - F_1 = 1, F_3 - F_2 = 1, F_4 - F_3 = 1, F_5 - F_1 = 2^2, \\ F_5 - F_2 &= 2^2, F_6 - F_4 = 5, F_7 - F_5 = 2^3, F_7 - F_6 = 5, \\ F_8 - F_5 &= 2^4, F_8 - F_7 = 2^3, F_9 - F_3 = 2^5, F_{11} - F_6 = 9^2, F_{13} - F_6 = 15^2, \\ F_{13} - F_{11} &= 12^2, F_{14} - F_9 = 7^3, F_{14} - F_{13} = 12^2, F_{15} - F_9 = 24^2. \end{aligned}$$

Consequently, it is true that the above conjecture holds for $y = 2, 3, 5$ by our result and the results in [13] and unpublished work of the second author. It is reasonable to conjecture that:

Conjecture 6 *If $a \geq 2$ and $p > 7$ is prime, then the equation $F_n - F_m = p^a$ has no solutions in nonnegative integers m, n .*

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