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Some Sufficient conditions for a group to be abelian

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Abstract: A group is said to satisfy a word w in the symbols $\{x, x^{-1}, y, y^{-1}\}$ provided that if the 'x' and 'y' are replaced by arbitrary elements of the group then the equation $w = 1$ is satisfied. This paper studies certain equations in words, as above, which together with other conditions imply that groups which satisfy these equations and conditions must be abelian.

Key words: Group laws, commutators, abelian groups

1. Introduction

A word w in the symbols of $X = \{x_1, x_2, \cdots\}$ is an element of the free group $\mathrm{Fr}(X)$. We say that a group G satisfies the word w provided whenever the elements $x_1, x_2, \cdots$ are replaced by arbitrary elements of G, we obtain the equation $w = 1$ in G. In that case, we say that w is a law for G. In this paper we often consider words in the symbols x, y, that is elements of $\mathrm{Fr}(\{x, y\})$, and sometimes we use x, y for elements of a group G. It should be clear from the context which is implied.

A variety is an "equationally defined" class of groups. If W is a set of words in the set $\{x_1, x_2, \cdots\}$ we denote the variety determined by W as $V(W) = \{G \mid G$ is a group and all the elements of G satisfy all the words of $W\}$. For example, if $W = \{[x, y] := x^{-1}y^{-1}xy\}$, then $V(W)$ is the class of all abelian groups. This paper is concerned with words and conditions that force a group to be abelian. Similar results are contained in [1, 2] and [4].

2. Some notation and some identities

We will use the following notations.

1. For all $x, y \in G, [x, y] := x^{-1}y^{-1}xy$
2. For all $x_1, x_2, \cdots x_n \in G, n \geq 3, [x_1, x_2, \cdots x_n] := [[x_1, x_2, \cdots x_{n-1}], x_n]$. This is a left-normed commutator of weight n.
3. For all $x, y \in G, x^y := y^{-1}xy$.
4. If A and B are words we write $A \iff B$ in a group G to mean G satisfies the law $[A, B]$.

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We will use the following identities.

5. For all $x, y, z \in G$ $[xy, z] = [x, z]^y[y, z]$ and $[x, yz] = [x, z][x, y]^z$.
6. For all $x, y \in G, [x, y] = x^{-1}x^y$.

We make frequent use of the following theorem by Higman [2].

Theorem 2.1 *Let G be a 2-generated group which satisfies the law*

$$[x, y] \iff [x, y^{-1}],$$

then G must be metabelian.

and the following lemma by Gupta [1]

Theorem 2.2 *Let G be a group satisfying the law*

$$[x, y] = C_n,$$

where C_n is a left-normed commutator of weight $n, n \geq 2$, with entries from the set $\{x, x^{-1}, y, y^{-1}\}$. Then, if G is solvable or if G is finite, then G is abelian.

3. Main results

We begin with the following lemma.

Lemma 3.1 *Suppose that G satisfies the law*

$$[x, y]^2 = [x, \underbrace{y, y, \cdots, y}_{n}], n \geq 2,$$

then

(i) every 2-generated subgroup of G is metabelian, and

(ii) for all $x, y \in G$, $[x, y]^4 = 1$.

In particular, if G' contains no elements of order 2, then G must be abelian.

Proof Now replacing $'x'$ in the above formula by $'[x, y]'$ gives

$$[x, y, y]^2 = [x, \underbrace{y, y, \cdots y}_{n+1}] = [[x, y]^2, y] = [x, y, y]^{[x,y]}[x, y, y]$$

It follows that $[x, y, y] = [x, y, y]^{[x,y]}$. Thus, we have the following:

$$[x, y, y] \iff [x, y]$$

$$[x, y]^{-1}[x, y]^y \iff [x, y]$$

$$[x,y]^y \iff [x,y] \text{ so}$$

$$[x,y] \iff ([x,y]^{y^{-1}})^{-1} = [x,y^{-1}].$$

Now by Higman's result (Theorem 2.1) all 2-generated subgroups of G must be metabelian.

Now assume that G is metabelian and let $z \in G'$. It follows that

$$[x,z]^2 = [x,z,\cdots,z] = 1$$

and so $[x,z^2] = [x,z][x,z]^z = [x,z]^2 = 1$. Therefore, $z^2 \in Z(G)$ and thus for all $x,y \in G$, we have $[x,y]^2 \in Z(G)$. It follows that

$$1 = [[x,y]^2, y, \cdots, y] = [x,y,\cdots,y]^2 = [x,y]^4,$$

as required. □

Theorem 3.2 *Let G be a group. Then, the following are equivalent:*

(i) G satisfies the law $[x,y]^2 = [x,y,y]$

(ii) $G/Z(G)$ is an elementary abelian 2-group and G' is an elementary abelian 2-group.

Proof By lemma 3.1 every 2-generated subgroup of G is metabelian and for all $x,y \in G$, we have $[x,y]^4 = 1$. From the above law $[x,y]^2 = [x,y]^{-1}[x,y]^y$. It follows that $[x,y]^{-1} = [x,y]^3 = [x,y]^y$ and thus, $[x,y^2] = [x,y][x,y]^y = [x,y][x,y]^{-1} = 1$. Thus, for all $y \in G$, we have $y^2 \in Z(G)$. Hence, $G/Z(G)$ is an elementary abelian 2-group. Thus, G has nilpotence class ≤ 2. Thus, $[x,y]^2 = [x,y,y] = 1$. Hence, $G' \subseteq Z(G)$ and hence G' has exponent 2.

The other direction is clear. □

The next result is similar to a result of Gupta [1]. It also follows from Higman's result (Theorem 2.1).

Theorem 3.3 *Let G be a group which satisfies*

$$[x,y] = [x, \underbrace{y^2, y^2, \cdots, y^2}_{n}] \; n \geq 2.$$

Then, G must be abelian.

Proof To make the notation easier to follow we define the commutator $[x,y;n] := [x, \underbrace{y, y, \cdots, y}_{n}]$ for $n \geq 1$. Using this notation our condition becomes $[x,y] = [x,y^2;n]$.Replacing $'x'$ by $'[x,y^2]'$ gives

$$[x,y^2,y] = [x,y^2;n+1].$$

It follows that

$$[x,y^2]^{-1}[x,y^2]^y = [[x,y^2;n],y^2] = [x,y,y^2].$$

Hence,

$$([x,y][x,y]^y)^{-1}[x,y^2]^y = [x,y]^{-1}[x,y]^{y^2}.$$

And thus,

$$([x,y]^y)^{-1}[x,y]^{-1}([x,y][x,y]^y)^y = [x,y]^{-1}[x,y]^{y^2}.$$

Hence,

$$([x,y]^y)^{-1}[x,y]^{-1}[x,y]^y[x,y]^{y^2} = [x,y]^{-1}[x,y]^{y^2}.$$

And thus,

$$([x,y]^y)^{-1}[x,y]^{-1}[x,y]^y = [x,y]^{-1}.$$

Thus, we see that

$$[x,y] \iff [x,y]^y$$

and hence

$$[x,y^{-1}] = ([x,y]^{y^{-1}})^{-1} \iff [x,y].$$

Again by Higman's result (Theorem 2.1) we must have $\langle x,y\rangle$ metabelian. Now let $y \in G'$. Then, $[x,y]=1$, as $[x,y^2,y^2]=1$. It follows that $\langle x,y\rangle$ is nilpotent of class ≤ 2. Hence, for all $x,y \in G$ we get that $[x,y]=1$ and G is abelian as required. □

A similar proof gives the following result.

Theorem 3.4 *Suppose that G is a finite group and that $\phi_1(x,y),\phi_2(x,y),\cdots,\phi_n(x,y)$ are words in $\{x,y,x^{-1},y^{-1}\}, n>2$ so that for all $a \in G, b \in G'$ we have $\phi_3(a,b) \in G'$. Then, if G satisfies the law*

$$[x,y] = [\phi_1(x,y),\phi_2(x,y),\cdots,\phi_n(x,y)],$$

then G is abelian.

Proof First, suppose that G is metabelian and $y \in G'$. Now for all $x \in G$, since $y \in G'$, arguing as in the above proof we get $[x,y]=1$ and thus, G is nilpotent of class ≤ 2. It follows that for all $x,y \in G$, we have $[x,y]=1$. Hence, in this case G must be abelian.

Next assume that G is solvable and satisfies the above law. It follows that $\frac{G}{G''}$ is metabelian and satisfies the given law. Hence, $\frac{G}{G''}$ must be abelian. Thus, $G' = G''$. As G is solvable, then $G'=1$ and G is abelian.

Now assume that G is a minimal counter-example to the theorem. It follows that all proper subgroups of G are abelian. Thus, G must be solvable and hence abelian by the above remarks. □

Note that if in the above result we had assumed that G is solvable, we could see that G is abelian without the assumption that G is finite.

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