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On positive periodic solutions of second-order semipositone differential equations

Fanglei WANG*, **Nannan YANG**
Depatment of Mathematics, College of Science, Hohai University, Nanjing, China

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Abstract: Using the Krasnosel'skii's fixed point theorem, we establish the existence and multiplicity of positive T-periodic solutions of second-order semipositone system

$$\begin{cases} x''(t) + a(t)x(t) = \lambda f(t, x(t)), \\ x(0) = x(\mathrm{T}), x'(0) = x'(\mathrm{T}), \end{cases}$$

where $x = (x_1, x_2, \cdots, x_n)$, $f(t, x) = (f_1(t, x), f_2(t, x), \cdots, f_n(t, x))$ is bounded below.

Key words: Non-autonomous system ; Periodic solutions ; Fixed point theorem; Cone

1. Introduction

In this paper, we are concerned with the existence and multiplicity of positive T-periodic solutions of the second-order semipositone differential equation

$$\begin{cases} x''(t) + a(t)x(t) = \lambda f(t, x(t)), \\ x(0) = x(\mathrm{T}), x'(0) = x'(\mathrm{T}), \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $a(t) = (a_1(t), \cdots, a_n(t))$, $x(t) = (x_1(t), \cdots, x_n(t))$, $f(t, x) = (f_1(t, x), f_2(t, x), \cdots, f_n(t, x))$. We say that the differential equation

$$x'' + a(t)x = \lambda f(t, x(t))$$

is semipositone if a vector-valued function f may be negative and bounded from below, which means that f satisfies the following condition:

(F_0) $f : [0, \mathrm{T}] \times \mathbb{R}^n_+ \to \mathbb{R}^n$ is continuous, periodic in t with period T, and

$$f_i(t, x) \geq -e_i(t), \ for \ (t, x) \in [0, \mathrm{T}] \times \mathbb{R}^n_+,$$

where $e_i : [0, \mathrm{T}] \to \mathbb{R}_+$ is continuous and $e_i(t) \not\equiv 0$ on $[0, \mathrm{T}]$.

*Correspondence: wang-fanglei@hotmail.com

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To the best of our knowledge, existence and multiplicity of nontrivial solutions of (1.1) have been widely studied using the variational method [2, 15, 26], the method of upper and lower solutions [3, 4], fixed point theorems [9, 10, 12, 16–18], alternative principle of Leray–Schauder [5] or topological degree theory [23]. In general, in order to ensure the positivity of the solutions of the boundary value problems, one of the crucial assumptions is that the nonlinearity f is nonnegative. We refer the readers to [7, 8, 19–21] and the references. For the BVP (1.1), there is little literature that has referred to the existence of positive solutions when the nonlinearity can take a negative value. In [22], Wang used a well-known fixed point theorem in a cone to establish the existence of T-periodic solution of a class of nonautonomous second-order systems

$$\begin{cases} x''(t) + \mu x(t) + V(t, x(t)) = 0, \\ x(0) = x(\mathrm{T}),\ x'(0) = x'(\mathrm{T}), \end{cases} \tag{1.2}$$

where $x = (x_1, x_2, \cdots, x_n)$, $V(t, x) = (v_1(t, x), v_2(t, x), \cdots, v_n(t, x))$. The main result is

Theorem 1.1 *[22] Assume that* $\lim_{|x|\to\infty} \frac{v_i(t,x)}{|x|} = 0$, $i = 1, \ldots, n$, *uniformly in* $t \in [0, \mathrm{T}]$.

(a) *If* $\mu \in (-\infty, 0)$ *and* $V(t, x)$ *is bounded below, then (1.2) has a solution* $x(t)$.

(b) *If* $\mu \in (0, (\frac{\pi}{\mathrm{T}})^2)$ *and* $V(t, x)$ *is bounded above, then (1.2) has a solution* $x(t)$.

However, the author finds sufficient conditions under which the semipositone BVP (1.2) has a solution, which is not necessarily positive.

Inspired by these references, using a well-known fixed point theorem, the purpose of this paper is to study the existence and multiplicity of positive periodic solutions of the semipositone BVP (1.1) under some suitable assumptions, which guarantee that a positive solution of the semipositone BVP (1.1) here simply requires that all component of the solution is positive. Compared to the results in [19–22], our work presented in this paper has the following new features. Firstly, we find some new conditions,which differ from those in the majority of papers as we know. Secondly, the nonlinear term f may take a negative value. Thirdly, instead of the constant M by any continuous function $e(t)$ on $[0, \mathrm{T}]$, which has been used in [6]. Fourthly, the existence and multiplicity of positive solutions obtained here for suitable $\lambda > 0$.

The paper is organized as follows: in Section 2, we give some preliminaries and an appropriate transformation, which are usually used in semipositone problems such as [1, 6, 13, 14, 24, 25]; in Section 3, we give the main results and the corresponding proof. In addition, some examples are given to illustrate the existence results.

2. Preliminaries

Different from the ordinary boundary value problems, we cannot write the specific Green function of the more general linear equation $x'' + k(t)x = 0$. Therefore, we first consider the following second-order linear differential equation

$$\begin{cases} x''(t) + k(t)x(t) = h(t), \\ x(0) = x(\mathrm{T}),\ x'(0) = x'(\mathrm{T}), \end{cases} \tag{2.1}$$

where $k(t)$, $h(t)$ are continuous, positive and T-periodic. In addition, $k(t)$ satisfies the following assumption

$$0 < k_* = \min_{t\in[0,\mathrm{T}]} k(t) < k^* = \max_{t\in[0,\mathrm{T}]} k(t) < (\frac{\pi}{\mathrm{T}})^2.$$

From [22], the scalar periodic boundary value problems

$$\begin{cases} x''(t) + k^* x(t) = h(t), \\ x(0) = x(\mathrm{T}),\ x'(0) = x'(\mathrm{T}) \end{cases}$$

has a unique positive solution

$$x(t) = \int_0^{\mathrm{T}} G(t,s)h(s)ds,$$

where

$$G(t,s) = \begin{cases} \frac{\sin\sqrt{k^*}(t-s) + \sin\sqrt{k^*}(\mathrm{T}-t+s)}{2\sqrt{k^*}(1-\cos\sqrt{k^*}t)}, \\ \frac{\sin\sqrt{k^*}(s-t) + \sin\sqrt{k^*}(\mathrm{T}-s+t)}{2\sqrt{k^*}(1-\cos\sqrt{k^*}t)}. \end{cases}$$

Furthermore,

$$0 < m = \min_{0\le s,t\le T} G(t,s) < M = \max_{0\le s,t\le T} G(t,s).$$

Secondly, let

$$Ah(t) = \int_0^{\mathrm{T}} G(t,s)h(s)ds,$$

$$Bx = (k^* - k(t))x.$$

It is obvious that $\|A\| = \frac{1}{k^*}$ and $\|B\| \le (k^* - k_*)$. Then, (2.1) can be rewritten as

$$x''(t) + k^* x(t) = [k^* - k(t)]x(t) + h(t).$$

Namely,

$$x(t) = Ah(t) + A \circ Bx(t).$$

Since $\|A \circ B\| \le \frac{1}{k^*}(k^* - k_*) < 1$, (2.1) has a unique positive solution

$$x(t) = (I - A \circ B)^{-1} Ah(t) = Ph(t).$$

Finally, we show that the operator P is completely continuous, positive and it satisfies

$$m|h|_{L^1} \le Ph(t) \le \frac{Mk^*}{k_*}|h|_{L^1}.$$

On the one hand, by the expansion of P

$$\begin{aligned} P &= (I - A \circ B)^{-1} A \\ &= (I + AB + (AB)^2 + \cdots + (AB)^n + \cdots)A \\ &= A + (AB)A + (AB)^2 A + \cdots + (AB)^n A + \cdots, \end{aligned}$$

it is clear that P is completely continuous since A is completely continuous ([22]). On the other hand, since the operators A and B are positive, we have

$$\begin{aligned}
m|h|_{L^1} \leq Ah(t) &\leq Ph(t) \\
&= (I - A\circ B)^{-1}Ah(t) \\
&= (I + AB + (AB)^2 + \cdots + (AB)^n + \cdots)Ah(t) \\
&= [A + (AB)A + (AB)^2A + \cdots + (AB)^nA + \cdots]h(t) \\
&\leq \|(I - A\circ B)^{-1}\| \cdot |Ah(t)|_\infty \\
&\leq \frac{1}{1 - \|A\circ B\|} \cdot |Ah(t)|_\infty \\
&\leq \frac{k^*}{k_*}|Ah|_{L^1} \leq \frac{Mk^*}{k_*}|h|_{L^1}.
\end{aligned}$$

Therefore, throughout this paper, we always assume that the following condition is satisfied:

(H) The function a_i is continuous, positive, T-periodic and the linear equation $x'' + a_i(t)x = 0$ has a positive Green's function $G_i(t,s)$, i.e.

$$G_i(t,s) > 0 \quad \text{for all } (t,s) \in [0,\mathrm{T}] \times [0,\mathrm{T}].$$

We denote

$$m_i = \min_{0\leq s,t\leq T} G_i(t,s), \quad M_i = \max_{0\leq s,t\leq T} G_i(t,s), \quad \delta_i = m_i/M_i.$$

Obviously, $M_i > m_i > 0$ and $0 < \delta_i < 1$ (see [19]).

Lemma 2.1 *[19] Assume that* (H) *and* (F_0) *hold. Then*

$$x'' + a(t)x = e(t). \tag{2.2}$$

has the unique solution $\omega(t) = (\omega_1(t), \cdots, \omega_n(t))$ *and each component of the solution* $\omega(t)$ *can be expressed by*

$$\omega_i(t) = \int_0^T G_i(t,s)e_i(s)ds.$$

Furthermore, $\omega_i(t)$ *satisfies the estimates*

$$m_i\int_0^T e_i(t)dt \leq \omega_i(t) \leq M_i\int_0^T e_i(t)dt.$$

From (1.1) and (2.2), we have

$$\frac{d^2}{dt^2}(x + \lambda\omega) + a(t)(x + \lambda\omega) = \lambda[f(t,x) + e(t)]. \tag{2.3}$$

Let $u = x + \lambda\omega$, then we rewrite (2.3) as

$$u'' + a(t)u = \lambda[f(t, u - \lambda\omega) + e(t)]. \tag{2.4}$$

Let $H(t)$ denote the Heaviside function of a single real variable:

$$H(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases}$$

Lemma 2.2 *Assume that* (H) *and* (F_0) *hold. Then* $x(t)$ *is a positive solution of (1.1) if only if* $u(t)$ *is a positive solution of the following system*

$$u'' + a(t)u = \lambda \tilde{f}(t, u - \lambda\omega) \tag{2.5}$$

with $u_i(t) \geq \lambda\omega_i(t)$. *Here*

$$\tilde{f}(t, u - \lambda\omega) = \bar{f}(t, u, \omega) + e(t),$$

$$\bar{f}(t, u, \omega) = f(t, H(u_1 - \lambda\omega_1)(u_1 - \lambda\omega_1), \dots, H(u_n - \lambda\omega_n)(u_n - \lambda\omega_n)).$$

Proof If $x(t)$ is a positive solution of (1.1), then from the transform process, it yields that $u(t) = x(t) + \lambda\omega(t)$ is a solution of (2.5). Since λ, $e_i(t)$, $G_i(t,s)$ and $x(t)$ are nonnegative, it is clear that each component of $u(t)$ is positive and satisfies $u_i(t) \geq \lambda\omega_i(t)$.

On the other hand, if $u(t)$ is a positive solution of (2.5) with $u_i(t) \geq \lambda\omega_i(t)$, then $H(u_i - \lambda\omega_i) \equiv 1$, which implies that

$$\tilde{f}(t, u - \lambda\omega) = f(t, u_1 - \lambda\omega_1, \dots, u_n - \lambda\omega_n) + e(t).$$

From the transform $x = u - \lambda\omega$, we have that $x(t)$ is a positive solution of (1.1). □

Let E denote the Banach space $\overbrace{C[0,\mathrm{T}] \times \cdots \times C[0,\mathrm{T}]}^{n}$ with the norm

$$\|x\| = \max_{i=1,\dots,n} \{|x_i|_\infty\},$$

where $|x_i|_\infty = \max_{t\in[0,\mathrm{T}]} |x_i(t)|$. Define a cone $K \subset E$ by

$$K = K_1 \times K_2 \times \cdots \times K_n,$$

where $K_i = \{x_i(t) \in C[0,\mathrm{T}] : x_i(t) \geq \delta_i |x_i|_\infty\}$. Also, for $r > 0$, define K_r and ∂K_r by

$$K_r = \{x(t) \in K : \|x\| < r\}, \quad \partial K_r = \{x(t) \in K : \|x\| = r\}.$$

Define an operator $\mathfrak{T}$ by $\mathfrak{T}(u)(t) = (\mathfrak{T}_1(u)(t), \cdots, \mathfrak{T}_n(u)(t))$, where

$$\mathfrak{T}_i(u)(t) = \lambda \int_0^{\mathrm{T}} G_i(t,s) \tilde{f}_i(s, u(s) - \lambda\omega(s)) ds.$$

Now solutions of (2.5) can be rewritten as fixed points of $\mathfrak{T}$ in Banach space E.

Lemma 2.3 *Assume that* (H) *and* (F_0) *hold. Then* $\mathfrak{T}(K) \subseteq K$ *and* $\mathfrak{T} : K \to K$ *is completely continuous.*

Proof Firstly, we show that $\mathfrak{T}(K) \subseteq K$.

For any $u(t) \in K$, we have

$$\begin{aligned}
\mathfrak{T}_i(u)(t) &= \lambda \int_0^{\mathrm{T}} G_i(t,s)\tilde{f}_i(s,u(s)-\lambda\omega(s))ds \\
&\geq \lambda m_i \int_0^{\mathrm{T}} \tilde{f}_i(s,u(s)-\lambda\omega(s))ds \\
&= \lambda \frac{m_i}{M_i} \int_0^{\mathrm{T}} M_i\tilde{f}_i(s,u(s)-\lambda\omega(s))ds \\
&= \lambda\delta_i \max_{t\in[0,\mathrm{T}]} \int_0^{\mathrm{T}} G_i(t,s)\tilde{f}_i(s,u(s)-\lambda\omega(s))ds,
\end{aligned}$$

namely, $\mathfrak{T}_i(u)(t) \geq \delta_i|\mathfrak{T}_i(u)|_\infty$, which implies that

$$\mathfrak{T}(K) \subseteq K.$$

Secondly, we show that $\mathfrak{T}$ maps bounded set into itself. Suppose that $c > 0$ is a constant and $u \in \overline{K}_c$. From the the continuity of e_i and f, there exists a constant L such that

$$\tilde{f}_i(t,u-\lambda\omega) = f_i(t,H(u_1-\lambda\omega_1)(u_1-\lambda\omega_1),\ldots,H(u_n-\lambda\omega_n)(u_n-\lambda\omega_n)) + e_i(t) \leq L,$$

for $t \in [0,\mathrm{T}]$, $i = 1,\ldots,n$. Let $M = \lambda L\mathrm{T}\max_{i=1,\ldots,n}\{M_i\}$. Then, we have

$$\begin{aligned}
|\mathfrak{T}_i(u)(t)|_\infty &= |\lambda \int_0^{\mathrm{T}} G_i(t,s)\tilde{f}_i(s,u(s)-\lambda\omega(s))ds|_\infty \\
&\leq \lambda M_i L\mathrm{T},
\end{aligned}$$

which implies that $\mathfrak{T}(\overline{K}_c)$ is uniformly bounded.

Thirdly, from the elementary properties of Green s function and discussion in [19], let $\Gamma = \max_{i=1,\ldots,n}\{\Gamma_i\}$, where $\Gamma_i = \max_{0\leq s,t\leq T}|\frac{\partial G_i(t,s)}{\partial t}|$. For $t_1, t_2 \in [0,\mathrm{T}]$, we have

$$\begin{aligned}
|\mathfrak{T}_i u(t_2) - \mathfrak{T}_i u(t_1)| &= |\lambda \int_0^{\mathrm{T}} G_i(t_2,s)\tilde{f}_i(s,u(s)-\lambda\omega(s))ds \\
&\quad -\lambda \int_0^{\mathrm{T}} G_i(t_1,s)\tilde{f}_i(s,u(s)-\lambda\omega(s))ds| \\
&= |\lambda \int_0^{\mathrm{T}} [G_i(t_2,s)-G_i(t_1,s)]\tilde{f}_i(s,u(s)-\lambda\omega(s))ds| \\
&\leq \lambda L\Gamma\mathrm{T}|t_2-t_1|.
\end{aligned}$$

Therefore, by applying the Arzela-Ascoli theorem [11], we can find that $\mathfrak{T}_{\mathrm{i}}(\overline{K}_c)$ is relatively compact, namely, $\mathfrak{T}(\overline{K}_c)$ is relatively compact.

Finally, we claim that $\mathfrak{T} : \overline{K}_c \to K$ is continuous. Assume that $\{u^n\}_{n=1}^\infty \subset \overline{K}_c$ which converges to $u^0(t)$ uniformly on $[0,\mathrm{T}]$. Since $\{(\mathfrak{T}u^n)(t)\}_{n=1}^\infty$ is uniformly bounded and equicontinuous on $[0,\mathrm{T}]$, from the

Arzela-Ascoli theorem it follows that there exists a uniformly convergent subsequence in $\{(\mathfrak{T}u^n)(t)\}_{n=1}^{\infty}$. Let $\{(\mathfrak{T}u^{n(m)})(t)\}_{m=1}^{\infty}$ be a subsequence which converges to $v(t)$ uniformly on $[0,\mathrm{T}]$. Observe that

$$\mathfrak{T}_i u^{n(m)}(t) = \lambda \int_0^{\mathrm{T}} G_i(t,s)\tilde{f}_i(s, u^{n(m)}(s) - \lambda\omega(s))ds.$$

Furthermore, by Lebesgue s dominated convergence theorem and letting $m \to \infty$, we have

$$v_i(t) = \lambda \int_0^{\mathrm{T}} G_i(t,s)\tilde{f}_i(s, u^0(s) - \lambda\omega(s))ds = \mathfrak{T}_i u^0(t),$$

namely, $v(t) = \mathfrak{T}u^0(t)$. This shows that each subsequence of $\{(\mathfrak{T}u^n)(t)\}_{n=1}^{\infty}$ uniformly converges to $\mathfrak{T}u^0(t)$. Therefore, the sequence $\{(\mathfrak{T}u^n)(t)\}_{n=1}^{\infty}$ uniformly converges to $\mathfrak{T}u^0(t)$. This means that $\mathfrak{T}$ is continuous at $u^0 \in \overline{K}_c$. So, $\mathfrak{T}$ is continuous on $\overline{K}_c$ since u^0 is arbitrary. Thus, $\mathfrak{T}$ is completely continuous. The proof is completed. $\square$

Lemma 2.4 *[11] Let E be a Banach space, and $K \subset E$ be a cone in E. Assume that Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that either*

(i) $\|Tu\| \leq \|u\|,\ \ u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \geq \|u\|,\ \ u \in K \cap \partial\Omega_2$*; or*

(ii) $\|Tu\| \geq \|u\|,\ \ u \in K \cap \partial\Omega_1$ *and* $\|Tu\| \leq \|u\|,\ \ u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. Existence results

For convenience, define the height functions

$$\Phi_i(\rho) = \min\{f_i(t,x) : (t,x) \in [0,\mathrm{T}] \times \mathbb{R}_+ \times \cdots \times \overbrace{[0,\rho]}^{i} \times \cdots \times \mathbb{R}_+\}.$$

$$\Psi_{i,j}(t,\rho) = \min\{f_i(t,x) : (x_1,\ldots,x_j,\ldots,x_n) \in \mathbb{R}_+ \times \cdots \times \overbrace{[\delta_j\rho,\rho]}^{j} \times \cdots \times \mathbb{R}_+\}.$$

Theorem 3.1 *Assume that* (H) *and* (F_0) *hold. In addition, the functions $f_i (i = 1,\ldots,n)$ satisfy the following assumptions:*

(F_1) $\Phi_i(r) \geq 0$*, where* $r = \max_{i\in\{1,\ldots,n\}} \frac{M_i^2 \mathrm{T}}{m_i}$.

(F_2)

$$\lim_{x_{k(i)}\to+\infty} \frac{f_i(t,x_1,\ldots,x_n)}{x_{k(i)}} = +\infty,\ \ \textit{uniformly}, t \in [0,\mathrm{T}], x_{k(j)},$$

where $k(i) \in \{1,\ldots,n\}$ and $k(i) \neq k(j)$ for $i \neq j$.

Then there exists a $\lambda^* > 0$ such that (1.1) has at least one positive solution for $0 < \lambda < \lambda^*$.

Remark 3.2 *(1) The index $k(i)$ is related to the index i of f_i;*

(2) The index $k(i)$ may be equal to the index i of f_i or not.

Proof of Theorem 3.1. Let

$$N_i = \sup\{f_i(t,x) + e(t) : t \in [0,\mathrm{T}], 0 \le x_1, \ldots, x_n \le r\}.$$

For any $u \in \partial K_r$, it is clear that

$$0 \le H(u_i(t) - \lambda\omega_i(t))(u_i(t) - \lambda\omega_i(t)) \le r.$$

Furthermore, we get

$$\begin{aligned}\mathfrak{T}_i(u)(t) &= \lambda \int_0^{\mathrm{T}} G_i(t,s)\tilde{f}_i(s, u(s) - \lambda\omega(s))ds \\ &\le \lambda N_i M_i \mathrm{T} \\ &\le M_i \mathrm{T}\frac{1}{\delta_i} = \frac{M_i^2 \mathrm{T}}{m_i} \le r,\end{aligned}$$

for $0 < \lambda < \lambda^* = \min_{i\in\{1,\ldots,n\}}\left\{\frac{1}{N_i}, \frac{\mathrm{T}}{\int_0^{\mathrm{T}} e_i(t)dt}\right\}$.

Thus, we have

$$\|\mathfrak{T}(u)\| = \max_{i\in\{1,\ldots,n\}} |\mathfrak{T}_{\mathrm{i}}(u)|_\infty < r = \|u\|, \text{for } u \in \partial K_r,\ 0 < \lambda < \lambda^*.$$

Since $\lim_{x_{k(i)}\to+\infty} \frac{f_i(t,x)}{x_{k(i)}} = +\infty$, then there exist two positive constants L and X such that

$$f_i(t,x) \ge L x_{k(i)},\ \forall\, x_{k(i)} > X,\ \text{uniformly},\ t \in [0,\mathrm{T}], x_{k(j)} \in \mathbb{R}_+, j \ne i,$$

where L satisfies $L\frac{\lambda \mathrm{T}}{2}\min_{i\in\{1,\ldots,n\}} m_i\delta_{k(i)} > 1$.

Take

$$R = 1 + r + 2\max_{i\in\{1,\ldots,n\}}\left\{\lambda\frac{M_i^2}{m_i}\int_0^{\mathrm{T}} e_i(s)ds, \frac{X}{\delta_i}\right\}.$$

For any $u \in \partial K_R$, by the definition of $\|u\|$, there exists an index $k(i)$ such that $\|u\| = |u_{k(i)}|_\infty = R$. Since

$$\begin{aligned}\lambda\omega_{k(i)}(t) &= \lambda\int_0^{\mathrm{T}} G_{k(i)}(t,s)e_{k(i)}(s)ds \\ &\le \lambda M_{k(i)}\int_0^{\mathrm{T}} e_{k(i)}(s)ds \\ &= \lambda M_{k(i)}\int_0^{\mathrm{T}} e_{k(i)}(s)ds\frac{M_{k(i)}}{m_{k(i)}}\delta_{k(i)} \\ &\le \lambda\int_0^{\mathrm{T}} e_{k(i)}(s)ds\frac{M_{k(i)}^2}{m_{k(i)}}\frac{u_{k(i)}}{R},\end{aligned}$$

we have

$$\begin{aligned} u_{k(i)} - \lambda\omega_{k(i)} &\geq u_{k(i)} - \lambda\int_0^{\mathrm{T}} e_{k(i)}(s)ds\frac{M_{k(i)}^2}{m_{k(i)}}\frac{u_{k(i)}}{R} \\ &= (1 - \lambda\int_0^{\mathrm{T}} e_{k(i)}(s)ds\frac{M_{k(i)}^2}{m_{k(i)}}\frac{1}{R})u_{k(i)} \\ &\geq \frac{1}{2}u_{k(i)} \geq \frac{1}{2}\delta_{k(i)}|u_{k(i)}|_\infty = \frac{1}{2}\delta_{k(i)}R \\ &> X. \end{aligned}$$

Furthermore, for any $u \in \partial K_R$ with $\|u\| = |u_{k(i)}|_\infty = R$, we have

$$\begin{aligned} \mathfrak{T}_i(u)(t) &= \lambda\int_0^{\mathrm{T}} G_i(t,s)\tilde{f}_i(s, u(s) - \lambda\omega(s))ds \\ &\geq \lambda m_i L\mathrm{T}[u_{k(i)} - \lambda\omega_{k(i)}] \\ &\geq \lambda m_i L\mathrm{T}\frac{1}{2}\delta_{k(i)}R > R = \|u\|. \end{aligned}$$

Thus, we have

$$\|\mathfrak{T}(u)\| = \max_{i\in\{1,\ldots,n\}} |\mathfrak{T}_{\mathrm{i}}(u)|_\infty > R = \|u\|, \text{for } u \in \partial K_R.$$

Therefore, by Lemma 2.4, $\mathfrak{T}$ has at least one fixed point $u = (u_1, \cdots, u_n)$ with $r < \|u\| < R$ for $0 < \lambda < \lambda^*$.

Finally, we verify that $u(t) \geq \lambda\omega(t)$, namely, $u_i(t) \geq \lambda\omega_i(t)$.

Without loss of generality, if $r < \|u\| = |u_1|_\infty < R$, we get

$$\begin{aligned} u_1(t) &\geq \delta_1|u_1|_\infty > \delta_1\frac{M_1^2\mathrm{T}}{m_1} \\ &> \lambda\frac{m_1}{M_1}\int_0^{\mathrm{T}} e_1(s)ds\frac{M_1^2}{m_1} = \lambda\int_0^{\mathrm{T}} M_1 e_1(s)ds \\ &> \lambda\int_0^{\mathrm{T}} G_1(t,s)e_1(s)ds = \lambda\omega_1(t). \end{aligned}$$

For $i \neq 1$, there are two cases :(I) $|u_i|_\infty > r$; (II) $|u_i|_\infty < r$.

Case (I): Since $|u_i|_\infty > r$, we also have

$$\begin{aligned} u_i(t) &\geq \delta_i|u_i|_\infty > \delta_i\frac{M_i^2\mathrm{T}}{m_i} \\ &> \lambda\frac{m_i}{M_i}\int_0^{\mathrm{T}} e_i(s)ds\frac{M_i^2}{m_i} = \lambda\int_0^{\mathrm{T}} M_i e_i(s)ds \\ &> \lambda\int_0^{\mathrm{T}} G_i(t,s)e_i(s)ds = \lambda\omega_i(t). \end{aligned}$$

Case (II): Let

$$\Omega_{1i} = \{t \in [0, \mathrm{T}] : u_i(t) \geq \lambda\omega_i(t)\},\ \Omega_{2i} = \{t \in [0, \mathrm{T}] : u_i(t) < \lambda\omega_i(t)\}.$$

It is clear that $\Omega_{11} = [0, \mathrm{T}]$. Observe that

$$\begin{aligned}
u_i(t) &= \lambda \int_0^{\mathrm{T}} G_i(t,s)\tilde{f}_i(t, u(s) - \lambda\omega(s))ds \\
&= \lambda \int_0^{\mathrm{T}} G_i(t,s) f_i(s, u_1 - \lambda\omega_1, H(u_2 - \lambda\omega_2)(u_2 - \lambda\omega_2), \ldots, H(u_n - \lambda\omega_n)(u_n - \lambda\omega_n))ds \\
&\quad + \lambda \int_0^{\mathrm{T}} G_i(t,s)e_i(s)ds \\
&= \lambda \left[\int_{\Omega_{11}\cap\Omega_{1i}} G_i(t,s)\bar{f}_i(s,u,\omega)ds + \int_{\Omega_{11}\cap\Omega_{2i}} G_i(t,s)\bar{f}_i(s,u,\omega)ds \right] \text{(For n=2,i=2)} \\
&\quad + \lambda \int_0^{\mathrm{T}} G_i(t,s)e_i(s)ds \\
&= \lambda \left[\int_{\Omega_{11}\cap\Omega_{1i}\cap\Omega_{12}} G_i(t,s)\bar{f}_i(s,u,\omega)ds + \int_{\Omega_{11}\cap\Omega_{1i}\cap\Omega_{22}} G_i(t,s)\bar{f}_i(s,u,\omega)ds \right. \\
&\quad \left. + \int_{\Omega_{11}\cap\Omega_{2i}\cap\Omega_{12}} G_i(t,s)\bar{f}_i(s,u,\omega)ds + \int_{\Omega_{11}\cap\Omega_{2i}\cap\Omega_{22}} G_i(t,s)\bar{f}_i(s,u,\omega)ds \right] \text{(For n=3,i=3)} \\
&\quad + \lambda \int_0^{\mathrm{T}} G_i(t,s)e_i(s)ds \\
&= \lambda \left[\int_{\Omega_{11}\cap\Omega_{1i}\cap\Omega_{12}\cap\Omega_{13}} G_i(t,s)\bar{f}_i(s,u,\omega)ds + \int_{\Omega_{11}\cap\Omega_{1i}\cap\Omega_{12}\cap\Omega_{23}} G_i(t,s)\bar{f}_i(s,u,\omega)ds \right. \\
&\quad + \int_{\Omega_{11}\cap\Omega_{1i}\cap\Omega_{22}\cap\Omega_{13}} G_i(t,s)\bar{f}_i(s,u,\omega)ds + \int_{\Omega_{11}\cap\Omega_{1i}\cap\Omega_{22}\cap\Omega_{23}} G_i(t,s)\bar{f}_i(s,u,\omega)ds \\
&\quad + \int_{\Omega_{11}\cap\Omega_{2i}\cap\Omega_{12}\cap\Omega_{13}} G_i(t,s)\bar{f}_i(s,u,\omega)ds + \int_{\Omega_{11}\cap\Omega_{2i}\cap\Omega_{12}\cap\Omega_{23}} G_i(t,s)\bar{f}_i(s,u,\omega)ds \\
&\quad \left. + \int_{\Omega_{11}\cap\Omega_{2i}\cap\Omega_{22}\cap\Omega_{13}} G_i(t,s)\bar{f}_i(s,u,\omega)ds + \int_{\Omega_{11}\cap\Omega_{2i}\cap\Omega_{22}\cap\Omega_{23}} G_i(t,s)\bar{f}_i(s,u,\omega)ds \right] \text{(For n=4,i=4)} \\
&\quad + \lambda \int_0^{\mathrm{T}} G_i(t,s)e_i(s)ds.
\end{aligned}$$

For $n = 4, i = 3$, The decomposition $[0, \mathrm{T}]$ is

$$\Omega_{11} \cap \Omega_{13} \cap \Omega_{12} \cap \Omega_{14},\ \Omega_{11} \cap \Omega_{13} \cap \Omega_{12} \cap \Omega_{24},$$

$$\Omega_{11} \cap \Omega_{13} \cap \Omega_{22} \cap \Omega_{14},\ \Omega_{11} \cap \Omega_{13} \cap \Omega_{22} \cap \Omega_{24},$$

$$\Omega_{11} \cap \Omega_{23} \cap \Omega_{12} \cap \Omega_{14},\ \Omega_{11} \cap \Omega_{23} \cap \Omega_{12} \cap \Omega_{24},$$

$$\Omega_{11} \cap \Omega_{23} \cap \Omega_{22} \cap \Omega_{14},\ \Omega_{11} \cap \Omega_{23} \cap \Omega_{22} \cap \Omega_{24}.$$

As the same decomposition rule, we have

$$\begin{aligned} u_i(t) &= \lambda[\int_{\Omega_1^i} G_i(t,s)\bar{f}_i(s,u,\omega)ds + \ldots + \int_{\Omega_{2^{n-1}}^i} G_i(t,s)\bar{f}_i(s,u,\omega)ds] \\ &\quad +\lambda \int_0^{\mathrm{T}} G_i(t,s)e_i(s)ds, \end{aligned}$$

where $\Omega_j^i = (j = 1, \ldots, 2^{n-1})$ denote the decomposition chain with $\sum_{j=1}^{2^{n-1}} \Omega_j^i = [0, \mathrm{T}]$. Furthermore, (F_1) can yield $\int_{\Omega_j^i} G_i(t,s)\bar{f}_i(s,u,\omega)ds > 0(j = 1, 2, \ldots, 2^{n-1})$, which implies that

$$u_i(t) > \lambda \int_0^{\mathrm{T}} G_i(t,s)e_i(s)ds = \lambda\omega_i(t).$$

Therefore, (1.1) has at least one positive solution $x = (x_1, \ldots, x_n) = (u_1 - \lambda\omega_1, \ldots, u_n - \lambda\omega_n)$. □

Remark 3.3 *The structure of Ω_j^i can be expressed by*

$$\Omega_j^i = \begin{cases} \Omega_{11} \cap \Omega_{1i} \cap \Omega_{k_2 l_2} \cap \Omega_{k_3 l_3} \cap \cdots \Omega_{k_{i-1} l_{i-1}} \cap \Omega_{k_{i+1} l_{i+1}} \cap \cdots \Omega_{k_n l_n},\ \textit{or} \\ \Omega_{11} \cap \Omega_{2i} \cap \Omega_{k_2 l_2} \cap \Omega_{k_3 l_3} \cap \cdots \Omega_{k_{i-1} l_{i-1}} \cap \Omega_{k_{i+1} l_{i+1}} \cap \cdots \Omega_{k_n l_n}, \end{cases}$$

where $k_p = \{1, 2\}$ and $l_p = p$, for $p = 2, 3 \ldots, i-1, i+1, \ldots, n$.

Theorem 3.4 *Assume that* (H) *and* (F_0) *hold. In addition, the functions $f_i (i = 1, \ldots, n)$ satisfy the following assumptions:*

(F_3) *there exists a $\overline{R} > \max_{i\in\{1,\ldots,n\}} \frac{M_i^3 \mathrm{T}}{m_i^2}$ such that*

$$\Phi_i(\overline{R}) \geq 0$$

and

$$\max_{i=1,\ldots,n} \frac{\overline{R}}{m_i \delta_i \int_0^{\mathrm{T}} \Psi_{i,j}(t, \overline{R})dt} < \min_{i=1,\ldots,n} \left\{1, \frac{1}{N_i}, \frac{\mathrm{T}}{\int_0^{\mathrm{T}} e_i(t)dt}\right\}.$$

(F_4)

$$\lim_{x_{k(i)} \to +\infty} \frac{f_i(t,x)}{x_{k(i)}} = 0, \textit{uniformly}, t \in [0, \mathrm{T}], x_{k(j)}.$$

Then there exist λ_*, λ^* such that (1.1) has at least two positive solutions for $\lambda_* < \lambda < \lambda^*$.

Proof From the assumption (F_3), it is clear that $\overline{R} > r = \max_{i\in\{1,\ldots,n\}} \frac{M_i^2 \mathrm{T}}{m_i}$. By the proof of Theorem 3.1, we have

$$\|\mathfrak{T}(u)\| = \max_{i\in\{1,\ldots,n\}} |\mathfrak{T}_{\mathfrak{i}}(u)|_\infty < r = \|u\|, \text{for } u \in \partial K_r,\ 0 < \lambda < \lambda^*,$$

where $\lambda^* = \min_{i\in\{1,\ldots,n\}} \left\{\frac{1}{N_i}, \frac{\mathrm{T}}{\int_0^{\mathrm{T}} e_i(t)dt}\right\}$.

For any $u \in \partial K_{\bar{R}}$, by the definition of $\|u\|$, there exists an index j_0 such that $\|u\| = |u_{j_0}|_\infty = \bar{R}$. From the assumption (F_3), we have

$$\begin{aligned}
\mathfrak{T}_{i_0}(u)(t) &= \lambda \int_0^{\mathrm{T}} G_{i_0}(t,s)\tilde{f}_{i_0}(s, u(s) - \lambda\omega(s))ds \\
&\geq \lambda m_{i_0} \int_0^{\mathrm{T}} \Psi_{i_0,j_0}(t,\bar{R})dt \\
&= \lambda \frac{m_{i_0} \int_0^{\mathrm{T}} \Psi_{i_0,j_0}(t,\bar{R})dt}{\bar{R}} \bar{R} \\
&> \bar{R} = \|u\|,
\end{aligned}$$

for $\lambda > \lambda_* = \max_{i=1,\dots,n} \frac{\overline{R}}{m_i \int_0^{\mathrm{T}} \Psi_{i,j}(t,\overline{R})dt}$. By (F_3) and λ^*, it is easy to see that $\lambda_* < \lambda^*$.

By the definition of $\lim_{x_{k(i)} \to +\infty} \frac{f_i(t,x)}{x_{k(i)}} = 0$, then there exist two positive constants $\epsilon > 0$ and $\tilde{X}$ such that

$$f_i(t,x) \leq \epsilon x_{k(i)},\ \forall\, x_{k(i)} > \tilde{X},\ \ \text{uniformly},\ t \in [0,\mathrm{T}], x_{k(j)} \in \mathbb{R}_+, j \neq i,$$

where ϵ satisfies $\epsilon\lambda\mathrm{T}\max_{i=1,\dots,n}\{M_i, \lambda M_i|\omega_{k(i)}|_\infty\} < \frac{1}{2}$.

Take

$$R = \bar{R} + 2 \max_{i \in \{1,\dots,n\}} \left\{ \lambda \frac{M_i^2}{m_i} \int_0^{\mathrm{T}} e_i(s)ds, \frac{X}{\delta_i} \right\}.$$

For any $u \in \partial K_R$, by the definition of $\|u\|$, there exists an index $k(i)$ such that $\|u\| = |u_{k(i)}|_\infty = R$. Since

$$\begin{aligned}
\lambda\omega_{k(i)}(t) &= \lambda \int_0^{\mathrm{T}} G_{k(i)}(t,s)e_{k(i)}(s)ds \\
&\leq \lambda M_{k(i)} \int_0^{\mathrm{T}} e_{k(i)}(s)ds \\
&= \lambda M_{k(i)} \int_0^{\mathrm{T}} e_{k(i)}(s)ds \frac{M_{k(i)}}{m_{k(i)}}\delta_{k(i)} \\
&\leq \lambda \int_0^{\mathrm{T}} e_{k(i)}(s)ds \frac{M_{k(i)}^2}{m_{k(i)}} \frac{u_{k(i)}}{R},
\end{aligned}$$

we have

$$\begin{aligned}
u_{k(i)} - \lambda\omega_{k(i)} &\geq u_{k(i)} - \lambda \int_0^{\mathrm{T}} e_{k(i)}(s)ds \frac{M_{k(i)}^2}{m_{k(i)}} \frac{u_{k(i)}}{R} \\
&= (1 - \lambda \int_0^{\mathrm{T}} e_{k(i)}(s)ds \frac{M_{k(i)}^2}{m_{k(i)}} \frac{1}{R})u_{k(i)} \\
&\geq \frac{1}{2}u_{k(i)} \geq \frac{1}{2}\delta_{k(i)}|u_{k(i)}|_\infty = \frac{1}{2}\delta_{k(i)}R \\
&> X.
\end{aligned}$$

Furthermore, for any $u \in \partial K_R$ with $\|u\| = |u_{k(i)}|_\infty = R$, we have

$$\begin{aligned}\mathfrak{T}_i(u)(t) &= \lambda \int_0^{\mathrm{T}} G_i(t,s)\tilde{f}_i(s, u(s) - \lambda\omega(s))ds \\ &\leq \lambda M_i \epsilon \mathrm{T}[u_{k(i)} - \lambda\omega_{k(i)}] \\ &\leq \epsilon\lambda \mathrm{T} M_i [R + \lambda|\omega_{k(i)}|_\infty] < R = \|u\|.\end{aligned}$$

So, we have

$$\|\mathfrak{T}(u)\| = \max_{i\in\{1,\dots,n\}} |\mathfrak{T}_{\mathrm{i}}(u)|_\infty < R = \|u\|, \text{ for } u \in \partial K_R.$$

Therefore, by Lemma 2.4, $\mathfrak{T}$ has at least one fixed point $u^1 \in K \cap (\overline{\Omega}_{\bar{R}} \setminus \Omega_r)$ and $u^2 \in K \cap (\overline{\Omega}_R \setminus \Omega_{\bar{R}})$ for $\lambda_* < \lambda < \lambda^*$. As the similar proof of theorem 3.1, we also verify that $u^i(t) \geq \lambda\omega(t)$. □

Corollary 3.5 *Assume that* (H), (F_0), (F_1), *and* (F_4) *hold. In addition, the functions* $f_i (i = 1, \dots, n)$ *satisfy the following assumptions:*

(F_5) *there exists a* $\overline{R} > \max_{i\in\{1,\dots,n\}} \frac{M_i^3 \mathrm{T}}{m_i^2}$ *such that*

$$\max_{i=1,\dots,n} \frac{\overline{R}}{m_i \delta_i \int_0^{\mathrm{T}} \Psi_{i,j}(t, \overline{R})dt} < \min_{i=1,\dots,n} \left\{1, \frac{1}{N_i}, \frac{\mathrm{T}}{\int_0^{\mathrm{T}} e_i(t)dt}\right\}.$$

Then there exist λ_*, λ^* *such that (1.1) has at least two solutions for* $\lambda_* < \lambda < \lambda^*$, *one is positive, the other is not necessarily positive.*

4. Examples

Now we give two examples to illustrate our main results.

Example 4.1 *Let us consider the following system*

$$\begin{cases} x_1''(t) + a(t)x_1(t) = \lambda[\cos^2(\frac{m}{4M^2}x_1) + x_2^3 + x_3^2 - \frac{1}{2}\sin^2 t], \\ x_2''(t) + a(t)x_2(t) = \lambda[x_1^{\frac{3}{2}} + \cot^2(\frac{m}{8M^2}x_2) + x_3 - \frac{1}{6}\sin^2 t \cos^4 t], \\ x_3''(t) + a(t)x_3(t) = \lambda[x_1^{\frac{3}{2}} + +x_2^3 + \cos^2(\frac{m}{16M^2}x_3) - \frac{1}{3}\sin^2 2t], \\ x(0) = x(\pi), x'(0) = x'(\pi). \end{cases} \tag{4.1}$$

Let

$$e_1(t) = \sin^2 t,\ e_2(t) = \frac{1}{3}\sin^2 t \cos^4 t,\ \ e_3(t) = \frac{1}{3}\sin^2 2t.$$

It is obvious that (F_0) *holds. Since* $r = \max_{i\in\{1,2,3\}} \frac{M_i^2 \mathrm{T}}{m_i} = \frac{M^2\pi}{m}$, *we have* $\frac{m}{4M^2}x_1 \in [0, \frac{\pi}{4}]$. *Furthermore, we get* $f_1(t, x_1, x_2, x_3) = \cos^2(\frac{m}{4M^2}x_1) + x_2^3 + x_3^2 - \frac{1}{2}\sin^2 t \geq 0$, *for* $(t, x_1, x_2, x_3) \in [0, \pi] \times [0, r] \times \mathbb{R}_+ \times \mathbb{R}_+$. *Thus,* f_1 *satisfies* (F_1). *In the similar way, we can also verify that* f_2 *and* f_3 *satisfy* (F_1). *Finally, it is easy to verify that*

$$\begin{aligned}\lim_{x_3\to+\infty} \frac{f_1(t, x_1, x_2, x_3)}{x_3} &= \lim_{x_3\to+\infty} \frac{\cos^2(\frac{m}{4M^2}x_1) + x_2^3 + x_3^2 - \frac{1}{2}\sin^2 t}{x_3} \\ &= +\infty,\ \textit{uniformly}, t \in [0, \mathrm{T}], x_1, x_2 \in \mathbb{R}_+,\end{aligned}$$

$$\begin{aligned}\lim_{x_1\to+\infty}\frac{f_2(t,x_1,x_2,x_3)}{x_1} &= \lim_{x_1\to+\infty}\frac{x_1^{\frac{3}{2}}+\cot^2(\frac{m}{8M^2}x_2)+x_3-\frac{1}{6}\sin^2 t\cos^4 t}{x_1}\\ &= +\infty,\ \textit{uniformly}, t\in[0,\mathrm{T}], x_2,x_3\in\mathbb{R}_+\end{aligned}$$

and

$$\begin{aligned}\lim_{x_2\to+\infty}\frac{f_3(t,x_1,x_2,x_3)}{x_2} &= \lim_{x_2\to+\infty}\frac{x_1^{\frac{3}{2}}+x_2^3+\cos^2(\frac{m}{16M^2}x_3)-\frac{1}{3}\sin^2 2t}{x_2}\\ &= +\infty,\ \textit{uniformly}, t\in[0,\mathrm{T}], x_1,x_3\in\mathbb{R}_+.\end{aligned}$$

Therefore, by Theorem 3.1, there exists a $\lambda^*>0$ *such that (3.1) has at least one positive solution for* $0<\lambda<\lambda^*$*.*

Example 4.2 *Let us consider the following system*

$$\begin{cases} x_1''(t)+a(t)x_1(t)=\lambda[\cos^2(\frac{m^2}{4M^3}x_1)+x_2^3+x_3^{\frac{2}{5}}-\frac{1}{3}\cos^2 t],\\ x_2''(t)+a(t)x_2(t)=\lambda[x_1^{\frac{1}{2}}+\cot^2(\frac{m^2}{8M^3}x_2)+x_3^2-\frac{1}{6}\sin^2 t],\\ x_3''(t)+a(t)x_3(t)=\lambda[x_1^{\frac{4}{3}}++x_2^{\frac{2}{3}}+\cos^2(\frac{m^2}{16M^3}x_3)-\frac{1}{8}\sin^2 2t],\\ x(0)=x(\pi), x'(0)=x'(\pi).\end{cases} \tag{4.2}$$

Let

$$e_1(t)=\cos^2 t,\ e_2(t)=\frac{1}{3}\sin^2 t,\ \ e_3(t)=\frac{1}{3}\sin^2 2t.$$

It is obvious that (F_0) *holds. Choose* $\bar{R}=\frac{M^3\pi}{m^2}>r$*. Then it is easy to verify that* $\Phi(\bar{R})\geq 0$*. Since*

$$\begin{aligned}\Psi_{1,2}(t,\bar{R}) &= \min\{f_1(t,x):(x_1,x_2,x_3)\in\mathbb{R}_+\times[\delta_2\bar{R},\bar{R}]\times\mathbb{R}_+\}\\ &= (\delta_2\bar{R}^3)-\frac{1}{3}\cos^2 t,\end{aligned}$$

for sufficiently large $\bar{R}$*, we have*

$$\frac{\overline{R}}{m_1\delta_1\int_0^{\mathrm{T}}(\delta_2\bar{R})^3-\frac{1}{3}\cos^2 t dt}<\min_{i=1,2,3}\left\{1,\frac{1}{N_i},\frac{\mathrm{T}}{\int_0^{\mathrm{T}}e_i(t)dt}\right\}.$$

In the similar way, we conclude that (F_3) *holds.*

Finally, it is easy to verify that

$$\begin{aligned}\lim_{x_3\to+\infty}\frac{f_1(t,x_1,x_2,x_3)}{x_3} &= \lim_{x_3\to+\infty}\frac{\cos^2(\frac{m^2}{4M^3}x_1)+x_2^3+x_3^{\frac{2}{5}}-\frac{1}{3}\cos^2 t}{x_3}\\ &= 0,\ \textit{uniformly}, t\in[0,\mathrm{T}], x_1,x_2\in\mathbb{R}_+,\end{aligned}$$

$$\begin{aligned}\lim_{x_1\to+\infty}\frac{f_2(t,x_1,x_2,x_3)}{x_1} &= \lim_{x_1\to+\infty}\frac{x_1^{\frac{1}{2}}+\cot^2(\frac{m^2}{8M^3}x_2)+x_3^2-\frac{1}{6}\sin^2 t}{x_1}\\ &= 0,\ \textit{uniformly}, t\in[0,\mathrm{T}], x_2,x_3\in\mathbb{R}_+\end{aligned}$$

and

$$\lim_{x_2\to+\infty}\frac{f_3(t,x_1,x_2,x_3)}{x_2} = \lim_{x_2\to+\infty}\frac{x_1^{\frac{4}{3}}++x_2^{\frac{2}{3}}+\cos^2(\frac{m^2}{16M^3}x_3)-\frac{1}{8}\sin^2 2t}{x_2}$$
$$= 0,\ \textit{uniformly}, t\in[0,\mathrm{T}], x_1,x_3\in\mathbb{R}_+.$$

Therefore, by Theorem 3.2, (1.1) has at least two positive solutions for $\lambda_*<\lambda<\lambda^*$.

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