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On Wiener's Tauberian theorems and convolution for oscillatory integral operators

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Abstract: The main aim of this work is to obtain Paley–Wiener and Wiener's Tauberian results associated with an oscillatory integral operator, which depends on cosine and sine kernels, as well as to introduce a consequent new convolution. Additionally, a new Young-type inequality for the obtained convolution is proven, and a new Wiener-type algebra is also associated with this convolution.

Key words: Wiener Tauberian theorem, Paley–Wiener theorem, convolution, asymptotic behavior, Young inequality

1. Introduction

We will build a new convolution associated with an integral operator of oscillatory nature which exhibits very distinctive properties. This is constructed in Section 3, after having a previous section where some auxiliary known results are recalled. Young's convolution inequalities are also obtained for the new convolution. Moreover, consequent Paley–Wiener type theorems will be derived in Section 4. The main aim of Section 5 is to obtain Tauberian-type results for our oscillatory integral operator. This gives rise to a nonclassic notion of translation which interconnects well with the kernel of the integral operator, therefore being a key ingredient in our construction of a Wiener–Tauberian-type theorem. To conclude the paper, a new Wiener–Pitt-type theorem is also deduced.

Having that plan in mind, we would like to start by recalling that for any $f \in L^1$ (or L^2), the span of translations $(\tau_{-a}f)(x) := f(x+a)$ is dense in such spaces if and only if the real zeros of the Fourier transform of f is the empty set (or a set of zero Lebesgue measure). This gives rise to the necessary and sufficient condition under which any function in L^1 (or L^2) can be approximated by linear combinations of translations of a given function. These facts are related with the classic Wiener's theorem. This theorem was taken into consideration by Gelfand when stating a theorem in terms of commutative C^*-algebras. For the convenience of our presentation, let us formulate the classic Wiener's Tauberian theorem: Suppose $h \in L^\infty$. If the convolution $(f * h)(x)$ tends to zero at infinity for some $f \in L^1$ whose Fourier transform $\hat{f}$ has no real zeros, then the convolution $(g * h)(x)$ tends to zero at infinity for any $g \in L^1$. More generally, if

$$\lim_{x\to\infty} (f * h)(x) = A \int f(x)\, dx$$

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2010 *AMS Mathematics Subject Classification:* Primary 42B10; Secondary 40E05, 43A32, 44A35, 47A05 for some $f \in L^1$, then also holds

$$\lim_{x\to\infty}(g * h)(x) = A\int g(x)\,dx$$

for any $g \in L^1$. A stronger conclusion that $h(x) \to A$ might not hold, but it is true if an additional condition about slow oscillation is imposed on h (see [13, Theorem 9.7]).

The crucial importance of Wiener's and Tauber's results is that the asymptotic behavior at infinity of some objects (with rare information) can be discovered through other ones whose properties are better known. For instance, invoking Wiener's Tauberian theorems, Ikehara presented a simple proof for the *Prime Number Theorem*. This event was seen as a clear identification of the Wiener's and Tauber's exceptionally original ideas, and caused a great attention, giving rise to a significant number of studies related to the theory of oscillatory integrals and their convolutions (see [6–8] and the interesting comprehensive analysis therein). As it is well known, the above issues are concerned with certain oscillatory integrals and convolutions. Together with other relevant theoretical and applied needs, a significant amount of recent investigations on oscillatory integrals and convolutions continue to be done in both analytical and numerical perspectives (see [4, 9–11, 14, 17–20]).

Within the just presented framework, we will consider the oscillatory integral operator $T_{\eta,i}$ defined by

$$\left(T_{\eta,i}f\right)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\left[\eta\cos(xy) + i\sin(xy)\right]f(y)\,dy, \tag{1}$$

with $\eta \in \mathbb{C}\backslash\{0\}$. A particular case of this operator, with $\eta = 2$ had been already considered in [3], but there the main purpose was focused on the analysis of consequent Heisenberg uncertainty principles where such operator takes an important role. In a sense, $T_{\eta,i}$ can be considered as a model oscillatory integral operator for the class which uses separately the cosine and sine integral kernels.

We will now indicate some of the notation used in this work. Let F and F^{-1} denote the Fourier and inverse Fourier transforms, given by

$$(F^{\pm 1}f)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{\mp ixy}f(y)\,dy,$$

respectively. Let

$$(T_c f)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\cos(xy)f(y)\,dy, \qquad (T_s f)(x) := \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\sin(xy)f(y)\,dy$$

stand for the cosine and sine Fourier transform, respectively. It is clear that $T_{\eta,i} = \eta T_c + iT_s$, and $2T_{\eta,i} = (\eta-1)F + (\eta+1)F^{-1}$. We will denote by $\widetilde{\varphi}$ the reflection of the function φ; i.e. $\widetilde{\varphi}(x) = \varphi(-x)$, $x \in \mathbb{R}^n$. Moreover, the reflection operator will be denoted by W. Thus, $(W\varphi)(x) := \widetilde{\varphi}(x) = \varphi(-x)$. We will denote by $\langle\cdot,\cdot\rangle_2$ the usual inner product of $L^2(\mathbb{R}^n)$ and by $\|\cdot\|_2$ the corresponding norm. More generally, we will be using $\|\cdot\|_p$ for the usual norm of $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$. Finally, $B(x_0,\delta) := \{x \in \mathbb{R}^n : |x - x_0| < \delta\}$ denotes the open ball centered at $x_0 \in \mathbb{R}^n$ with radius δ, and $\mathcal{S}$ stands for the Schwartz space.

2. Auxiliary results

In this section, we introduce some results associated with the operator $T_{\eta,i}$, whose special case with $\eta = 2$ were obtained by the authors in [3], which will be used either to prove or to interpret some further results in the present work. We shall use the multidimensional Hermite functions defined by $\Phi_\alpha(x) := (-1)^{|\alpha|}e^{\frac{1}{2}|x|^2}D_x^\alpha e^{-|x|^2}$.

Theorem 1 *The Hermite functions are eigenfunctions of the operator $T_{\eta,i}$ with eigenvalues $\pm\eta, \pm i$. Indeed, the following formula holds*

$$T_{\eta,i}\Phi_\alpha = \begin{cases} (-1)^{\frac{|\alpha|}{2}}\,\eta\Phi_\alpha, & if\ |\alpha| \equiv 0,2 \pmod 4 \\ (-1)^{\frac{|\alpha|-1}{2}}\,i\,\Phi_\alpha, & if\ |\alpha| \equiv 1,3 \pmod 4. \end{cases} \tag{2}$$

Proof If we consider $L^2(\mathbb{R}^n)$ as the domain of the Fourier and inverse Fourier transform F and F^{-1}, respectively, then the domain of $T_{\eta,i}$ is also $L^2(\mathbb{R}^n)$. In fact, we can rewrite $T_{\eta,i}$ as $T_{\eta,i} = \frac{\eta-1}{2}F + \frac{\eta+1}{2}F^{-1}$. Since $F\Phi_\alpha = (-i)^{|\alpha|}\Phi_\alpha$ and $F^{-1}\Phi_\alpha = i^{|\alpha|}\Phi_\alpha$ (see [16]), we have $T_{\eta,i}\Phi_\alpha = \left[\frac{\eta-1}{2}(-i)^{|\alpha|} + \frac{\eta+1}{2}i^{|\alpha|}\right]\Phi_\alpha$. Calculating the coefficient on the right-hand side of this equality, we obtain (2). □

It is worth remarking that if $\eta = \pm i$, then the eigenvalues are just $\pm\, i$. This case of $T_{\eta,i}$ corresponds to the well-known Hartley integral transform $\mathcal{H}$ with the kernel $\mathrm{cas}(xy)$ scaled by the constant i, which is an involution operator in $L^2(\mathbb{R}^n)$ as $\mathcal{H}^2 = I$ (see [1]). As the reader will see and may make some comparison between the polynomial identity in Theorem 3, the 2-involution of $\mathcal{H}$, and the 4-involution of the Fourier operator $F^4 = I$ (and also between almost all the statements in this work and those of the well-known Hartley and Fourier transforms), the Hartley integral transform is a very special case in some sense. Hence, for simplicity and to avoid eventual confusion, throughout this paper we will assume that $\eta \notin \{0, \pm i\}$.

The following auxiliary lemma is very useful for proving some of our further theorems.

Lemma 1 (cf. [16]) *The formula*

$$\frac{1}{2}\{f(x+0) + f(x-0)\} = \lim_{\lambda\to\infty}\frac{1}{\pi}\int_{-\infty}^{+\infty} f(t)\frac{\sin(\lambda(x-t))}{x-t}\,dt$$

holds, provided $\frac{f(x)}{1+|x|} \in L^1(\mathbb{R})$.

Theorem 2 (Riemann-Lebesgue lemma) *$T_{\eta,i}$ is a bounded linear operator from $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$. Namely, if $f \in L^1(\mathbb{R}^n)$, then $T_{\eta,i}f \in C_0(\mathbb{R}^n)$ and*

$$\|T_{\eta,i}f\|_\infty \le \frac{\delta_0}{(2\pi)^{\frac{n}{2}}}\|f\|_1, \quad with\ \ \delta_0 := \sqrt{\frac{|\eta|^2+1}{2} + \left[\left(\frac{|\eta|^2-1}{2}\right)^2 + \Im(\eta)^2\right]^{\frac{1}{2}}}. \tag{3}$$

Proof By the Riemann–Lebesgue lemma for cosine and sine Fourier transforms T_c and T_s, as defined above, we deduce that if $f \in L^1(\mathbb{R}^n)$, then $T_{\eta,i}f \in C_0(\mathbb{R}^n)$. We shall prove the norm inequality. Let us write $\eta = \eta_1 + i\eta_2$, with $\eta_1, \eta_2 \in \mathbb{R}$. By using the identity

$$\begin{aligned} |\eta\cos(xy) + i\sin(xy)|^2 &= (\eta_1^2 + \eta_2^2 - 1)\cos^2(xy) + 1 + \eta_2\sin(2xy) \\ &= \frac{(\eta_1^2+\eta_2^2-1)(1+\cos(2xy))}{2} + 1 + \eta_2\sin(2xy) = \frac{|\eta|^2+1}{2} + \left[\left(\frac{|\eta|^2-1}{2}\right)^2 + \Im(\eta)^2\right]^{\frac{1}{2}}\cos(2xy-\theta), \end{aligned} \tag{4}$$

for some $\theta \in \mathbb{R}$, we deduce that $|\eta\cos(xy) + i\sin(xy)| \le \delta_0$. We then have

$$\|T_{\eta,i}f\|_\infty = \sup_{x\in\mathbb{R}^n}\left|\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}[\eta\cos(xy) + i\sin(xy)]\,f(y)\,dy\right|$$

$$\leq \sup_{x\in\mathbb{R}^n} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} |\eta\cos(xy) + i\sin(xy)|\ |f(y)|\, dy \leq \frac{\delta_0}{(2\pi)^{\frac{n}{2}}} \|f\|_1. \tag{5}$$

□

Theorem 3 *$T_{\eta,i}$ is a continuous linear operator of $\mathcal{S}$ into itself, and fulfills the reflection and polynomial identities:*

$$T_{\eta,i}^2 = \frac{\eta^2-1}{2} I + \frac{\eta^2+1}{2} W, \quad \text{and} \quad T_{\eta,i}^4 - (\eta^2-1)T_{\eta,i}^2 - \eta^2 I = 0.$$

Moreover, the operator $T_{\eta,i}$ is invertible in $\mathcal{S}$.

Proof Clearly, $T_{\eta,i}$ is a continuous operator in $\mathcal{S}$. We shall prove the polynomial identity. Let us first prove the identity $(T_{\eta,i}^2 f)(x) = \frac{\eta^2-1}{2} f(x) + \frac{\eta^2+1}{2} f(-x)$, for every $f \in \mathcal{S}$. For $\lambda > 0$, consider the n-dimensional box in $\mathbb{R}^n$: $B(0,\lambda) := \{y = (y_1,\dots,y_n) \in \mathbb{R}^n : |y_k| \leq \lambda,\ k = 1,\dots,n\}$. Obviously, $\int_{B(0,\lambda)} \cos(xy)\sin(xv)\, dx = 0$. Acting inductively on n, we obtain

$$\int_{B(0,\lambda)} \cos\left(y(x-t)\right)\, dy = \frac{2^n \sin\left(\lambda(x_1-t_1)\right)\cdots\sin\left(\lambda(x_n-t_n)\right)}{(x_1-t_1)\cdots(x_n-t_n)}. \tag{6}$$

Since $f \in \mathcal{S}$, we may use the Fubini's theorem, Lemma 1, and (6) to calculate $(T_{\eta,i}^2 f)(x)$, with $x \in \mathbb{R}^n$, as follows

$$\begin{aligned}
(T_{\eta,i}^2 f)(x) &= \lim_{\lambda\to\infty} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} [\eta\cos(xy) + i\sin(xy)]\, dy \int_{B(0,\lambda)} [\eta\cos(yt) + i\sin(yt)]\, f(t)\, dt \\
&= \lim_{\lambda\to\infty} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(t) \int_{B(0,\lambda)} \left[\frac{\eta^2+1}{2}\cos\left(y(x+t)\right) + \frac{\eta^2-1}{2}\cos\left(y(x-t)\right) + \eta i \sin\left(y(x+t)\right)\right] dy\, dt \\
&= \lim_{\lambda\to\infty} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} f(t) \int_{B(0,\lambda)} \left[\frac{\eta^2+1}{2}\cos\left(y(x+t)\right) + \frac{\eta^2-1}{2}\cos\left(y(x-t)\right)\right] dy\, dt \\
&= \lim_{\lambda\to\infty} \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left[\frac{\eta^2-1}{2} \frac{2^n \sin\left(\lambda(y_1-t_1)\right)\cdots\sin\lambda(y_n-t_n)}{(y_1-t_1)\cdots(y_n-t_n)}\right. \\
&\left. + \frac{\eta^2+1}{2} \frac{2^n \sin\left(\lambda(y_1+t_1)\right)\cdots\sin\left(\lambda(y_n+t_n)\right)}{(y_1+t_1)\cdots(y_n+t_n)}\right] f(t)\, dt = \frac{\eta^2-1}{2} f(x) + \frac{\eta^2+1}{2} f(-x), \quad x \in \mathbb{R}^n.
\end{aligned} \tag{7}$$

Thus, we have

$$\begin{aligned}
(T_{\eta,i}^4 f)(x) =& T_{\eta,i}^2 \left[(T_{\eta,i}^2 f)(x)\right] \\
=& T_{\eta,i}^2 \left[\frac{\eta^2-1}{2} f(x) + \frac{\eta^2+1}{2} f(-x)\right] \\
=& \frac{\eta^2-1}{2} \left[\frac{\eta^2-1}{2} f(x) + \frac{\eta^2+1}{2} f(-x)\right] + \frac{\eta^2+1}{2} \left[\frac{\eta^2-1}{2} f(-x) + \frac{\eta^2+1}{2} f(x)\right] \\
=& \frac{\eta^4+1}{2} f(x) + \frac{\eta^4-1}{2} f(-x), \quad x \in \mathbb{R}^n.
\end{aligned} \tag{8}$$

Combining (7) and (8), we get $T_{\eta,i}^4 f - (\eta^2-1)T_{\eta,i}^2 f - \eta^2 f = 0$, for every $f \in \mathcal{S}$. The polynomial identity for $T_{\eta,i}$ is proved.

Moreover, by this polynomial identity, we obtain $T_{\eta,i}\left[\frac{1}{\eta^2}T^3_{\eta,i}-\frac{\eta^2-1}{\eta^2}T_{\eta,i}\right]=\left[\frac{1}{\eta^2}T^3_{\eta,i}-\frac{\eta^2-1}{\eta^2}T_{\eta,i}\right]T_{\eta,i}=I$, which implies that $T_{\eta,i}$ is invertible by $T^{-1}_{\eta,i}=\frac{1}{\eta^2}T^3_{\eta,i}-\frac{\eta^2-1}{\eta^2}T_{\eta,i}$. □

Actually, the inversion formula of $T_{\eta,i}$ can be expressed in an explicit way by Theorem 4.

Theorem 4 (inversion and uniqueness theorem) *If $f\in L^1(\mathbb{R}^n)$, and if $T_{\eta,i}f\in L^1(\mathbb{R}^n)$, then*

$$\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(T_{\eta,i}f)(y)\left[\frac{1}{\eta}\cos(xy)-i\sin(xy)\right]dy=f(x), \tag{9}$$

for almost every $x\in\mathbb{R}^n$. Consequently, if $f\in L^1(\mathbb{R}^n)$ and if $T_{\eta,i}f=0$, then $f=0$.

Proof Let us first prove the inversion formula in $\mathcal{S}$, i.e.

$$g(x)=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(T_{\eta,i}g)(y)\left[\frac{1}{\eta}\cos(xy)-i\sin(xy)\right]dy,\quad \text{for every } x\in\mathbb{R}^n,\quad g\in\mathcal{S}. \tag{10}$$

Indeed, since $T_{\eta,i}g\in\mathcal{S}$, the inner function on the right-hand side of (10) belongs to $\mathcal{S}$. This means that the integral (10) is uniformly convergent on $\mathbb{R}^n$ according to each variable $x_1,\dots,x_d$. Let us calculate the right-hand side of (10), using Fubini's theorem, Lemma 2 and (6). Thus, we have

$$\begin{aligned}
\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(T_{\eta,i}g)(y)\left[\frac{1}{\eta}\cos(xy)-i\sin(xy)\right]dy \;&=\; \lim_{\lambda\to\infty}\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{B(0,\lambda)}(T_{\eta,i}g)(y)\left[\frac{1}{\eta}\cos(xy)-i\sin(xy)\right]dy\\
&=\lim_{\lambda\to\infty}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\left[\frac{1}{\eta}\cos(xy)-i\sin(xy)\right]\int_{B(0,\lambda)}\left[\eta\cos(yt)+i\sin(yt)\right]g(t)\,dtdy\\
&=\lim_{\lambda\to\infty}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}g(t)\int_{B(0,\lambda)}\left[\frac{1}{\eta}\cos(xy)-i\sin(xy)\right]\left[\eta\cos(yt)+i\sin(yt)\right]dy\,dt\\
&=\lim_{\lambda\to\infty}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}g(t)\left(\int_{B(0,\lambda)}(\cos y(x-t))\,dy\,dt\right)\\
&=\frac{1}{(2\pi)^n}\lim_{\lambda\to\infty}\int_{\mathbb{R}^n}g(t)\frac{2^n\sin(\lambda(x_1-t_1))\cdots\sin(\lambda(x_n-t_n))}{(x_1-t_1)\cdots(x_n-t_n)}dt=g(x),
\end{aligned}$$

for every $x\in\mathbb{R}^n$ (having in mind that $g\in\mathcal{S}$). Identity (10) is proved. Now let $f\in L^1(\mathbb{R}^n)$, and let $g\in\mathcal{S}$. Clearly, $\int_{\mathbb{R}^n}f(x)(T_{\eta,i}g)(x)dx=\int_{\mathbb{R}^n}g(y)(T_{\eta,i}f)(y)dy$. Hence,

$$\begin{aligned}
\int_{\mathbb{R}^n}f(x)(T_{\eta,i}g)(x)\,dx&=\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n}(T_{\eta,i}g)(x)\left[\frac{1}{\eta}\cos(xy)-i\sin(xy)\right]dx\right)(T_{\eta,i}f)(y)\,dy\\
&=\int_{\mathbb{R}^n}(T_{\eta,i}g)(x)\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}(T_{\eta,i}f)(y)\left[\frac{1}{\eta}\cos(xy)-i\sin(xy)\right]dy\right)dx=\int_{\mathbb{R}^n}f_0(x)(T_{\eta,i}g)(x)\,dx.
\end{aligned}$$

By (10), the functions $T_{\eta,i}g$ cover all $\mathcal{S}$. Therefore, $\int_{\mathbb{R}^n}(f_0(x)-f(x))\,\Psi(x)\,dx=0$, for every $\Psi\in\mathcal{S}$. Since $\mathcal{S}$ is dense in $L^1(\mathbb{R}^n)$, it follows $f_0(x)-f(x)=0$ for almost every $x\in\mathbb{R}^n$. □

The operator $T_{\eta,i}$ is continuously extended onto $L^2(\mathbb{R}^n)$ and fulfills the following Parseval-type identities that are significantly different from the usual integral transforms.

Theorem 5 (Parseval-type identities) *The following identities hold for any* $f, g \in L^2(\mathbb{R}^n)$*:*

$$\langle T_{\eta,i}f, T_{\eta,i}g\rangle_2 = \frac{|\eta|^2+1}{2}\langle f, g\rangle_2 + \frac{|\eta|^2-1}{2}\langle f, Wg\rangle_2 \tag{11}$$

$$\langle T^{-1}_{\eta,i}f, T^{-1}_{\eta,i}g\rangle_2 = \frac{|\eta|^2+1}{2|\eta|^2}\langle f, g\rangle_2 + \frac{1-|\eta|^2}{2|\eta|^2}\langle f, Wg\rangle_2 \tag{12}$$

$$\langle T_{\eta,i}f, T^{-1}_{\eta,i}g\rangle_2 = \frac{\eta-\bar{\eta}}{2\bar{\eta}}\langle f, g\rangle_2 + \frac{\eta+\bar{\eta}}{2\bar{\eta}}\langle f, Wg\rangle_2. \tag{13}$$

Proof Let us write $T_{\eta,i} = \frac{\eta-1}{2}F + \frac{\eta+1}{2}F^{-1}$. The well-known identities $\langle Ff, g\rangle_2 = \langle f, F^{-1}g\rangle_2$ and $\langle F^{-1}f, g\rangle_2 = \langle f, Fg\rangle_2$, and a straightforward computation yields the just presented identities (11)–(13). □

Remark 1 *If* $\eta \in \mathbb{R}\backslash\{0\}$*, then the identity (13) becomes*

$$\langle T_{\eta,i}f, T^{-1}_{\eta,i}g\rangle_2 = \langle f, Wg\rangle_2. \tag{14}$$

Corollary 1 *The point spectrum of the operator* $T_{\eta,i}$ *defined on the Hilbert space* $L^2(\mathbb{R}^n)$ *is given by* $\sigma = \{\pm\eta, \pm i\}$*. Moreover, if* $|\eta| \neq 1$*, then* $T_{\eta,i}$ *is not a unitary operator.*

Proof For any $\eta \in \mathbb{C}\setminus\{0, \pm i\}$ given, the polynomial $P(t) := t^4 - (\eta^2-1)t^2 - \eta^2$ has four distinct roots within $\{\pm\eta, \pm i\}$. Hence, if $\lambda \notin \{\pm\eta, \pm i\}$, then the inverse operator of $(\lambda I + T_{\eta,i})$ is given by

$$(\lambda I + T_{\eta,i})^{-1} = \frac{\lambda\big[\lambda^2-\eta^2+1\big]I - \big[\lambda^2-\eta^2+1\big]T_{\eta,i} + \lambda T^2_{\eta,i} - T^3_{\eta,i}}{\lambda^4 - (\eta^2-1)\lambda^2 - \eta^2}.$$

The corollary follows from Theorem 1, as $\{\pm\eta, \pm i\}$ are the eigenvalues of $T_{\eta,i}$. Furthermore, if $|\eta| \neq 1$, then the spectrum of $T_{\eta,i}$ does not lie on the unit circle. It implies that the operator $T_{\eta,i}$ is not unitary. □

So, in contrast to the Fourier and Hartley cases, where we may recognize unitary operators defined in $L^2(\mathbb{R}^n)$ (after performing a normalization), the operator $T_{\eta,i}$ is not unitary, provided $|\eta| \neq 1$ (and in such case also cannot be normalized in view to be transformed into a unitary operator; cf. (11)).

3. Convolution and Young-type inequality

In this section, we propose a new convolution operation and prove a Young-type inequality for the constructed convolution.

Theorem 6 *If* $f, g \in L^1(\mathbb{R}^n)$,

$$(f \overset{T_{\eta,i}}{\star} g)(x) := \frac{1}{4\eta(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}\big[(3\eta^2+1)f(x-u) + (\eta^2-1)f(x+u) + (\eta^2-1)f(-x+u) - (\eta^2-1)f(-x-u)\big]\, g(u)du, \tag{15}$$

defines a convolution operation for $T_{\eta,i}$*, which satisfies the following factorization identity and norm inequality:*

$$T_{\eta,i}(f \overset{T_{\eta,i}}{\star} g)(x) = (T_{\eta,i}f)(x)(T_{\eta,i}g)(x); \qquad \|f \overset{T_{\eta,i}}{\star} g\|_1 \leq \frac{3|\eta|^2+2}{2|\eta|(2\pi)^{\frac{n}{2}}}\|f\|_1\|g\|_1. \tag{16}$$

Proof We start by the factorization identity, deducing it in a direct way as

$$
\begin{aligned}
(T_{\eta,i}f)(x)(T_{\eta,i}g)(x) =& \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[\eta\cos(xu)+i\sin(xu)]\,[\eta\cos(xv)+i\sin(xv)]\,f(u)g(v)dudv \\
=& \frac{3\eta^2+1}{4\eta(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[\eta\cos(x(u+v))+i\sin(x(u+v))]\,f(u)g(v)dudv \\
&+\frac{\eta^2-1}{4\eta(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[\eta\cos(x(u-v))+i\sin(x(u-v))]\,f(u)g(v)dudv \\
&+\frac{\eta^2-1}{4\eta(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[\eta\cos(x(-u+v))+i\sin(x(-u+v))]\,f(u)g(v)dudv \\
&-\frac{\eta^2-1}{4\eta(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[\eta\cos(x(-u-v))+i\sin(x(-u-v))]\,f(u)g(v)dudv \qquad (17)\\
=& \frac{3\eta^2+1}{4\eta(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[\eta\cos(xy)+i\sin(xy)]\,f(y-v)g(v)dydv \\
&+\frac{\eta^2-1}{4\eta(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[\eta\cos(xy)+i\sin(xy)]\,f(y+v)g(v)dydv \\
&+\frac{\eta^2-1}{4\eta(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[\eta\cos(xy)+i\sin(xy)]\,f(-y+v)g(v)dydv \\
&-\frac{\eta^2-1}{4\eta(2\pi)^n}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}[\eta\cos(xy)+i\sin(xy)]\,f(-y-v)g(v)dydv \\
=& \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}[\eta\cos(xy)+i\sin(xy)]\,(f\overset{T_{\eta,i}}{\star}g)(y)dy = T_{\eta,i}(f\overset{T_{\eta,i}}{\star}g)(x),
\end{aligned}
$$

which proves the factorization identity. The norm inequality is a special case of the possibilities of choice of p,q,r in Theorem 7, as showed by Corollary 2 below. □

The decomposition of the trigonometric kernel as in (17) takes a preponderant role in the above proof of the factorization property. This technique continues to be helpful in several proofs below.

Similarly to the Young-type inequality based on the Fourier transform, we will also obtain a consequent result associated with our convolution for the $T_{\eta,i}$ transform.

Theorem 7 (Young-type inequality, see [2, 11, 15]) *The convolution* $\overset{T\eta,i}{\star}$*, given in (15), is a continuous bilinear map between suitable* $L^s(\mathbb{R}^n)$ *spaces in the sense that, if* $1 \le p,q,r \le \infty$ *satisfy*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1,$$

then

$$
\begin{aligned}
&\overset{T_{\eta,i}}{\star}:\ L^p(\mathbb{R}^n)\times L^q(\mathbb{R}^n)\longrightarrow L^r(\mathbb{R}^n),\\
&\|f\overset{T_{\eta,i}}{\star}g\|_r \le \frac{3|\eta|^2+2}{2|\eta|(2\pi)^{\frac{n}{2}}}\|f\|_p\|g\|_q, \qquad (18)
\end{aligned}
$$

for any $f\in L^p(\mathbb{R}^n)$*,* $g\in L^q(\mathbb{R}^n)$*.*

Proof We have two different cases associated with the parameter r.
Case 1: $1 \leq r < \infty$. Put

$$M(x) := \int_{\mathbb{R}^n} f(x-u)g(u)du; \quad N(x) := \int_{\mathbb{R}^n} f(x+u)g(u)du;$$
$$P(x) := \int_{\mathbb{R}^n} f(-x+u)g(u)du; \quad Q(x) := \int_{\mathbb{R}^n} f(-x-u)g(u)du.$$

The convolution (15) contains four terms M, N, P and Q, and each one of those is scaled by a constant. By the Minkowski inequality it suffices to prove the Young inequality for each term. Indeed, if $f \in L^p(\mathbb{R}^n)$, $g \in L^q(\mathbb{R}^n)$,

$$N(x) = \int_{\mathbb{R}^n} f(x+u)g(u)du = \int_{\mathbb{R}^n} f(x-u)\widetilde{g}(u)du = (f \overset{F}{*} \widetilde{g})(x),$$

where $(\cdot \overset{F}{*} \cdot)$ stands for the usual Fourier convolution. Clearly, $\widetilde{g} \in L^q(\mathbb{R}^n)$, and $\|\widetilde{g}\|_q = \|g\|_q$ for any $q \geq 1$. By the known Young's convolution inequality, $\|N(x)\|_r = \|f \overset{F}{*} \widetilde{g}\|_r \leq \|f\|_p\|\widetilde{g}\|_q = \|f\|_p\|g\|_q$. Similarly,

$$\|M(x)\|_r = \|(f \overset{F}{*} g)(x)\|_r \leq \|f\|_p\|g\|_q,$$
$$\|P(x)\|_r = \|(f \overset{F}{*} \widetilde{g})(-x)\|_r = \|(f \overset{F}{*} \widetilde{g})(x)\|_r \leq \|f\|_p\|\widetilde{g}\|_q = \|f\|_p\|g\|_q,$$
$$\|Q(x)\|_r = \|(f \overset{F}{*} g)(-x)\|_r = \|(f \overset{F}{*} g)(x)\|_r \leq \|f\|_p\|g\|_q.$$

Case 2: $r = \infty$. By the Hölder inequality with $1/p + 1/q = 1$, we have

$$\|h_{\pm}(\pm x)\|_\infty \leq \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \int |f(\pm x \pm y)||g(y)|\, dy \leq \operatorname{ess\,sup}_{x\in\mathbb{R}^n} \|f(\pm x \pm y)\|_p\|g\|_q = \|f\|_p\|g\|_q.$$

Now, the result follows from the Minkowski inequality. □

Corollary 2 *The convolution operation given by (15) is continuous in the Banach space $L^1(\mathbb{R}^n)$. Namely, we have*

$$\|f \overset{T_{\eta,i}}{\star} g\|_1 \leq \frac{3|\eta|^2 + 2}{2|\eta|(2\pi)^{\frac{n}{2}}}\|f\|_1\|g\|_1.$$

The norm inequality (16) in Theorem 6 has been proved by Theorem 7 with $p = q = r = 1$.

Theorem 8 *The space $\mathcal{X} := L^1(\mathbb{R}^n)$, equipped with the convolution multiplication (15), becomes a commutative algebra without unit.*

Proof By Theorem 6, we derive that $L^1(\mathbb{R}^n)$, equipped with the convolution multiplication (15), has a commutative ring structure. In addition, we have the multiplicative inequality in Corollary 2.

What remains is to prove that $\mathcal{X}$ has no unit. Suppose that there exits an element $e \in \mathcal{X}$ such that $f = f \overset{T_{\eta,i}}{\star} e = e \overset{T_{\eta,i}}{\star} f$, for every $f \in \mathcal{X}$. We choose the Hermite function $\Phi_0(x) := e^{-\frac{1}{2}|x|^2} \in L^1(\mathbb{R}^n)$ for which $T_{\eta,i}\Phi_0 = \eta\Phi_0$, by Theorem 1. Using the factorization identity and the fact that $\Phi_0 = \Phi_0 \overset{T_{\eta,i}}{\star} e$, we obtain

$T_{\eta,i}(\Phi_0) = T_{\eta,i}(\Phi_0)T_{\eta,i}(e)$ or, equivalently, $\eta\Phi_0 = \eta\Phi_0 T_{\eta,i}(e)$. Since $\Phi_0(x) \neq 0$, for every $x \in \mathbb{R}^n$, we derive $(T_{\eta,i}e)(x) = 1$, for every $x \in \mathbb{R}^n$, which contradicts the fact that $\lim_{|x|\to\infty}(T_{\eta,i}e)(x) = 0$, which is deduced from the Riemann-Lebesgue lemma as showed by Theorem 2. Hence, $\mathcal{X}$ has no unit. □

4. Paley–Wiener theorems

This section is devoted to some Paley-Wiener theorems. As it is well-known, these type of theorems have different applications such as in the theory of differential equations (see, for example, [13, Chapter 8]), and in harmonic analysis, e.g., in the sense of uncertainty principles; namely, Corollary 3 below can be considered to be an elementary form of that principle: the Fourier transform of a nonzero function (or, a nonzero distribution) of compact support is never compactly supported (see also [5]).

Theorem 9 (Paley–Wiener theorem) (i) *If $\phi \in \mathcal{D}(\mathbb{R}^n)$ has its support in $B(0,R)$, $R > 0$, then the function defined as follows*

$$g(z) := (T_{\eta,i}\phi)(z) = \int_{\mathbb{R}^n} (\eta\cos(yz) + i\sin(yz))\phi(z)\,dy, \quad z \in \mathbb{C}^n \tag{19}$$

is entire and there are constants $\gamma_p < \infty$ such that

$$|g(z)| \leq \gamma_p(1 + |z|)^{-p} e^{R|\Im(z)|}, \tag{20}$$

for all $z \in \mathbb{C}^n$ and for every $p = 0, 1, 2, \ldots$.
(ii) *Conversely, assume that g is an entire function satisfying (20) with $R > 0$, for all $z \in \mathbb{C}^n$ and every $p = 0, 1, 2, \ldots$, then there exists a function $\phi \in \mathcal{D}(\mathbb{R}^n)$, supported in $\overline{B(0,R)}$, such that (19) holds.*

Proof (i) To prove this theorem, we will use a similar result for the Fourier transform because our operator can be written in terms of the Fourier transform and its inverse. In fact, we have

$$g_1(z) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{-iyz}\phi(y)\,dy = (F\phi)(z),$$

and

$$g_2(z) := \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} e^{iyz}\phi(y)\,dy = (F^{-1}\phi)(z) \quad (\text{with } z \in \mathbb{C}^n)$$

are entire functions, and there are constants $\gamma_p^{(1)}, \gamma_p^{(2)} < \infty$ such that

$$|g_1(z)| \leq \gamma_p^{(1)}(1 + |z|)^{-p} e^{R|\Im(z)|},$$

and

$$|g_2(z)| \leq \gamma_p^{(2)}(1 + |z|)^{-p} e^{R|\Im(z)|},$$

for every $p = 0, 1, 2, \ldots$ and for all $z \in \mathbb{C}^n$. As $T_{\eta,i} = \frac{\eta-1}{2}F + \frac{\eta+1}{2}F^{-1}$, we have that

$$g(z) = \frac{\eta - 1}{2}g_1(z) + \frac{\eta + 1}{2}g_2(z) = (T_{\eta,i}\phi)(z)$$

is an entire function for which

$$|g(z)| \leq \left|\frac{\eta-1}{2}\gamma_p^{(1)} + \frac{\eta+1}{2}\gamma_p^{(2)}\right| (1+|z|)^{-p} e^{R|\Im(z)|},$$

for every $p = 0, 1, 2, \ldots$ and for all $z \in \mathbb{C}^n$. For the proof of the item (ii), we can use a similar reasoning, because of the existence of the inverse of the operator, which ensures the existence of such a function ϕ. □

Let $C_c^\infty(\mathbb{R}^n)$ denote the set of all infinitely differentiable functions with compact support.

Corollary 3 *Let $f \in C_c^\infty(\mathbb{R}^n)$. If $T_{\eta,i}f$ has compact support, then $f \equiv 0$.*

Proof By contradiction, let us suppose that f is a nonzero infinitely differentiable function with compact support and that the support of $T_{\eta,i}f$ is also compact. Thus, there exists an $R > 0$ such that $f(x) = (T_{\eta,i}f)(x) = 0$ for all $|x| > R$. By Theorem 9, $T_{\eta,i}f$ can be extended to an entire holomorphic function on $\mathbb{C}^n$ (see [13]). By the uniqueness theorem of holomorphic functions, for a nonzero holomorphic function g on an open domain D, if $g(z_0) = 0$, then there exists a $\delta > 0$ such that $g(z) \neq 0$, for any $z \in B(z_0, \delta)$ (except for z_0, of course). Since $(T_{\eta,i}f)(2R) = 0$ and, for any $\delta > 0$, we have $2R + \delta/2 \in B(2R, \delta)$ and $(T_{\eta,i}f)(2R + \delta/2) = 0$. Thus, we deduce that $T_{\eta,i}f = 0$ on all $\mathbb{C}^n$, and therefore also on all $\mathbb{R}^n$. By the uniqueness theorem of $T_{\eta,i}$, $f(x) = 0$ for almost every $x \in \mathbb{R}^n$. Since $f \in C_c^\infty(\mathbb{R}^n)$, $f = 0$ everywhere on $\mathbb{R}^n$ – which contradicts the initial assumption. □

We will denote by $C_0^m(\mathbb{R}^n)$ the space of all m-differentiable complex-valued functions on $\mathbb{R}^n$ that vanish at infinity. The relation between the differential operator and the convolution multiplication is given by the following proposition.

Proposition 1 *Let $1 \leq p \leq \infty$ and let $0 \leq m \leq \infty$. If $f \in L^p(\mathbb{R}^n)$ and $g \in C_0^m(\mathbb{R}^n)$, then $f \overset{T_{\eta,i}}{\star} g \in C_0^m(\mathbb{R}^n)$ and*

$$D^k(f \overset{T_{\eta,i}}{\star} g) = f \overset{T_{\eta,i}}{\star} (D^k g), \quad \text{for any multi-index } k \in \mathbb{N}^n \text{ with } \ |k| \leq m. \tag{21}$$

Proof We know that

$$\lim_{|x|\to\infty} \int_{\mathbb{R}^n} f(\pm x \pm u) g(u)\, du = 0.$$

Thus, $f \overset{T_{\eta,i}}{\star} g \in C_0(\mathbb{R}^n) := C_0^0(\mathbb{R}^n)$. Let $q = p/(p-1)$ be the conjugate exponent of p, and for a given function g, let $(\tau_h g)(x) = g(x-h)$ denote the translation by $h \in \mathbb{R}^n$.

Consider $k = 0$. For $h \in \mathbb{R}^n$, by changing variables and then applying the Hölder inequality, we have

$$\begin{aligned}\left|(f \overset{T_{\eta,i}}{\star} g)(x+h) - (f \overset{T_{\eta,i}}{\star} g)(x)\right| \leq & \frac{1}{4|\eta|(2\pi)^{\frac{n}{2}}} \left[|3\eta^2+1| \int_{\mathbb{R}^n} |f(x-u)\,(g(u+h)-g(u))\, du| \right.\\ & + |\eta^2-1| \int_{\mathbb{R}^n} |f(x+u)\,(g(u-h)-g(u))|\, du \\ & + |\eta^2-1| \int_{\mathbb{R}^n} |f(-x+u)\,(g(u+h)-g(u))|\, du \\ & \left. + |\eta^2-1| \int_{\mathbb{R}^n} |f(-x-u)\,(g(u-h)-g(u))|\, du \right]\end{aligned}$$

$$\leq \frac{1}{4|\eta|(2\pi)^{\frac{n}{2}}}\left[|3\eta^2+1|\left(\int_{\mathbb{R}^n}|f(x-u)|^p\;du\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}|g(u+h)-g(u)|^q\;du\right)^{\frac{1}{q}}\right.$$
$$+|\eta^2-1|\left(\int_{\mathbb{R}^n}|f(x+u)|^p\;du\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}|g(u-h)-g(u)|^q\;du\right)^{\frac{1}{q}}$$
$$+|\eta^2-1|\left(\int_{\mathbb{R}^n}|f(-x+u)|^p\;du\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}|g(u+h)-g(u)|^q\;du\right)^{\frac{1}{q}}$$
$$\left.+|\eta^2-1|\left(\int_{\mathbb{R}^n}|f(-x-u)|^p\;du\right)^{\frac{1}{p}}\left(\int_{\mathbb{R}^n}|g(u-h)-g(u)|^q\;du\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{1}{4|\eta|(2\pi)^{\frac{n}{2}}}\left[(4|\eta|^2+2)\|f\|_p\|\tau_{-h}g-g\|_q+(2|\eta|^2+2)\|f\|_p\|\tau_h g-g\|_q\right].$$

Case 1: $q=\infty$. The terms $\|\tau_{\pm h}g-g\|_\infty$ tend to zero when $h\to 0$ since g is uniformly continuous.

Case 2: $q<\infty$. We know that the translations are continuous in $L^p(\mathbb{R}^n)$ in the sense that, for all $g\in L^p(\mathbb{R}^n)$, we have $\|\tau_h g-g\|_p\to 0$ as $h\to 0\in\mathbb{R}^n$.

Combining these two cases, we have $\left|(f\overset{T_{\eta,i}}{\star}g)(x+h)-(f\overset{T_{\eta,i}}{\star}g)(x)\right|\to 0$, as $h\to 0$. Thus, $f\overset{T_{\eta,i}}{\star}g\in C_0(\mathbb{R}^n)$.

We now prove (21) for the case $|k|=1$. Let $e_j:=(0,\dots,1,\dots,0)$ denote the j-th unit vector of $\mathbb{R}^n$ and let $t>0$ be given. Using some changes of variables and applying the mean-value theorem, we have that there are constants $s_1,s_2,s_3,s_4\in[0,t]$ such that

$$(f\overset{T_{\eta,i}}{\star}g)(x+te_j)-(f\overset{T_{\eta,i}}{\star}g)(x)=\frac{1}{4\eta(2\pi)^{\frac{n}{2}}}\left[(3\eta^2+1)\int_{\mathbb{R}^n}(g(x+te_j-u)-g(x-u))\,f(u)\,du\right.$$
$$+(\eta^2-1)\int_{\mathbb{R}^n}(g(x+te_j+u)-g(x+u))\,f(u)\,du$$
$$+(\eta^2-1)\int_{\mathbb{R}^n}(g(-x-te_j+u)-g(-x+u))\,f(u)\,du$$
$$\left.-(\eta^2-1)\int_{\mathbb{R}^n}(g(-x-te_j-u)-g(-x-u))\,f(u)\,du\right]$$
$$=\frac{1}{4\eta(2\pi)^{\frac{n}{2}}}\left[(3\eta^2+1)\int_{\mathbb{R}^n}(g(x+te_j-u)-g(x-u))\,f(u)\,du\right.$$
$$+(\eta^2-1)\int_{\mathbb{R}^n}(g(x+te_j+u)-g(x+u))\,f(u)\,du$$
$$+(\eta^2-1)\int_{\mathbb{R}^n}(\widetilde{g}(x+te_j-u)-\widetilde{g}(x-u))\,f(u)\,du$$
$$\left.-(\eta^2-1)\int_{\mathbb{R}^n}(\widetilde{g}(x+te_j+u)-\widetilde{g}(x+u))\,f(u)\,du\right]$$
$$=\frac{1}{4\eta(2\pi)^{\frac{n}{2}}}\left[(3\eta^2+1)t\int_{\mathbb{R}^n}f(u)\frac{\partial}{\partial x_j}g(x-u+s_1e_j)\,du\right.$$

$$+(\eta^2-1)t\int_{\mathbb{R}^n} f(u)\frac{\partial}{\partial x_j}g(x+u+s_2e_j)\,du$$
$$+(\eta^2-1)t\int_{\mathbb{R}^n} f(u)\frac{\partial}{\partial x_j}\widetilde{g}(x-u+s_3e_j)\,du$$
$$-(\eta^2-1)t\int_{\mathbb{R}^n} f(u)\frac{\partial}{\partial x_j}\widetilde{g}(x+u+s_4e_j)\,du\Bigg].$$

By hypothesis, $\frac{\partial}{\partial x_j}g, \frac{\partial}{\partial x_j}\widetilde{g} \in C_0(\mathbb{R}^n)$. By the above proof (for $k=0$), we derive that each term on the right-hand side of the last identity belongs to $C_0(\mathbb{R}^n)$. Then,

$$\begin{aligned}
\frac{\partial}{\partial x_j}(f \overset{T_{\eta,i}}{\star} g)(x) &= \lim_{t\to 0}\frac{(f \overset{T_{\eta,i}}{\star} g)(x+te_j)-(f \overset{T_{\eta,i}}{\star} g)(x)}{t}\\
&= \frac{1}{4\eta(2\pi)^{\frac{n}{2}}}\Bigg[(3\eta^2+1)\int_{\mathbb{R}^n} f(u)\frac{\partial}{\partial x_j}g(x-u)\,du+(\eta^2-1)\int_{\mathbb{R}^n} f(u)\frac{\partial}{\partial x_j}g(x+u)\,du\\
&\quad +(\eta^2-1)\int_{\mathbb{R}^n} f(u)\frac{\partial}{\partial x_j}g(-x+u)\,du-(\eta^2-1)\int_{\mathbb{R}^n} f(u)\frac{\partial}{\partial x_j}g(-x-u)\,du\Bigg]\\
&= \frac{1}{4\eta(2\pi)^{\frac{n}{2}}}\Bigg[(3\eta^2+1)\int_{\mathbb{R}^n} f(x-u)\frac{\partial}{\partial x_j}g(u)\,du+(\eta^2-1)\int_{\mathbb{R}^n} f(x+u)\frac{\partial}{\partial x_j}g(u)\,du\\
&\quad +(\eta^2-1)\int_{\mathbb{R}^n} f(-x+u)\frac{\partial}{\partial x_j}(u)\,du-(\eta^2-1)\int_{\mathbb{R}^n} f(-x-u)\frac{\partial}{\partial x_j}\widetilde{g}(u)\,du\Bigg]\\
&= \left(f \overset{T_{\eta,i}}{\star} \frac{\partial}{\partial x_j}g\right)(x).
\end{aligned}$$

This implies that $f \overset{T_{\eta,i}}{\star} g \in C_0^1(\mathbb{R}^n)$. Identity (21) follows now by induction on $|k|$. □

Proposition 2 *If $f,g\in L^1(\mathbb{R}^n)$, then we have $f \overset{T_{\eta,i}}{\star} g = 0$ on*

$$\Omega^c = \left[\overline{(\mathrm{supp}(f)+\mathrm{supp}(g))}\right]^c \cap \left[\overline{(\mathrm{supp}(f)+\mathrm{supp}(-g))}\right]^c \cap \left[\overline{(\mathrm{supp}(-f)+\mathrm{supp}(g))}\right]^c \cap \left[\overline{(\mathrm{supp}(-f)+\mathrm{supp}(-g))}\right]^c.$$

In particular, $\mathrm{supp}(f \overset{T_{\eta,i}}{\star} g) \subseteq \Omega$.

Proof We will analyze each of the four integral terms in the definition of our convolution $\overset{T_{\eta,i}}{\star}$. For the first and second ones, we note that

$$\int_{\mathbb{R}^n} f(x\mp u)g(u)\,du = \int_{\{(x\mp \mathrm{supp}(f))\cap \mathrm{supp}(g)\}} f(x\mp u)g(u)\,du, \quad x\in\mathbb{R}^n.$$

If $x\notin \mathrm{supp}(f)\pm\mathrm{supp}(g)$, then $f(x\mp u)=0$ for any $u\in\mathrm{supp}(g)$. It follows

$$\int_{\{(x\mp \mathrm{supp}(f))\cap \mathrm{supp}(g)\}} f(x\mp u)g(u)\,dy = 0,$$

for any $x \in (\operatorname{supp}(f) \pm \operatorname{supp}(g))^c$. Hence,

$$\operatorname{supp}\left(\int_{\mathbb{R}^n} f(x \mp u)g(u)\,du\right) \subseteq \overline{\operatorname{supp}(f) \pm \operatorname{supp}(g)}.$$

Similarly, for the third and fourth integrals, we have

$$\int_{\mathbb{R}^n} f(-x \pm u)g(u)\,du = \int_{\operatorname{supp}(g)} f(-x \pm u)g(u)\,du, \quad x \in \mathbb{R}^n.$$

If $x \notin -\operatorname{supp}(f) \pm \operatorname{supp}(g)$, then $f(-x \pm u) = 0$, for any $u \in \operatorname{supp}(g)$. This implies

$$\int_{\operatorname{supp}(g)} f(-x \pm u)g(u)\,dy = 0,$$

for any $x \in (-\operatorname{supp}(f) \pm \operatorname{supp}(g))^c$. Therefore,

$$\operatorname{supp}\left(\int_{\mathbb{R}^n} f(-x \pm u)g(u)\,du\right) \subseteq \overline{-\operatorname{supp}(f) \pm \operatorname{supp}(g)}.$$

We deduce that $\operatorname{supp}(f \overset{T_{\eta,i}}{\star} g) \subseteq \Omega$. Consequently, $f \overset{T_{\eta,i}}{\star} g = 0$ on Ω^c. □

Remark 2 *For elements f and g such that* $\operatorname{supp}(f) = -\operatorname{supp}(f)$ *and* $\operatorname{supp}(g) = -\operatorname{supp}(g)$*, the last result coincides with the known one for the classical Fourier convolution.*

5. Wiener's Tauberian theorems

The Wiener's theorem obtained in 1932 states that the closed linear hull of translations of a function $f \in L^1(\mathbb{R})$ is the whole space $L^1(\mathbb{R})$ if and only if its Fourier transform never vanishes, i.e. $(Ff)(x) \neq 0$, for every $x \in \mathbb{R}^n$. This theorem, together with the generalizations of Gelfand for normed rings and Banach algebras, plays an important role in many fields of mathematics (and, in particular, has direct consequences in Tauberian theorems).

In this section, we will prove some versions of Wiener's Tauberian [12, 13] theorems related with $T_{\eta,i}$ and the convolution constructed previously. The next two lemmas below are typical features of the integral operator $T_{\eta,i}$ and will be also helpful to prove the forthcoming Wiener-type Tauberian theorems.

Lemma 2 (local constant approximation) *Suppose that $f \in L^1(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$ and $\epsilon > 0$ are given. Then, there exists an $h \in L^1(\mathbb{R}^n)$, with $\|h\|_1 < \epsilon$, such that*

$$(T_{\eta,i}h)(x) = (T_{\eta,i}f)(x_0) - (T_{\eta,i}f)(x), \tag{22}$$

for all x in some neighborhood of x_0.

In other words, an arbitrary function $f \in L^1(\mathbb{R}^n)$ can be approximated by a function $f + h$ such that $T_{\eta,i}(f + h)$ is constant in a neighborhood of a point $x_0 \in \mathbb{R}^n$.

Proof We shall present a proof of the sufficiently explicit approximation of h in some sense. We are able to freely choose a one-variable bump function $\theta(x) \in \mathcal{S}$ which satisfies the condition

$$\theta(x) = \theta(-x) = \begin{cases} 1, & \text{if} \quad |x| < 1 \\ 0, & \text{if} \quad |x| > 2. \end{cases}$$

A bump function in n variables is obtained by taking the product of n copies of the above bump function in one variable; thus,

$$\theta(x_1, x_2, \ldots, x_n) = \theta(x_1)\theta(x_2)\cdots\theta(x_n).$$

By [3, Theorem 4], $T_{\eta,i}$ is an invertible operator in $\mathcal{S}$. Thus, the function g_* given by

$$\begin{aligned} g_*(x) := (T^{-1}_{\eta,i}\theta)(x) =& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left[\frac{1}{\eta}\cos(xy) - i\sin(xy)\right] \theta(y)dy \\ =& \frac{1}{\eta(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \cos(xy)\theta(y)dy \quad (\text{as } \theta(y) = \theta(-y)) \end{aligned} \tag{23}$$

belongs to $\mathcal{S} \subset L^1(\mathbb{R}^n)$, with $(T_{\eta,i}g_*)(x) = \theta(x) = 1$ on $B(0,1)$. For short, we put

$$g_\lambda(x) := \lambda^{-n} \left[\eta\cos(x_0x) + i\sin(x_0x)\right] g_*(\lambda^{-1}x),$$

which is also in $\mathcal{S}$, for any $\lambda > 1$. Changing the variable $\lambda^{-1}y := y$ and decomposing the kernel as in (17), we obtain

$$\begin{aligned} (T_{\eta,i}g_\lambda)(x) =& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left[\eta\cos(x\lambda y) + i\sin(x\lambda y)\right]\left[\eta\cos(x_0\lambda y) + i\sin(x_0\lambda y)\right] g_*(y)dy \\ =& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left\{ \frac{3\eta^2+1}{4\eta} \left[\eta\cos(\lambda y(x+x_0)) + i\sin(\lambda y(x+x_0))\right] \right. \\ &+ \frac{\eta^2-1}{4\eta}\left[\eta\cos(\lambda y(x-x_0)) + i\sin(\lambda y(x-x_0))\right] \\ &+ \frac{\eta^2-1}{4\eta}\left[\eta\cos(\lambda y(-x+x_0)) + i\sin(\lambda y(-x+x_0))\right] \\ &\left. - \frac{\eta^2-1}{4\eta}\left[\eta\cos(\lambda y(-x-x_0)) + i\sin(\lambda y(-x-x_0))\right] \right\} g_*(y)\,dy \\ =& \frac{3\eta^2+1}{4\eta}\left(T_{\eta,i}g_*\right)(\lambda(x+x_0)) + \frac{\eta^2-1}{4\eta}\left(T_{\eta,i}g_*\right)(\lambda(x-x_0)) + \frac{\eta^2-1}{4\eta}\left(T_{\eta,i}g_*\right)(\lambda(-x+x_0)) \\ &- \frac{\eta^2-1}{4\eta}\left(T_{\eta,i}g_*\right)(\lambda(-x-x_0)) \quad (\text{as } T_{\eta,i}g_*(x) = \theta(x) = 1 \text{ on } B(0,1)) \\ \equiv& \begin{cases} \eta & \text{on} \;\; B(0,1/\lambda), \;\; \text{if} \;\; x_0 = 0, \lambda > 1 \\ \frac{\eta^2-1}{2\eta} & \text{on} \;\; B(x_0,1/\lambda), \;\; \text{if} \;\; x_0 \neq 0, \lambda^{-1} < \|x_0\|. \end{cases} \end{aligned}$$

We now consider $h_\lambda(x) := (T_{\eta,i}f)(x_0)g_\lambda(x) - (g_\lambda \overset{T_{\eta,i}}{\star} f)(x)$, which is a function in $L^1(\mathbb{R}^n)$ by Theorem 6. Corresponding to the above two cases of x_0, we deduce

$$(T_{\eta,i}h_\lambda)(x) = (T_{\eta,i}f)(x_0)(T_{\eta,i}g_\lambda)(x) - (T_{\eta,i}f)(x)(T_{\eta,i}g_\lambda)(x)$$

$$= \begin{cases} \eta\left[(T_{\eta,i}f)(0) - (T_{\eta,i}f)(x)\right] & \text{on } B(0,1/\lambda) \text{ if } x_0 = 0,\ \lambda > 1; \\ \frac{\eta^2-1}{2\eta}\left[(T_{\eta,i}f)(x_0) - (T_{\eta,i}f)(x)\right] & \text{on } B(x_0,1/\lambda) \text{ if } x_0 \neq 0, \lambda^{-1} < \|x_0\|, \end{cases}$$

which proves identity (22) with: (i) $\frac{1}{\eta}h_\lambda$ in place of h for $x_0 = 0$ and $x \in B(0,1/\lambda)$; and (ii) $\frac{2\eta}{\eta^2-1}h_\lambda$ in place of h for $x_0 \neq 0,\ \lambda^{-1} < \|x_0\|$.

We will justify that $\|h_\lambda\|_1 \to 0$ as $\lambda \to \infty$. For this purpose, we again split the kernel as in (17) and note that $|\eta\cos(t) + i\sin(t)| \leq \delta_0$ for all $t \in \mathbb{R}$ (as indicated in the proof of Theorem 2), to have

$$\begin{aligned}
|h_\lambda(x)| = &\left|(T_{\eta,i}f)(x_0)g_\lambda(x) - (g_\lambda \overset{T_{\eta,i}}{\star} f)(x)\right| \\
= &\frac{\lambda^{-n}}{4|\eta|(2\pi)^{\frac{n}{2}}}\left|\int_{\mathbb{R}^n}\Big\{(3\eta^2+1)\left[\eta\cos(x_0(x+y)) + i\sin(x_0(x+y))\right]\right. \\
&+ (\eta^2-1)\left[\eta\cos(x_0(x-y)) + i\sin(x_0(x-y))\right] \\
&+ (\eta^2-1)\left[\eta\cos(x_0(-x+y)) + i\sin(x_0(-x+y))\right] \\
&- (\eta^2-1)\left[\eta\cos(x_0(-x-y)) + i\sin(x_0(-x-y))\right]\Big\}g_*(\lambda^{-1}x)f(y)dy \\
&- \int_{\mathbb{R}^n}\Big\{(3\eta^2+1)\left[\eta\cos(x_0(x-y)) + i\sin(x_0(x-y))\right]g_*\left(\frac{x-y}{\lambda}\right) \\
&+ (\eta^2-1)\left[\eta\cos(x_0(x+y)) + i\sin(x_0(x+y))\right]g_*\left(\frac{x+y}{\lambda}\right) \\
&+ (\eta^2-1)\left[\eta\cos(x_0(-x+y)) + i\sin(x_0(-x+y))\right]g_*\left(\frac{-x+y}{\lambda}\right) \\
&\left.-(\eta^2-1)\left[\eta\cos(x_0(-x-y)) + i\sin x_0(-x-y)\right]g_*\left(\frac{-x-y}{\lambda}\right)\Big\}f(y)dy\right| \\
= &\frac{\lambda^{-n}}{4|\eta|(2\pi)^{\frac{n}{2}}}\left|\int_{\mathbb{R}^n}\Big\{(3\eta^2+1)\left[\eta\cos(x_0v) + i\sin(x_0v)\right]g_*\left(\frac{v-y}{\lambda}\right)\right. \\
&+ (\eta^2-1)\left[\eta\cos(x_0v) + i\sin(x_0v)\right]g_*\left(\frac{v+y}{\lambda}\right) \\
&+ (\eta^2-1)\left[\eta\cos(x_0v) + i\sin(x_0v)\right]g_*\left(\frac{y-v}{\lambda}\right) - (\eta^2-1)\left[\eta\cos(x_0v) + i\sin(x_0v)\right]g_*\left(\frac{-y-v}{\lambda}\right) \\
&\left.-4\eta^2\left[\eta\cos(x_0v) + i\sin(x_0v)\right]g_*(v/\lambda)\Big\}f(y)dy\right|.
\end{aligned}$$

Thanks to (23), $g_*(x) = g_*(-x)$ for every $x \in \mathbb{R}^n$. Thus, we have

$$\begin{aligned}
|h_\lambda(x)| = \frac{\lambda^{-n}}{4|\eta|(2\pi)^{\frac{n}{2}}}&\left|\int_{\mathbb{R}^n}\Big\{4\eta^2\left[\eta\cos(x_0v) + i\sin(x_0v)\right]g_*\left(\frac{v-y}{\lambda}\right)\right. \\
&\left.-4\eta^2\left[2\cos(x_0v) + i\sin(x_0v)\right]g_*(v/\lambda)\Big\}f(y)dy\right|.
\end{aligned}$$

Since $|\eta\cos(t)+i\sin(t)|\le\delta_0$ for all $t\in\mathbb{R}$, as shown in the proof of Theorem 2, we obtain

$$\begin{aligned}\|h_\lambda\|_1 \le& \frac{|\eta|\lambda^{-n}}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}|f(y)|\,dy\int_{\mathbb{R}^n}|\eta\cos(x_0v)+i\sin(x_0v)|\,|g_*((v-y)/\lambda)-g_*(v/\lambda)|\,dv\\ \le& \frac{|\eta|\delta_0\lambda^{-n}\|f\|_1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}|g_*((v-y)/\lambda)-g_*(v/\lambda)|\,dv\\ =& \frac{|\eta|\delta_0\|f\|_1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n}|g_*(s-y/\lambda)-g(s)|\,ds\xrightarrow{\lambda\to\infty}0.\end{aligned}$$

Therefore, we can choose λ large enough such that $\|h_\lambda\|_1<\epsilon$, for all $x\in B(0,1/\lambda)$. □

We introduce the following definition.

Definition 1 *Let $f\in L^1(\mathbb{R}^n)$. Then,*

$$(\mathcal{T}_af)(x):=\frac{3\eta^2+1}{4\eta}f(x+a)+\frac{\eta^2-1}{4\eta}f(x-a)+\frac{\eta^2-1}{4\eta}f(-x+a)-\frac{\eta^2-1}{4\eta}f(-x-a), \tag{24}$$

is called a translation by $a\in\mathbb{R}^n$.

Note that for the well-known translation defined by $(\tau_af)(x)=f(x-a)$, the asymptotic behaviors at $+\infty$ as well as at $-\infty$ of f and τ_af do not change. However, this fact can be different for the above one. Given a function $f\in L^1(\mathbb{R}^n)$, let $\mathcal{T}(f)$ denote the linear hull of all translations $\mathcal{T}_af$, with $a\in\mathbb{R}^n$. Then, $\mathrm{span}_{\mathcal{T}}(f)$ denotes the closed linear hull of translations of f, i.e., $\mathrm{span}_{\mathcal{T}}(f):=\overline{\mathcal{T}(f)}$. Since

$$\mathrm{span}_{\mathcal{T}}(f)=\overline{\left\{\sum_{j=1}^{N}(\mathcal{T}_{a_j}f)(x)\lambda_j : a_j\in\mathbb{R}^n\ N=1,2,\dots\right\}},$$

the $\mathrm{span}_{\mathcal{T}}(f)$ contains the convolution presented in Theorem 6, for $g\in L^1(\mathbb{R}^n)$.

We notice that the remarkable difference between the translation introduced in Definition 1, which we need, and the usual one, $(\tau_{-a}f)_a(x)=f(x+a)$, arises from the different structure of the kernel $\eta\cos(xu)+i\sin(xu)$ in comparison with the kernel of the Fourier transform. In particular, the subset $\mathcal{E}:=\{z=e^{ix}: x\in\mathbb{R}\}\subset\mathbb{C}$ assumes an abelian group structure for the usual multiplication as the set $\mathcal{E}$ is the unit circle, in contrast with the subset $\{\eta\cos(x)+i\sin(x): x\in\mathbb{R}\}\subset\mathbb{C}$, since $|\eta\cos(x)+i\sin(x)|\ne1$ for some $x\in\mathbb{R}$, provided $\eta\ne\pm1$.

For the Banach space $L^1(\mathbb{R}^n)$ with the usual norm, consider $\mathcal{W}:=T_{\eta,i}(L^1(\mathbb{R}^n))$. As for $\mathcal{W}$, we endow it with the norm of $L^1(\mathbb{R}^n)$, that is, $\|T_{\eta,i}f\|_{\mathcal{W}}=\|f\|_1$. Equipped with that norm, $L^1(\mathbb{R}^n)$ and $\mathcal{W}$ are topological vector spaces. Moreover, $\mathcal{W}$ has the pointwise multiplication structure with

$$\begin{aligned}\|(T_{\eta,i}f)(T_{\eta,i}g)\|_{\mathcal{W}}=&\|T_{\eta,i}(f\overset{T_{\eta,i}}{\star}g)\|_{\mathcal{W}}=\|f\overset{T_{\eta,i}}{\star}g\|_1\\ &\le\frac{3|\eta|^2+2}{2|\eta|(2\pi)^{\frac{n}{2}}}\|f\|_1\|g\|_1=\frac{3|\eta|^2+2}{2|\eta|(2\pi)^{\frac{n}{2}}}\|T_{\eta,i}f\|_{\mathcal{W}}\|T_{\eta,i}g\|_{\mathcal{W}}.\end{aligned}$$

Thus, $L^1(\mathbb{R}^n)$ and $\mathcal{W}$ are Banach algebras equipped with the convolution and pointwise multiplications, respectively. Let us consider also the spaces

$$X_0 := \left\{ f \in L^1(\mathbb{R}^n) : T_{\eta,i} f \text{ has compact support} \right\} \subset L^1(\mathbb{R}^n);$$
$$\mathcal{W}_0 := T_{\eta,i}(X_0) \subset \mathcal{W} = T_{\eta,i}(L^1(\mathbb{R}^n)).$$

Similar to Lemma 2, the proof of the following lemma should come from the convolution (15), as we will see in (27)–(30). Here, the dependence of the kernel $\eta\cos(xy) + i\sin(xy)$, the convolution (15), and the translation (24) is significant.

Lemma 3 *$\mathcal{W}_0$ is dense in $\mathcal{W}$.*

Proof We start by noticing that $\Upsilon_a : f(x) \mapsto (\eta\cos(ax) + i\sin(ax))f(x)$, for any $a \in \mathbb{R}^n$ fixed, and the translation $\mathcal{T}_a : f \mapsto \mathcal{T}_a f$, defined as in Definition 1, are bounded linear operators in $L^1(\mathbb{R}^n)$, with $\|\Upsilon_a f\|_1 \le \delta_0 \|f\|_1$ and

$$\|\mathcal{T}_a f\|_1 \le \frac{6|\eta|^2 + 4}{4|\eta|(2\pi)^{\frac{n}{2}}} \|f\|_1.$$

Remark that $\mathcal{T}_a(\mathcal{W}_0) \subset \mathcal{W}_0$, for every $a \in \mathbb{R}^n$ and $T_{\eta,i}\Upsilon_a = \mathcal{T}_a T_{\eta,i}$, which can be expressed by the following diagram

$$\begin{array}{ccc} L^1(\mathbb{R}^n) & \xrightarrow{T_{\eta,i}} & \mathcal{W} \supset \mathcal{W}_0 \\ \Upsilon_a \big\downarrow & & \big\downarrow \mathcal{T}_a \\ L^1(\mathbb{R}^n) & \xrightarrow[T_{\eta,i}]{} & \mathcal{W} \supset \mathcal{W}_0 \end{array}$$

that is commutative. Moreover, X_0 is invariant with respect to those operators. Indeed, suppose that $f \in X_0$, which implies $(T_{\eta,i} f)(\pm x \pm a) \in \mathcal{W}_0$ for $a \in \mathbb{R}^n$. We obtain

$$\begin{aligned}(T_{\eta,i}\Upsilon_a f)(x) =& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} [\eta\cos(xu) + i\sin(xu)]\,[\eta\cos(au) + i\sin(au)]\, f(u)du \\ =& \frac{1}{4\eta}\left[(3\eta^2+1)(T_{\eta,i}f)(x+a) + (\eta^2-1)(T_{\eta,i}f)(x-a) + (\eta^2-1)(T_{\eta,i}f)(-x+a)\right. \\ & \left. -(\eta^2-1)(T_{\eta,i}f)(-x-a)\right], \end{aligned} \tag{25}$$

that belongs to $\mathcal{W}_0$, and

$$\begin{aligned}(T_{\eta,i}\mathcal{T}_a f)(x) =& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} [\eta\cos(xy) + i\sin(xy)] \left[\frac{3\eta^2+1}{4\eta} f(y+a) + \frac{\eta^2-1}{4\eta} f(y-a) + \frac{\eta^2-1}{4\eta} f(-y+a)\right. \\ & \left. - \frac{\eta^2-1}{4\eta} f(-y-a)\right] dy \\ =& \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \left[\frac{3\eta^2+1}{4\eta}\left[\eta\cos(x(s-a)) + i\sin(x(s-a))\right]\right. \\ & + \frac{\eta^2-1}{4\eta}\left[\eta\cos(x(s+a)) + i\sin(x(s+a))\right] + \frac{\eta^2-1}{4\eta}\left[\eta\cos(x(-s+a)) + i\sin(x(-s+a))\right] \end{aligned}$$

$$- \frac{\eta^2-1}{4\eta}\left[\eta\cos(x(-s-a))+i\sin(x(-s-a))\right]\Big] f(s)\,ds$$
$$= \left[\eta\cos(ax)+i\sin(ax)\right]\left(T_{\eta,i}f\right)(x), \tag{26}$$

which belongs to $\mathcal{W}_0$. Thus, X_0 is invariant as mentioned above.

Let $0 \neq f \in X_0$. Clearly, $\widetilde{f} \in X_0$. By (25) and (26), we have that $\mathcal{T}_{\pm a}f$, $\mathcal{T}_{\pm a}f$, $(\eta\cos(bx) + i\sin(bx))(\mathcal{T}_{\pm a}f)(x)$ and $(\eta\cos(bx)+i\sin(bx))(\mathcal{T}_{\pm a}\widetilde{f})(x)$ are in X_0, for $a, b \in \mathbb{R}^n$. To prove the lemma, we can use the fact that a bounded linear operator has dense range if and only if its adjoint is injective (see Theorem 4.12 and its corollaries in [13]). Thus, we can consider the (continuous) adjoint operator of the bounded linear operator acting between two normed spaces $T_{\eta,i} : L^1(\mathbb{R}^n) \to \mathcal{W}$, that is $T^*_{\eta,i} : \mathcal{W}^* \to \left(L^1(\mathbb{R}^n)\right)^*$. Recall that the dual space $\left(L^1(\mathbb{R}^n)\right)^*$ is $L^\infty(\mathbb{R}^n)$. We shall prove that $T^*_{\eta,i}$ is injective. Let us consider

$$\mathbb{S} := \Big\{ \left(\eta\cos(bx)+i\sin(bx)\right)(\mathcal{T}_a f)(x), \left(\eta\cos(bx)+i\sin(bx)\right)(\mathcal{T}_{-a} f)(x),$$
$$\left(\eta\cos(bx)+i\sin(bx)\right)(T_a \widetilde{f})(x), \left(\eta\cos(bx)+i\sin(bx)\right)(\mathcal{T}_{-a} \widetilde{f})(x)\Big\} \subset X_0.$$

If a function $\phi \in L^\infty(\mathbb{R}^n)$ is orthogonal to X_0, then $\phi \perp \mathbb{S}$. Hence,

$$\int_{\mathbb{R}^n} \left[\eta\cos(bx)+i\sin(bx)\right] \left[\frac{3\eta^2+1}{4\eta}f(x+a)+\frac{\eta^2-1}{4\eta}f(x-a)+\frac{\eta^2-1}{4\eta}f(-x+a)-\frac{\eta^2-1}{4\eta}f(-x-a)\right]\phi(x)\,dx = 0 \tag{27}$$

$$\int_{\mathbb{R}^n} \left[\eta\cos(bx)+i\sin(bx)\right] \left[\frac{3\eta^2+1}{4\eta}f(x-a)+\frac{\eta^2-1}{4\eta}f(x+a)+\frac{\eta^2-1}{4\eta}f(-x-a)-\frac{\eta^2-1}{4\eta}f(-x+a)\right]\phi(x)\,dx = 0 \tag{28}$$

$$\int_{\mathbb{R}^n} \left[\eta\cos(bx)+i\sin(bx)\right] \left[\frac{3\eta^2+1}{4\eta}f(-x-a)+\frac{\eta^2-1}{4\eta}f(-x+a)+\frac{\eta^2-1}{4\eta}f(x-a)-\frac{\eta^2-1}{4\eta}f(x+a)\right]\phi(x)\,dx = 0 \tag{29}$$

$$\int_{\mathbb{R}^n} \left[\eta\cos(bx)+i\sin(bx)\right] \left[\frac{3\eta^2+1}{4\eta}f(-x+a)+\frac{\eta^2-1}{4\eta}f(-x-a)+\frac{\eta\ 2-1}{4\eta}f(x+a)-\frac{\eta^2-1}{4\eta}f(x-a)\right]\phi(x)\,dx = 0 \tag{30}$$

for all $a, b \in \mathbb{R}^n$. The identities (27)–(30) can be seen as an homogeneous system of four equations with four unknowns:

$$\int_{\mathbb{R}^n} (\eta\cos(bx)+i\sin(bx))f(\pm x \pm a)\phi(x)\,dx,$$

whose determinant is given by

$$\det(A) = \begin{vmatrix} \frac{3\eta^2+1}{4\eta} & \frac{\eta^2-1}{4\eta} & \frac{\eta^2-1}{4\eta} & -\frac{\eta^2-1}{4\eta} \\ \frac{\eta^2-1}{4\eta} & \frac{3\eta^2+1}{4\eta} & -\frac{\eta^2-1}{4\eta} & \frac{\eta^2-1}{4\eta} \\ -\frac{\eta^2-1}{4\eta} & \frac{\eta^2-1}{4\eta} & \frac{\eta^2-1}{4\eta} & \frac{3\eta^2+1}{4\eta} \\ \frac{\eta^2-1}{4\eta} & -\frac{\eta^2-1}{4\eta} & \frac{3\eta^2+1}{4\eta} & \frac{\eta^2-1}{4\eta} \end{vmatrix} = -\eta^2 \neq 0. \tag{31}$$

This implies that the system of equations (27)–(30) has only the trivial solution, i.e.

$$\int_{\mathbb{R}^n} [\eta\cos(bx) + i\sin(bx)]\, f(\pm x \pm a)\phi(x)\, dx = 0,$$

for every $a, b \in \mathbb{R}^n$. By the uniqueness theorem (Theorem 4) for $T_{\eta,i}$, we have that $f(\pm x \pm a)\phi(x) = 0$, for almost every x and every $a \in \mathbb{R}^n$. Integrating with respect to the variable $a \in \mathbb{R}^n$, we obtain

$$0 = \int_{\mathbb{R}^n} f(\pm x \pm a)\phi(x) da = \phi(x)\|f\|_1,$$

which follows $\phi(x) = 0$, almost everywhere, once $\|f\|_1 \neq 0$. Therefore, the operator $T^*_{\eta,i}$ is injective. Since $f \neq 0$, we derive that $\phi(x) = 0$ almost everywhere. Therefore, the operator $T^*_{\eta,i}$ has a trivial kernel, i.e. is injective. □

Theorem 10 (Wiener-type Tauberian theorem) *Let $K \in L^1(\mathbb{R}^n)$. The set $(T_{\eta,i}K)\mathcal{W}$ is dense in $\mathcal{W}$ if and only if $(T_{\eta,i}K)(x) \neq 0$, for all $x \in \mathbb{R}^n$.*

Proof Suppose that $(T_{\eta,i}K)(x_0) = 0$, for some $x_0 \in \mathbb{R}^n$. By the factorization identity (16), $T_{\eta,i}(K \overset{T_{\eta,i}}{\star} f)(x_0) = 0$, for all $f \in L^1(\mathbb{R}^n)$. Having in mind that $T_{\eta,i}(K \overset{T_{\eta,i}}{\star} f)$, for all $f \in L^1(\mathbb{R}^n)$, are uniformly continuous functions on $\mathbb{R}^n$, we deduce that $(T_{\eta,i}K)\mathcal{W}$ is not dense in $\mathcal{W}$, which contradicts the assumption.

Conversely, for any $\Psi \in L^1(\mathbb{R}^n)$, such that $T_{\eta,i}\Psi \in \mathcal{W}$, we need to prove that $T_{\eta,i}\Psi \in \overline{(T_{\eta,i}K)\mathcal{W}}$. By Lemma 3, we only need to prove that the above inclusion holds for $T_{\eta,i}\Psi \in \mathcal{W}_0$, with $0 \neq \Psi \in X_0$. By rescaling with the coefficient $(\|\Psi\|_1)^{-1}$, we may assume that $\|\Psi\|_1 = 1$. On the other hand, we put $(T_{\eta,i}K)(0) := \beta \neq 0$. Observe that $(T_{\eta,i}K)\mathcal{W}$ is dense in $\mathcal{W}$ if and only if $(T_{\eta,i}K')\mathcal{W}$ is, where $K' = \beta^{-1}K$. Therefore, we may assume that $(T_{\eta,i}K)(0) = 1$. By Lemma 2, there exists an $h \in L^1(\mathbb{R}^n)$ with $\|h\|_1 < \frac{2|\eta|}{3|\eta|^2+2}$ such that

$$1 - (T_{\eta,i}h)(x) = (T_{\eta,i}K)(x) \quad \text{on} \quad B(0,\delta). \tag{32}$$

Note that $T_{\eta,i}h$ is uniformly continuous on $\mathbb{R}^n$ and $(T_{\eta,i}h)(x)$ tends to 0 as $x \to 0$ [3, Theorem 3]. Therefore, we can choose δ small enough such that $|(T_{\eta,i}h)(x)| < 1$ on $B(0,\delta)$. Combining (32) with the factorization identity for the convolution, we have

$$\frac{1}{1 - (T_{\eta,i}h)(x)} = \sum_{m\geq 0} [(T_{\eta,i}h)(x)]^m = 1 + \sum_{m\geq 1} T_{\eta,i}\left[(h)^{\overset{T_{\eta,i}}{\star} m}\right](x) \quad \text{on} \quad B(0,\delta), \tag{33}$$

where we assume that

$$(h)^{\overset{T_{\eta,i}}{\star}1} := h, \quad (h)^{\overset{T_{\eta,i}}{\star}m} := \underbrace{h \overset{T_{\eta,i}}{\star} h \overset{T_{\eta,i}}{\star} \cdots \overset{T_{\eta,i}}{\star} h}_{m \text{ times } h}, \quad \text{for } m \geq 2.$$

Now, we can construct a sequence of functions in $L^1(\mathbb{R}^n)$ by choosing $g_0 := \Psi$ and $g_m := h \overset{T_{\eta,i}}{\star} g_{m-1}$, for $m \geq 1$. In fact, $g_m = \Psi \overset{T_{\eta,i}}{\star} (h)^{\overset{T_{\eta,i}}{\star}m}$, for any $m \geq 1$. By the norm inequality of the convolution, we have

$$\|g_m\|_1 \leq \left(\frac{3|\eta|^2+2}{2|\eta|(2\pi)^{\frac{n}{2}}}\right)^m \|\Psi\|_1 \|h\|_1^m = \left(\frac{3|\eta|^2+2}{2|\eta|(2\pi)^{\frac{n}{2}}}\right) \|h\|_1^m,$$

since $\|\Psi\|_1 = 1$. As $\|h\|_1 < \frac{2|\eta|}{3|\eta|^2+2}$, we have $\|g_m\|_1 \leq \left(\frac{1}{(2\pi)^{n/2}}\right)^m < 1$. Hence, the series $\sum_{m\geq 0} g_m$ is convergent in $L^1(\mathbb{R}^n)$ and defines a function $G \in L^1(\mathbb{R}^n)$,

$$G(x) = \sum_{m\geq 0} g_m(x). \tag{34}$$

There are two cases for the support of $T_{\eta,i}\Psi$.

Case 1: $\operatorname{supp}(T_{\eta,i}\Psi) \subset B(0,\delta)$. Combining (32), (33), (34) and the previous assumption on $T_{\eta,i}\Psi$, we have

$$[1-(T_{\eta,i}h)(x)]\, T_{\eta,i}\Psi(x) = (T_{\eta,i}K)(x)(T_{\eta,i}\Psi)(x), \quad \text{for every } x \in \mathbb{R}^n.$$

Thus,

$$\begin{aligned}
(T_{\eta,i}\Psi)(x) =& \frac{1}{1-(T_{\eta,i}h)(x)}(T_{\eta,i}K)(x)(T_{\eta,i}\Psi)(x) \\
=& (T_{\eta,i}K)(x)(T_{\eta,i}\Psi)(x)\left[1+\sum_{m\geq 0} T_{\eta,i}\left[(h)^{\overset{T_{\eta,i}}{\star}m}\right](x)\right] \\
=& (T_{\eta,i}K)(x)\left[(T_{\eta,i}g_0)(x)+\sum_{m\geq 1} T_{\eta,i}\big(\Psi \overset{T_{\eta,i}}{\star} (h)^{\overset{T_{\eta,i}}{\star}m}\big)(x)\right] \\
=& (T_{\eta,i}K)(x) T_{\eta,i}\left(\sum_{m\geq 0} g_m\right)(x) \\
=& (T_{\eta,i}K)(x)(T_{\eta,i}G)(x),
\end{aligned}$$

for every $x \in \mathbb{R}^n$. Thus, $(T_{\eta,i}\Psi)(x) = (T_{\eta,i}K)(x)(T_{\eta,i}G)(x)$ that belongs to $\overline{(T_{\eta,i}K)\mathcal{W}}$.

Case 2: $\operatorname{supp}(T_{\eta,i}\Psi) \not\subseteq B(0,\delta)$. To prove this case, we will use the fact that we can scale any function that has compact support, in order to obtain a function for which its image by the operator $T_{\eta,i}$ is supported in a ball centered at the origin and with an arbitrarily small radius. In particular, suppose that $\operatorname{supp}(T_{\eta,i}\Psi)$ is contained in the ball $B(0,M)$, for some $M>0$. Putting $(\Psi_*)(x) := (\delta/M)\Psi(\delta x/M)$, we obtain $(T_{\eta,i}\Psi_*) = (T_{\eta,i}\Psi)(Mx/\delta)$. Since $\operatorname{supp}[(T_{\eta,i}\Psi)(Mx/\delta)] \subset B(0,\delta)$, we have that $\operatorname{supp}(T_{\eta,i}\Psi_*) \subset B(0,\delta)$.

Thus, we are in the first case and we have that $T_{\eta,i}\Psi_* \in \overline{(T_{\eta,i}K)\mathcal{W}}$. By the fact $\Psi \in L^1(\mathbb{R}^n)$ if and only if $\Psi_* \in L^1(\mathbb{R}^n)$, we have that $T_{\eta,i}\Psi \in \overline{(T_{\eta,i}K)\mathcal{W}}$. □

Theorem 11 (Wiener-type theorem) *Let $K \in L^1(\mathbb{R}^n)$. Then,* $\mathrm{span}_{\mathcal{T}}(K) = L^1(\mathbb{R}^n)$ *if and only if* $(T_{\eta,i}K)$ *has no real zeros.*

Proof Suppose that $(T_{\eta,i}K)(x) \neq 0$, for every $x \in \mathbb{R}^n$. By the factorization identity in (16), we have

$$(T_{\eta,i}K)\mathcal{W} = \left\{(T_{\eta,i}K)(T_{\eta,i}g) : g \in L^1(\mathbb{R}^n)\right\} = \left\{T_{\eta,i}(K \overset{T_{\eta,i}}{\star} g) : g \in L^1(\mathbb{R}^n)\right\}.$$

Combining this with the previous theorem and the fact that $\mathrm{span}_{\mathcal{T}}(K)$ contains all the convolutions $K \overset{T_{\eta,i}}{\star} g$, for $g \in L^1(\mathbb{R}^n)$, we derive that $\mathrm{span}_{\mathcal{T}}(K) = L^1(\mathbb{R}^n)$.

Conversely, suppose that $\mathrm{span}_{\mathcal{T}}(K) = L^1(\mathbb{R}^n)$. If $(T_{\eta,i}K)(x_0) = 0$, for some x_0, then $T_{\eta,i}(K \overset{T_{\eta,i}}{\star} g)(x_0) = 0$, for all $g \in L^1(\mathbb{R}^n)$. Since the functions $T_{\eta,i}(K \overset{T_{\eta,i}}{\star} g)(x)$, for $g \in L^1(\mathbb{R}^n)$, are uniformly continuous on $\mathbb{R}^n$, we derive that $\mathrm{span}_{\mathcal{T}}(K) \subsetneq L^1(\mathbb{R}^n)$, which contradicts the assumption of the theorem. □

Definition 2 (see [13]) *A function $\phi \in L^\infty(\mathbb{R}^n)$ is said to be slowly oscillating if, for every $\epsilon > 0$, there exist an $A < \infty$ and a $\delta > 0$ such that*

$$|\phi(x) - \phi(y)| < \epsilon \quad \text{if} \quad |x| > A, |y| > A, |x - y| < \delta. \tag{35}$$

The slowly oscillating continuous functions are very attractive because they are closely related to several different topics in functional and harmonic analysis. The following theorem incorporates the Wiener's and Pitt's ideas, but our operator presents an asymmetric kernel, in contrast with the original idea of Pitt.

Theorem 12 (Wiener–Pitt type Tauberian theorem) *Let $K \in L^1(\mathbb{R}^n)$ satisfies the condition $(T_{\eta,i}K)(x) \neq 0$, for every $x \in \mathbb{R}^n$. Suppose that for $\phi \in L^\infty(\mathbb{R}^n)$ given, it holds*

$$\lim_{|x|\to\infty} (K \overset{T_{\eta,i}}{\star} \phi)(x) = \zeta (T_{\eta,i}K)(0),$$

for some $\zeta \in \mathbb{C}$. Then,

$$\lim_{|x|\to\infty} (f \overset{T_{\eta,i}}{\star} \phi)(x) = \zeta (T_{\eta,i}f)(0), \tag{36}$$

for every $f \in L^1(\mathbb{R}^n)$.

Moreover, if we assume that ϕ is slowly oscillating, then

$$\lim_{|x|\to\infty} \phi(x) = \zeta. \tag{37}$$

Proof We start by observing that

$$\zeta(T_{\eta,i}K)(0) = \frac{\zeta\eta}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} K(y)\, dy, \qquad \zeta(T_{\eta,i}f)(0) = \frac{\zeta\eta}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(y)\, dy.$$

Put $\psi(x) = \phi(x) - \zeta$ and consider a subset of the Banach space $L^1(\mathbb{R}^n)$

$$P := \left\{ f \in L^1(\mathbb{R}^n) : \lim_{|x|\to\infty} (f \overset{T_{\eta,i}}{\star} \psi)(x) = 0 \right\}.$$

We shall prove that P is a closed linear subspace of $L^1(\mathbb{R}^n)$. Indeed, suppose that $f_n \in P$ with $\|f_n - f\|_1 \to 0$, as $n \to \infty$. By the Young type inequality (18), we have

$$\|f_n \overset{T_{\eta,i}}{\star} \psi - f \overset{T_{\eta,i}}{\star} \psi\|_\infty = \|(f_n - f) \overset{T_{\eta,i}}{\star} \psi\| \leq C\|f_n - f\|_1 \|\psi\|_\infty \to 0,$$

as $n \to \infty$, which implies that $\left\{ f_n \overset{T_{\eta,i}}{\star} \psi \right\}$ converges uniformly on $\mathbb{R}^n$ to $f \overset{T_{\eta,i}}{\star} \psi \in L^1(\mathbb{R}^n)$. In particular, there is some n_0 such that, for $n > n_0$, we have $\|f_n - f\|_1 < \epsilon$, and, as $f_{n_0} \in P$, there is some N such that $|x| > N$ implies $|(f_{n_0} \overset{T_{\eta,i}}{\star} \psi)(x)| < \epsilon$. Then, for $|x| > N$, it holds

$$\begin{aligned} |(f \overset{T_{\eta,i}}{\star} \psi)(x)| \leq& |(f \overset{T_{\eta,i}}{\star} \psi)(x) - (f_{n_0} \overset{T_{\eta,i}}{\star} \psi)(x)| + |(f_{n_0} \overset{T_{\eta,i}}{\star} \psi)(x)| \\ \leq& C\|\psi\|_\infty \|f_n - f\|_1 + |(f_{n_0} \overset{T_{\eta,i}}{\star} \psi)(x)| < \epsilon\,(C\|\psi\|_\infty + 1)\,. \end{aligned}$$

So, P is a closed space in $L^1(\mathbb{R}^n)$. Clearly, $K \in P$. Moreover, note that P contains all the translations in $T_{\eta,i}(K)$. By Theorem 11, we have that $P \equiv L^1(\mathbb{R}^n)$, which proves (36).

Now, let us prove (37). Let $\epsilon > 0$. As ϕ is a slowly oscillating function, we have $A > 0$ and $\delta > 0$ for which ϕ satisfies (35). For such δ, we will consider the function

$$g(x) := \begin{cases} e^{-\frac{1}{2}|x|^2}, & \text{if } |x| < \delta \\ 0, & \text{if } |x| \geq \delta. \end{cases}$$

Clearly, g is supported on the box $|x| < \delta$ and then, $g \in L^1(\mathbb{R}^n)$. Moreover, $g \geq 0$ and $g(x) = g(-x)$, for all x. Let us consider $Q := (T_{\eta,i} g)(0)$. Clearly, $Q > 0$. We will choose $f_0(x) = g(x)/Q$ which has the following properties:

(i) f_0 is supported in the box $|x| \leq \delta$, $f_0(x) = f_0(-x)$, for all x;

(ii) $f_0(x) \geq 0$, for every $x \in \mathbb{R}^n$ and

$$(T_{\eta,i} f_0)(0) = \frac{\eta}{(2\pi)^{n/2}} \int_{|u|\leq\delta} f_0(u)\, du = 1.$$

Applying (36) proved above and having in mind that $f \overset{T_{\eta,i}}{\star} \phi = \phi \overset{T_{\eta,i}}{\star} f$ and $f_0(-x) = f_0(x)$, for all x, we have

$$\begin{aligned} \lim_{|x|\to\infty} \Bigg\{ \frac{1}{4\eta(2\pi)^{n/2}} \int_{|u|\leq\delta} \Big[&(3\eta^2 + 1)\phi(x-u) + (\eta^2 - 1)\phi(x+u) + (\eta^2 - 1)\phi(-x+u) \\ &- (\eta^2 - 1)\phi(-x-u) \Big] f_0(u)\, du \Bigg\} = \zeta. \end{aligned}$$

Replacing $x := -x$ and $u := -u$, we have

$$\lim_{|x|\to\infty}\left\{\frac{1}{4\eta(2\pi)^{n/2}}\int_{|u|\le\delta}\Big[(\eta^2-1)\phi(x-u)+(3\eta^2+1)\phi(x+u)-(\eta^2-1)\phi(-x+u)\right.$$
$$\left.+(\eta^2-1)\phi(-x-u)\Big]f_0(u)\,du\right\}=\zeta;$$

$$\lim_{|x|\to\infty}\left\{\frac{1}{4\eta(2\pi)^{n/2}}\int_{|u|\le\delta}\Big[(\eta^2-1)\phi(x-u)-(\eta^2-1)\phi(x+u)+(3\eta^2+1)\phi(-x+u)\right.$$
$$\left.+(\eta^2-1)\phi(-x-u)\Big]f_0(u)\,du\right\}=\zeta;$$

$$\lim_{|x|\to\infty}\left\{\frac{1}{4\eta(2\pi)^{n/2}}\int_{|u|\le\delta}\Big[-(\eta^2-1)\phi(x-u)+(\eta^2-1)\phi(x+u)+(\eta^2-1)\phi(-x+u)\right.$$
$$\left.+(3\eta^2+1)\phi(-x-u)\Big]f_0(u)\,du\right\}=\zeta.$$

Defining

$$X:=\lim_{|x|\to\infty}\left\{\frac{1}{4\eta(2\pi)^{n/2}}\int_{|u|\le\delta}\phi(x-u)f_0(u)\,du\right\},\qquad Y:=\lim_{|x|\to\infty}\left\{\frac{1}{4\eta(2\pi)^{n/2}}\int_{|u|\le\delta}\phi(x+u)f_0(u)\,du\right\},$$
$$Z:=\lim_{|x|\to\infty}\left\{\frac{1}{4\eta(2\pi)^{n/2}}\int_{|u|\le\delta}\phi(-x+u)f_0(u)\,du\right\},\quad W:=\lim_{|x|\to\infty}\left\{\frac{1}{4\eta(2\pi)^{n/2}}\int_{|u|\le\delta}\phi(-x-u)f_0(u)\,du\right\},$$

we obtain the following system of linear equations

$$\begin{cases}(3\eta^2+1)X+(\eta^2-1)Y+(\eta^2-1)Z-(\eta^2-1)W=\zeta\\(\eta^2-1)X+(3\eta^2+1)Y-(\eta^2-1)Z+(\eta^2-1)W=\zeta\\(\eta^2-1)X-(\eta^2-1)Y+(3\eta^2+1)Z+(\eta^2-1)W=\zeta\\-(\eta^2-1)X+(\eta^2-1)Y+(\eta^2-1)Z+(3\eta^2+1)W=\zeta.\end{cases}$$

The determinant of this system, calculated in (31), is different from zero, provided $\eta\neq 0$, which implies that the unique solution of the system is $X=Y=Z=W=\zeta/4\eta^2$. Thus, for $|x|>A+\delta$ and $|u|\le\delta$, it follows that $|x-u|\ge A$. By the properties (i), (ii), (35), and the solution of the above system, we have

$$\left|\phi(x)-\frac{\eta}{(2\pi)^{n/2}}\int_{|u|\le\delta}\phi(x-u)f_0(u)\,du\right|=\left|\frac{\eta}{(2\pi)^{n/2}}\int_{|u|\le\delta}[\phi(x)-\phi(x-u)]f_0(u)\,du\right|$$
$$\le\frac{|\eta|}{(2\pi)^{n/2}}\int_{|u|\le\delta}|\phi(x)-\phi(x-u)|\,|f_0(u)|\,du\le\epsilon,$$

which means that

$$\lim_{|x|\to\infty}\phi(x)=4\eta^2\lim_{|x|\to\infty}\left(\frac{1}{4\eta(2\pi)^{n/2}}\int_{|u|\le\delta}\phi(x-u)f_0(u)\,du\right)=\zeta.$$

□

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