Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 1148 – 1160
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doi:10.3906/mat-1811-103

Some unpublished Recław theorems and their applications to Baire-star-one functions

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Received: 23.11.2018 • **Accepted/Published Online:** 08.03.2019 • **Final Version:** 29.05.2019

Abstract: Lunina's 7-tuples $\langle E^1, \ldots, E^7 \rangle$ of sets of pointwise convergence, divergence to ∞, divergence to $-\infty$, etc. for sequences of Baire-star-one functions are characterized. Generalization on ideal convergence of such sequences is discussed. Limits and ideal limits of sequences of Baire-star-one functions are considered in the last part of the article.

Key words: Pointwise convergence, ideal convergence, F_σ-ideal, $\mathcal{I}^\star$-universal set, ω-diagonalizability, Lunina 7-tuples, Baire-one-star functions

1. Introduction

Let $\vec{f} = (f_n)_n$ be a sequence of real-valued functions defined on a metric space X. It is not difficult to show that if $\vec{f}$ is a sequence of continuous functions then the set $L(\vec{f})$, of all points $x \in X$ such that $(f_n(x))_n$ converges, is of the type $F_{\sigma\delta}(X)$. On the other hand, Hahn [10] and Sierpiński [27] proved independently that if A is an $F_{\sigma\delta}$-subset of a Polish space X, then there is a sequence $\vec{f}$ of continuous functions for which $A = L(\vec{f})$. Further research (see, e.g., Kornfel'd [13] and Lipiński [17]) also involved the sets of points with the sequence divergent to infinity and the like. The full description of these sets was given by Lunina [18]; see Theorem 3.1 below. We will expand this result in two directions. First, the class of continuous functions will be replaced by $\mathcal{B}_1^*(X)$: the class of all real Baire-star-one mappings defined on X (see Theorem 3.4). In 2010 Borzestowski and Recław proved [1] an ideal version of Lunina's theorem for sequences of continuous functions. The second kind of extension of Lunina's theorem consists of characterization of sets of ideal (instead of pointwise) convergence points for sequences of Baire-star-one functions with respect to the ideal of F_σ-type (see Theorem 4.5 and Corollary 4.6). In the last part of the paper we discuss ideal limits of sequences of Baire-star-one functions (see Theorem 5.6).

2. Preliminaries

2.1. Notations

Let X be a topological space. $\mathcal{P}(X)$ denotes the family of all subsets of X. The symbol $\mathbf{\Pi}^0_\alpha(X)$ ($\mathbf{\Sigma}^0_\alpha(X)$, resp.) denotes the multiplicative (additive, resp.) class α of Borel subsets of X and $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Pi}^0_\alpha(X) \cap \mathbf{\Sigma}^0_\alpha(X)$.

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2010 *AMS Mathematics Subject Classification:* Primary 40A35; Secondary 40A30, 26A21, 54C30

The class of all continuous functions $f : X \to \mathbb{R}$ is denoted by $\mathcal{C}(X)$. Let Γ be a family of subsets of X. We say that a mapping $f\colon X \to \mathbb{R}$ is Γ-measurable iff $f^{-1}(U) \in \Gamma$ for any open set $U \subset \mathbb{R}$. In particular, for $1 \leqslant \alpha < \omega_1$, the family of $\mathbf{\Sigma}^0_{\alpha+1}(X)$-measurable functions is called the α Borel class. Recall that if X is a perfectly normal topological space then the α Borel class coincides with the α Baire class $\mathcal{B}_\alpha(X)$; see, e.g., [3, Proposition 3.14]. It is easy to see that for each α, a function $f : X \to \mathbb{R}$ is $\mathbf{\Pi}^0_\alpha(X)$-measurable iff it is $\mathbf{\Delta}^0_\alpha(X)$-measurable. Hence, the class of $\mathbf{\Pi}^0_{\alpha+1}(X)$-measurable functions is a subclass of the α Borel class and if X is an uncountable Polish space then this inclusion is proper. $\mathbf{\Pi}^0_2(X)$-measurable functions are called first-level Borel functions (see [11]) or Baire-.5 functions (see [26]). If X is a complete metric space then the class of $\mathbf{\Pi}^0_2(X)$-measurable functions is equal to the Baire-star-one class $\mathcal{B}^*_1(X)$ introduced by O'Malley [12, Theorem 2.3].

Definition 2.1 *$f \in \mathcal{B}^*_1(X)$ if for any nonempty closed set $F \subset X$ there is an open set $U \subset X$ such that $F \cap U \neq \emptyset$ and the restriction $f|_F$ is continuous on $F \cap U$.*

2.2. Lunina's 7-tuples

For a sequence $\vec{f} = (f_n)_n$ we define seven types of sets of convergence and divergence points:

$$\begin{aligned}
E^1(\vec{f}) &= \{x\colon (f_n(x)) \text{ is convergent}\};\\
E^2(\vec{f}) &= \{x\colon \lim f_n(x) = -\infty\};\\
E^3(\vec{f}) &= \{x\colon \lim f_n(x) = +\infty\};\\
E^4(\vec{f}) &= \{x\colon -\infty < \underline{\lim}\, f_n(x) < \overline{\lim}\, f_n(x) < +\infty\};\\
E^5(\vec{f}) &= \{x\colon -\infty = \underline{\lim}\, f_n(x) < \overline{\lim}\, f_n(x) < +\infty\};\\
E^6(\vec{f}) &= \{x\colon -\infty < \underline{\lim}\, f_n(x) < \overline{\lim}\, f_n(x) = +\infty\};\\
E^7(\vec{f}) &= \{x\colon -\infty = \underline{\lim}\, f_n(x) \;\&\; \overline{\lim}\, f_n(x) = +\infty\}.
\end{aligned}$$

Observe that the sets $E^1(\vec{f}), \ldots, E^7(\vec{f})$ form a partition of X. Moreover,

$$E^1(\vec{f}) \cup E^4(\vec{f}) = \{x\colon (f_n(x)) \text{ is bounded}\}.$$

Let $\mathcal{F} \subset \mathbb{R}^X$ be a family of real-valued functions defined on X. A sequence $\langle E^1, \ldots, E^7\rangle$ is called a Lunina's 7-tuple for $\mathcal{F}$ if there is a sequence $\vec{f} = (f_n)_n \in \mathcal{F}$ such that $E^i = E^i(\vec{f})$ for $i = 1, \ldots, 7$; see [1] or [6]. The family of all Lunina's 7-tuples for $\mathcal{F}$ is denoted by $\Lambda^7(\mathcal{F})$.

3. Lunina's 7-tuples for the families of Borel functions

Theorem 3.1 ([18]) *Let X be a metrizable space. $\langle E^1, \ldots, E^7\rangle \in \Lambda^7(\mathcal{C}(X))$ iff the following conditions hold:*

1. *$E^1, \ldots, E^7$ form a partition of X;*
2. *E^1, E^2, E^3 are $F_{\sigma\delta}$ in X;*
3. *$E^3 \cup E^5 \cup E^7$ and $E^2 \cup E^6 \cup E^7$ are G_δ in X.*

Wesołowska in her dissertation [29] considered sets $E^1(\vec{f})$, $E^2(\vec{f})$, and $E^3(\vec{f})$ for sequences $\vec{f} = (f_n)_n$ of Baire-alpha mappings $f_n \colon X \to \mathbb{R}$ with $\alpha < \omega_1$. She characterized each family $\{E^i(\vec{f}) \colon \vec{f} \subset \mathcal{B}_\alpha(X)\}$ separately and formulated the problem of characterization of pairs $\langle E^2(\vec{f}), E^3(\vec{f})\rangle$ for $\vec{f} \subset \mathcal{B}_\alpha(X)$ [29, Problem 2.14]. This problem was solved in a more general form by Irek Recław shortly before his death. Unfortunately, he did not manage to publish this result. We will do it now, believing that his proof is worth saving.

Theorem 3.2 (Recław *) *Let (X,τ) be a metrizable space and let $\alpha < \omega_1$. Then $\langle E^1, \ldots, E^7\rangle \in \Lambda^7(\mathcal{B}_\alpha(X))$ iff the following conditions hold:*

1. *$E^1, \ldots, E^7$ form a partition of X;*
2. *E^1, E^2, E^3 are $\mathbf{\Pi}^0_{\alpha+3}$ in X;*
3. *$E^3 \cup E^5 \cup E^7$ and $E^2 \cup E^6 \cup E^7$ are $\mathbf{\Pi}^0_{\alpha+2}$ in X.*

Proof The case $\alpha = 0$ follows from Lunina's theorem. Let $\alpha > 0$. Clearly, if $\vec{f} \subset \mathcal{B}_\alpha(X)$, then the sets $E^i(\vec{f})$, $i = 1, \ldots, 7$, satisfy conditions (1)–(3). Assume now that the sets $E^1, \ldots, E^7$ satisfy conditions (1)–(3). Then, for $i = 1, 2, 3$, there exists a sequence

$$\{H^i_{m,n,k} \colon m,n,k \in \mathbb{N}\} \subset \mathbf{\Delta}^0_{\alpha+1}(X)$$

with

$$E^i = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} H^i_{m,n,k}.$$

Similarly, for $i = 4, 5$, there is a sequence

$$\{H^i_{m,n} \colon m,n \in \mathbb{N}\} \subset \mathbf{\Delta}^0_{\alpha+1}(X)$$

with

$$E^2 \cup E^5 \cup E^7 = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} H^4_{m,n} \quad \text{and} \quad E^3 \cup E^6 \cup E^7 = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} H^5_{m,n}.$$

Let τ' be a metrizable topology on X generated by the family $\tau \cup \{H^i_{m,n,k} \colon i \leqslant 3;\ m,n,k \in \mathbb{N}\} \cup \{H^i_{m,n} \colon i = 4,5;\ m,n \in \mathbb{N}\} \cup \{X \setminus H^i_{m,n,k} \colon i \leqslant 3;\ m,n,k \in \mathbb{N}\} \cup \{X \setminus H^i_{m,n} \colon i = 4,5;\ m,n \in \mathbb{N}\}$; see [28, Lemma 3.2.1, Remark 3.2.2]. Observe that the sets E^i, $i = 1,2,3$, are of $F_{\sigma\delta}$-type, and the sets $E^2 \cup E^5 \cup E^7$ and $E^3 \cup E^6 \cup E^7$ are of G_δ-type in topology τ'. Hence, by Lunina's theorem, there is a sequence $\vec{f} = (f_n)_n$ of functions $f_n \colon X \to \mathbb{R}$, which are continuous in topology τ' and such that $E^i = E^i(\vec{f})$ for $i = 1, \ldots, 7$. Finally, notice that for each $n \in \mathbb{N}$ and any open set $V \subset \mathbb{R}$, the set $f_n^{-1}(V)$ is of type $\mathbf{\Sigma}^0_{\alpha+1}$ in topology τ. Thus, $\vec{f} \subset \mathcal{B}_\alpha(X)$, and consequently, $\langle E^1, \ldots, E^7\rangle \in \Lambda^7(\mathcal{B}_\alpha(X))$. □

Lemma 3.3 *Let $\mathcal{F} \subset \mathbb{R}^X$ and let $\mathcal{F}_0$ be a dense (in topology of uniform convergence) subset of $\mathcal{F}$. Then $\Lambda^7(\mathcal{F}_0) = \Lambda^7(\mathcal{F})$.*

Proof The inclusion "$\subset$" is clear. To show that $\Lambda^7(\mathcal{F}) \subset \Lambda^7(\mathcal{F}_0)$, fix $\langle E^1,\dots,E^7\rangle \in \Lambda^7(\mathcal{F})$. Then there is a sequence $\vec{f} = (f_n)_n \subset \mathcal{F}$ such that $E^i(\vec{f}) = E^i$, $i = 1,\dots,7$. For every $n \in \mathbb{N}$, choose a function $g_n \in \mathcal{F}_0$ with $|f_n - g_n| < \frac{1}{n}$ and observe that for each $x \in X$ the equalities $\underline{\lim}_n f_n(x) = \underline{\lim}_n g_n(x)$ and $\overline{\lim}_n f_n(x) = \overline{\lim}_n g_n(x)$ hold. Thus, $E^i(\vec{f}) = E^i((g_n)_n)$ for $i = 1,\dots,7$, so $\Lambda^7(\mathcal{F}) \subset \Lambda^7(\mathcal{F}_0)$. □

The following theorem characterizes Lunina's 7-tuples for the family of $\mathbf{\Pi}^0_{\alpha+1}(X)$-measurable functions.

Theorem 3.4 *Let X be a metric space, $1 \leqslant \alpha < \omega_1$, and let $\mathcal{F}$ be the class of $\mathbf{\Pi}^0_{\alpha+1}(X)$-measurable functions. Then $\langle E^1,\dots,E^7\rangle \in \Lambda^7(\mathcal{F})$ iff the following conditions hold:*

1. *$E^1,\dots,E^7$ form a partition of X;*
2. *E^1, E^2, E^3 are $\mathbf{\Pi}^0_{\alpha+3}$ in X;*
3. *$E^3 \cup E^5 \cup E^7$ and $E^2 \cup E^6 \cup E^7$ are $\mathbf{\Pi}^0_{\alpha+2}$ in X.*

Proof Clearly, $\mathcal{F} \subset \mathcal{B}_\alpha(X)$. Moreover, $\mathcal{F}$ is dense (in topology of uniform convergence) in $\mathcal{B}_\alpha(X)$. Indeed, every $f \in \mathcal{B}_\alpha(X)$ is a uniform limit of a sequence $(f_n)_n \subset \mathcal{B}_\alpha(X)$ such that the range of every f_α is an isolated set; see [14, §31, VIII, Theorem 3, p. 388]). Clearly, each f_n is $\mathbf{\Pi}^0_{\alpha+1}(X)$-measurable. By Lemma 3.3,

$$\Lambda^7(\mathcal{F}) = \Lambda^7(\mathcal{B}_\alpha(X)).$$

Applying now Theorem 3.2, we get the assertion. □

Corollary 3.5 *Let X be a complete metric space. Then $\langle E^1,\dots,E^7\rangle \in \Lambda^7(\mathcal{B}^*_1(X))$ iff the following conditions hold:*

1. *$E^1,\dots,E^7$ form a partition of X;*
2. *E^1, E^2, E^3 are $G_{\delta\sigma\delta}$ in X;*
3. *$E^3 \cup E^5 \cup E^7$ and $E^2 \cup E^6 \cup E^7$ are $F_{\sigma\delta}$ in X.*

The following example shows that Theorem 3.4 does not happen in any topological space.

Example 3.6 *Let $X = \mathbb{R}$ be a topological space with the topology τ of co-countable sets:*

$$\tau = \{A \subset \mathbb{R} \colon \mathbb{R} \setminus A \textit{ is countable}\} \cup \{\emptyset\}.$$

Then:

- *$\mathcal{C}(X)$ is equal to the family of constant functions;*
- *$\mathcal{B}^*_1(X) = \mathbb{R}^X$;*
- *$G_{\delta\sigma\delta}(X) = \tau$;*
- *$\mathbf{\Delta}^0_1(X) = \{\emptyset, \mathbb{R}\}$.*

Note that for a given $\alpha < \omega_1$ we can also consider the αth Sierpiński class of mappings $\mathcal{S}_\alpha(X) \subset \mathbb{R}^X$. Recall that $\mathcal{S}_\alpha(X) = \mathcal{L}_\alpha(X) + \mathcal{U}_\alpha(X)$, where $\mathcal{L}_\alpha(X)$ and $\mathcal{U}_\alpha(X)$ are the classes of limits of, respectively, nondecreasing and nonincreasing sequences of functions from the class $\mathcal{B}_\alpha(X)$ and the sign "+" denotes the algebraic sum of two sets of reals, i.e. $A + B = \{a + b \colon a \in A, b \in B\}$. Clearly,

$$\mathcal{B}_\alpha(X) \subset \mathcal{S}_\alpha(X) \subset \mathcal{B}_{\alpha+1}(X),$$

and if X is an uncountable Polish space then all above inclusions are proper. Moreover, for each $\alpha < \omega_1$, the closure of $\mathcal{S}_\alpha(X)$ (with topology of uniform convergence) is equal to $\mathcal{B}_{\alpha+1}(X)$; see [22]. Hence (and by Lemma 3.3), we get the following:

Corollary 3.7 *Let X be a Polish space. Then $\Lambda^7(\mathcal{S}_\alpha(X)) = \Lambda^7(\mathcal{B}_{\alpha+1}(X))$ for every ordinal $\alpha < \omega_1$.*

4. Ideal Lunina's 7-tuples for the families of Borel functions

4.1. Ideals on $\mathbb{N}$

An ideal on $\mathbb{N}$ is a nonempty family of subsets of $\mathbb{N}$ closed under taking finite unions and subsets of its elements. We assume moreover that $\mathcal{I}$ is proper ($\mathbb{N} \notin \mathcal{I}$) and contains all finite sets. By FIN we denote the ideal of all finite subsets of $\mathbb{N}$. By identifying subsets of $\mathbb{N}$ with their characteristic functions, we can identify the family $\mathcal{P}(\mathbb{N})$ with the Cantor space $\mathbb{C}$. In this sense, ideals can be F_σ sets or have the Baire property in the space $\mathbb{C}$. In particular, F_σ ideals have a nice property in terms of submeasures.

A map $\phi \colon \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a submeasure if

- $\phi(\emptyset) = 0$,
- ϕ is monotone (i.e. $\phi(A) \leqslant \phi(B)$ whenever $A \subset B$), and
- ϕ is subadditive (i.e. $\phi(A \cup B) \leqslant \phi(A) + \phi(B)$).

We will assume also that $\phi(\mathbb{N}) > 0$. For a submeasure ϕ, set

$$\mathrm{FIN}(\phi) = \{A \subset \mathbb{N} \colon \phi(A) < \infty\}.$$

For example, $\mathrm{FIN} = \mathrm{FIN}(\phi)$ for $\phi(A) = |A|$ – the number of elements of A. A submeasure ϕ is lower semicontinuous (lsc, in short) if

$$\phi(A) = \lim_{n\to\infty} \phi(A \cap \{0, 1, \ldots, n-1\})$$

holds for all $A \subset \mathbb{N}$. For example, $\mathrm{FIN} = \mathrm{FIN}(\phi)$ for $\phi(A) = |A|$ – the number of elements of A.

Now we recall the above-mentioned property of F_σ ideals.

Theorem 4.1 ([21]) *If $\mathcal{I}$ is an F_σ ideal then there exists an lsc submeasure ϕ such that $\mathcal{I} = FIN(\phi)$.*

The ideal FIN is F_σ. Similarly, summable ideals are F_σ. The ideal of sets of asymptotic density zero is $F_{\sigma\delta}$ but not F_σ (see, e.g., [6] for definitions and more details).

The following lemma follows directly from the definition.

Lemma 4.2 *[[1, Lemma 6]] Assume that $\mathcal{I} = FIN(\phi)$ for a lower semicontinuous submeasure ϕ. Then $A \in \mathcal{I}$ iff there exists a natural number n_A so that for every $B \in FIN$, if $B \subset A$ then $\phi(B) \leqslant n_A$.*

4.2. Ideal convergence

Fix an ideal $\mathcal{I}$. For a sequence $(x_n)_n$ of reals and $x \in \mathbb{R}$ we define:

$$\begin{array}{rcl} \mathcal{I}-\lim x_n = x & \text{iff} & \{n \in \mathbb{N}: \ |x_n - x| \geqslant \varepsilon\} \in \mathcal{I} \text{ for any } \varepsilon > 0; \\ \mathcal{I}-\lim x_n = +\infty & \text{iff} & \{n \in \mathbb{N}: \ x_n < M\} \in \mathcal{I} \text{ for any } M \in \mathbb{R}; \\ \mathcal{I}-\lim x_n = -\infty & \text{iff} & \{n \in \mathbb{N}: \ x_n > M\} \in \mathcal{I} \text{ for any } M \in \mathbb{R}; \\ \mathcal{I}-\overline{\lim}\ x_n & = & \inf\{\alpha\colon \{n\colon x_n > \alpha\} \in \mathcal{I}\}; \\ \mathcal{I}-\underline{\lim}\ x_n & = & \sup\{\alpha\colon \{n\colon x_n < \alpha\} \in \mathcal{I}\}. \end{array}$$

We say that $(x_n)_n$ is $\mathcal{I}$-convergent if $\mathcal{I}-\lim x_n = x$ for some $x \in \mathbb{R}$. Note that this fact is equivalent to the following Cauchy-like condition (see, e.g., [5]):

$$\forall_{\varepsilon>0}\ \exists_k\ \{n\colon |x_n - x_k| > \varepsilon\} \in \mathcal{I}.$$

4.3. Ideal Lunina's 7-tuples

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and let $\vec{f} = (f_n)_n$ be a sequence of functions $f_n\colon X \to \mathbb{R}$. We define seven types of sets of $\mathcal{I}$-convergence and divergence points for the sequence $\vec{f}$:

$$\begin{array}{rcl} E^1_{\mathcal{I}}(\vec{f}) & = & \{x\colon (f_n(x)) \text{ is } \mathcal{I}\text{-convergent}\}; \\ E^2_{\mathcal{I}}(\vec{f}) & = & \{x\colon \mathcal{I}-\lim f_n(x) = -\infty\}; \\ E^3_{\mathcal{I}}(\vec{f}) & = & \{x\colon \mathcal{I}-\lim f_n(x) = +\infty\}; \\ E^4_{\mathcal{I}}(\vec{f}) & = & \{x\colon -\infty < \mathcal{I}-\underline{\lim}\ f_n(x) < \mathcal{I}-\overline{\lim}\ f_n(x) < +\infty\}; \\ E^5_{\mathcal{I}}(\vec{f}) & = & \{x\colon -\infty = \mathcal{I}-\underline{\lim}\ f_n(x) < \mathcal{I}-\overline{\lim}\ f_n(x) < +\infty\}; \\ E^6_{\mathcal{I}}(\vec{f}) & = & \{x\colon -\infty < \mathcal{I}-\underline{\lim}\ f_n(x) < \mathcal{I}-\overline{\lim}\ f_n(x) = +\infty\}; \\ E^7_{\mathcal{I}}(\vec{f}) & = & \{x\colon -\infty = \mathcal{I}-\underline{\lim}\ f_n(x)\ \&\ \mathcal{I}-\overline{\lim}\ f_n(x) = +\infty\}. \end{array}$$

Let $\mathcal{F} \subset \mathbb{R}^X$. A sequence $\langle E^1, \ldots, E^7 \rangle$ is called an $\mathcal{I}$-Lunina's 7-tuple for $\mathcal{F}$ if there is a sequence $\vec{f} = (f_n)_n \subset \mathcal{F}$ such that $E^i = E^i_{\mathcal{I}}(\vec{f})$, $i = 1, \ldots, 7$; see [1] or [6]. The family of all $\mathcal{I}$-Lunina's 7-tuples for $\mathcal{F}$ is denoted by $\Lambda^7_{\mathcal{I}}(\mathcal{F})$. In our considerations we use the following fact, which originally was proved (see [1, Theorem 3]) for $\mathcal{F}$ being the class of continuous functions.

Lemma 4.3 *[[24, Lemma 1]] The inclusion*

$$\Lambda^7(\mathcal{F}) \subset \Lambda^7_{\mathcal{I}}(\mathcal{F})$$

holds for each family $\mathcal{F} \subset \mathbb{R}^X$ and any ideal $\mathcal{I}$ with the Baire property.

Borzestowski and Recław [1] considered the sets $E^i_{\mathcal{I}}(\vec{f})$, $i = 1, \ldots, 7$, for the sequences $\vec{f}$ of continuous functions and they proved the following:

Theorem 4.4 *Let X be a metric space and let $\mathcal{I}$ be an F_σ ideal. Then*

$$\Lambda^7_{\mathcal{I}}(\mathcal{C}(X)) = \Lambda^7(\mathcal{C}(X)).$$

4.4. Results

Theorem 4.5 (Recław †) *Let X be a metric space and let $\mathcal{I}$ be an F_σ ideal. Then*

$$\Lambda^7_{\mathcal{I}}(\mathcal{B}_\alpha(X)) = \Lambda^7(\mathcal{B}_\alpha(X)) \text{ for any } \alpha < \omega_1.$$

Proof The inclusion "$\supset$" follows from Lemma 4.3. To show that $\Lambda^7_{\mathcal{I}}(\mathcal{B}_\alpha(X)) \subset \Lambda^7(\mathcal{B}_\alpha(X))$, fix $\alpha < \omega_1$ and $\langle E^1, \dots, E^7 \rangle \in \Lambda^7_{\mathcal{I}}(\mathcal{B}_\alpha(X))$. Then there is $\vec{f} = (f_n)_n$ with $E^i = E^i_{\mathcal{I}}(\vec{f})$ for $i = 1, \dots, 7$. By Lemma 4.2 we have

$$E^1_{\mathcal{I}}(\vec{f}) = \left\{ x \in X \colon \forall_{k\in\mathbb{N}}\ \exists_{j\in\mathbb{N}}\ \left\{ n \colon |f_n(x) - f_j(x)| > \frac{1}{k} \right\} \in \mathcal{I} \right\} =$$

$$\left\{ x \in X \colon \forall_{k\in\mathbb{N}}\ \exists_{j\in\mathbb{N}}\ \exists_{m\in\mathbb{N}}\ \forall_{B\in\mathrm{FIN}}\ \left(\phi(B) > m \Rightarrow \exists_{i\in B} |f_j(x) - f_i(x)| \leqslant \frac{1}{k} \right) \right\} =$$

$$\bigcap_{k\in\mathbb{N}} \bigcup_{j,m\in\mathbb{N}} \bigcap_{\substack{B\in\mathrm{FIN}\\ \phi(B)>m}} \bigcup_{i\in B} \left\{ x \colon |f_j(x) - f_i(x)| \leqslant \frac{1}{k} \right\}$$

and thus $E^1_{\mathcal{I}}(\vec{f}) \in \mathbf{\Pi}^0_{\alpha+3}(X)$. Similarly, since

$$E^2_{\mathcal{I}}(\vec{f}) = \{ x \in X \colon \forall_{k\in\mathbb{N}}\ \{ n \colon f_n(x) > -k \} \in \mathcal{I} \} =$$

$$\{ x \in X \colon \forall_{k\in\mathbb{N}}\ \exists_{m\in\mathbb{N}}\ \forall_{B\in\mathrm{FIN}}\ (\phi(B) > m \Rightarrow \exists_{i\in B} f_i(x) \leqslant -k) \} =$$

$$\bigcap_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{\substack{B\in\mathrm{FIN}\\ \phi(B)>m}} \bigcup_{i\in B} \{ x \colon f_i(x) \leqslant -k \}$$

and

$$E^3_{\mathcal{I}}(\vec{f}) = \bigcap_{k\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} \bigcap_{\substack{B\in\mathrm{FIN}\\ \phi(B)>m}} \bigcup_{i\in B} \{ x \colon f_i(x) \geqslant k \},$$

we have $E^2_{\mathcal{I}}(\vec{f}), E^3_{\mathcal{I}}(\vec{f}) \in \mathbf{\Pi}^0_{\alpha+3}(X)$. Moreover, since

$$E^3 \cup E^5 \cup E^7 = \left\{ x \in X \colon \mathcal{I} - \overline{\lim}_n(\vec{f}) = +\infty \right\} =$$

$$\{ x \in X \colon \forall_{k\in\mathbb{N}}\ \{ n \in \mathbb{N} \colon f_n(x) > k \} \notin \mathcal{I} \} =$$

$$\bigcap_{k\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \bigcup_{\substack{B\in\mathrm{FIN}\\ \phi(B)>m}} \bigcap_{i\in B} \{ x \in X \colon f_i(x) > k \},$$

$E^3 \cup E^5 \cup E^7 \in \mathbf{\Pi}^0_{\alpha+2}(X)$. Similarly,

$$E^2 \cup E^6 \cup E^7 = \{ x \in X \colon \mathcal{I} - \underline{\lim}_n(\vec{f}) = -\infty \} \in \mathbf{\Pi}^0_{\alpha+2}(X).$$

Observe that the sets $E^1, \dots, E^7$ satisfy conditions (1)–(3) from Theorem 3.2. Therefore, $\langle E^1, \dots, E^7 \rangle \in \Lambda^7(\mathcal{B}_\alpha(X))$. □

Corollary 4.6 *Let X be a complete metric space and let $\mathcal{I}$ be an F_σ ideal. Then*

$$\Lambda^7_{\mathcal{I}}(\mathcal{B}^*_1(X)) = \Lambda^7(\mathcal{B}^*_1(X)).$$

Proof We have

$$\Lambda^7(\mathcal{B}_1^*(X)) \subset \Lambda^7_{\mathcal{I}}(\mathcal{B}_1^*(X)) \subset \Lambda^7_{\mathcal{I}}(\mathcal{B}_1(X)) = \Lambda^7(\mathcal{B}_1(X)) = \Lambda^7(\mathcal{B}_1^*(X)).$$

□

Corollary 4.7 *Let X be a Polish space, $\mathcal{I}$ be an F_σ ideal, and α be a countable ordinal. Then*

$$\Lambda^7_{\mathcal{I}}(\mathcal{S}_\alpha(X)) = \Lambda^7(\mathcal{S}_\alpha(X)) = \Lambda^7(\mathcal{B}_{\alpha+1}(X)).$$

Proposition 4.8 *Let X be a complete metric space containing a subspace homeomorphic to the Cantor space $\mathcal{P}(\mathbb{N})$. If $\Lambda^7_{\mathcal{I}}(\mathcal{B}_1^*(X)) = \Lambda^7(\mathcal{B}_1^*(X))$, then $\mathcal{I}$ is an $F_{\sigma\delta}$-ideal.*

Proof We use an idea of the proof of [1, Theorem 8]. Consider a sequence of continuous functions $f_n : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ given by the formula

$$f_n(A) = \begin{cases} 0 & \text{if } n \notin A, \\ n & \text{otherwise.} \end{cases}$$

Observe that

- if $A \in \mathcal{I}$ then $\mathcal{I} - \lim f_n(A) = 0$, and hence $A \in E^1_{\mathcal{I}}(\vec{f})$;
- if $A \notin \mathcal{I}$ then $\{n : f_n(A) > k\} = A \setminus \{0, \dots, k\} \notin \mathcal{I}$ for each $k \in \mathbb{N}$, so $A \in E_{\mathcal{I}}(\vec{f}) = E^3_{\mathcal{I}}(\vec{f}) \cup E^5_{\mathcal{I}}(\vec{f}) \cup E^7_{\mathcal{I}}(\vec{f})$.

Therefore, $E_{\mathcal{I}}(\vec{f}) = \mathcal{P}(\mathbb{N}) \setminus \mathcal{I}$. Assume now that $h : \mathcal{P}(\mathbb{N}) \to X$ is a homeomorphic embedding and $X_0 = h[\mathcal{P}(\mathbb{N})]$. Then X_0 is closed in X and for each $n \in \mathbb{N}$, $g_n = f_n \circ h^{-1} \in \mathcal{B}_1^*(X_0)$ (even it is continuous). Let $\vec{g} = (g_n)_n$. Observe that

$$E_{\mathcal{I}}(\vec{g}) = E^3_{\mathcal{I}}(\vec{g}) \cup E^5_{\mathcal{I}}(\vec{g}) \cup E^7_{\mathcal{I}}(\vec{g}) = h[E_{\mathcal{I}}(\vec{f})] = h[\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}].$$

We can extend each g_n to a continuous function $\tilde{g}_n$ defined on whole X. Put $E_{\mathcal{I}}((\tilde{g}_n)_n) = E^3_{\mathcal{I}}((\tilde{g}_n)_n) \cup E^5_{\mathcal{I}}((\tilde{g}_n)_n) \cup E^7_{\mathcal{I}}((\tilde{g}_n)_n)$, and then $E_{\mathcal{I}}(\vec{g}) = E_{\mathcal{I}}((\tilde{g}_n)_n) \cap X_0$. By the assumption, $E_{\mathcal{I}}((\tilde{g}_n)_n) \in G_{\delta\sigma}(X)$; hence,

$$h(\mathcal{I}) = X_0 \setminus h[\mathcal{P}(\mathbb{N}) \setminus \mathcal{I}] = X_0 \setminus E_{\mathcal{I}}(\vec{g}) = X_0 \setminus E_{\mathcal{I}}((\tilde{g}_n)_n) \in F_{\sigma\delta}(X_0).$$

Thus, $\mathcal{I}$ is of $F_{\sigma\delta}$-type. □

Problem 4.9 *Assume that $\mathcal{I}$ is an $F_{\sigma\delta}$ ideal. Does the equality $\Lambda^7_{\mathcal{I}}(\mathcal{B}_1^*(\mathbb{R})) = \Lambda^7(\mathcal{B}_1^*(\mathbb{R}))$ hold? Is it true for the ideal of sets of asymptotic density zero?*

5. Limits of sequences of Baire-star-one functions

5.1. ω-diagonalizable filters

We say that a sequence $\vec{f} = (f_n)_n \subset \mathbb{R}^X$ is $\mathcal{I}$-convergent to a function $f : X \to \mathbb{R}$ if $\mathcal{I} - \lim_n f_n(x) = f(x)$ for every $x \in X$. Then f is called an $\mathcal{I}$-limit of a sequence $\vec{f}$. For a family $\mathcal{F} \subset \mathbb{R}^X$, the symbol $\mathcal{I} - \mathrm{LIM}\,(\mathcal{F})$ denotes the family of all $\mathcal{I}$-limits of sequences $\vec{f} \subset \mathcal{F}$. Clearly, if $\mathcal{I} \subset \mathcal{J}$ then $\mathcal{I} - \mathrm{LIM}\,(\mathcal{F}) \subset \mathcal{J} - \mathrm{LIM}\,(\mathcal{F})$ for any family $\mathcal{F} \subset \mathbb{R}^X$. In particular:

Lemma 5.1 *The inclusion*

$$\mathrm{LIM}\,(\mathcal{F}) \subset \mathcal{I} - \mathrm{LIM}\,(\mathcal{F})$$

holds for every family $\mathcal{F} \subset \mathbb{R}^X$ *and all ideals* $\mathcal{I}$*.*

We say that the set $\mathcal{I}^\star = \{A \colon \mathbb{N} \setminus A \in \mathcal{I}\}$ is a dual filter to an ideal $\mathcal{I}$. A set $\mathcal{Z} = \{A_m \colon m \in \mathbb{N}\}$ of nonempty finite subsets of $\mathbb{N}$ is called $\mathcal{I}^\star$-universal if for each $F \in \mathcal{I}^\star$ there is $m \in \mathbb{N}$ such that $A_m \subset F$. We say that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets if there exists a sequence $(\mathcal{Z}_n)_n$ of $\mathcal{I}^\star$-universal sets such that for every $F \in \mathcal{I}^\star$ there is $n \in \mathbb{N}$ with $A \cap F = \emptyset$ only for finitely many sets $A \in \mathcal{Z}_n$ (see [16]). Clearly, the filter $\mathrm{FIN}^\star$ of cofinite sets is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. Moreover, each $F_{\sigma\delta}$ ideal has this property (see [15]). The following lemma gives us a useful criterion of ω-diagonalizability.

Lemma 5.2 ([7]) $\mathcal{I}^\star$ *is* ω*-diagonalizable by* $\mathcal{I}^\star$*-universal sets iff there exists a sequence* $(\mathcal{Z}_n)_n$ *such that*

1. $\mathcal{Z}_n \subset [\mathbb{N}]^{<\mathbb{N}}$ *for each* $n \in \mathbb{N}$*;*
2. $|\{A \in \mathcal{Z}_n \colon A \subset F\}| = \omega$ *for each* $n \in \mathbb{N}$ *and* $F \in \mathcal{I}^\star$*;*
3. *for each* $F \in \mathcal{I}^\star$ *there is* $n \in \mathbb{N}$ *such that* $A \cap F \neq \emptyset$ *for all* $A \in \mathcal{Z}_n$*.*

We say that a filter $\mathcal{I}^\star$ is ω-+-diagonalizable if there are sets $\{X_n \in \mathcal{I}^+ \colon n \in \mathbb{N}\}$ such that for each $A \in \mathcal{I}$ there is $n \in \mathbb{N}$ with $X_n \cap A = \emptyset$ (equivalently, for each $B \in \mathcal{I}^\star$ there is n with $X_n \subset B$) [16]. Clearly, the filter $\mathrm{FIN}^\star$ is ω-+-diagonalizable.

Lemma 5.3 ([16]) *Every* ω*-+-diagonalizable filter* $\mathcal{I}^\star$ *is* ω*-diagonalizable by* $\mathcal{I}^\star$*-universal sets.*

The family of F_σ ideals $\mathcal{I}$ for which $\mathcal{I}^\star$ is ω-+-diagonalizable is a proper subclass of all F_σ ideals. The ideal of sets of asymptotic density zero is not ω-+-diagonalizable (see [23]).

5.2. Results

For an ideal $\mathcal{I}$ and a topological space X we define $\mathcal{I}$-Baire classes of mappings:

- $\mathcal{B}_0^{\mathcal{I}}(X) = \mathcal{C}(X)$ and
- $\mathcal{B}_\alpha^{\mathcal{I}}(X) = \mathcal{I} - \mathrm{LIM}\left(\bigcup_{\beta<\alpha} \mathcal{B}_\beta^{\mathcal{I}}(X)\right)$, for $\alpha > 0$.

In our considerations we use the following result, which, in the case of functions defined on a Polish space X, belongs to Laczkovich and Recław [15] and independently to Debs and Saint Raymond [4]. In general form it has been proved by Filipów and Szuca.

Theorem 5.4 *[[7, Theorem 3.2]] Let* X *be a perfectly normal topological space. If* $\mathcal{I}$ *is an ideal such that* $\mathcal{I}^\star$ *is* ω*-diagonalizable by* $\mathcal{I}^\star$*-universal sets then* $\mathcal{B}_\alpha^{\mathcal{I}}(X) = \mathcal{B}_\alpha(X)$*.*

Lemma 5.5 *Let* $\mathcal{F} \subset \mathbb{R}^X$ *and let* $\mathcal{F}_0$ *be a dense (in topology of uniform convergence) subset of* $\mathcal{F}$*. Then for any ideal* $\mathcal{I}$ *we have*

$$\mathcal{I} - \mathrm{LIM}\,(\mathcal{F}_0) = \mathcal{I} - \mathrm{LIM}\,(\mathcal{F})\,.$$

Proof The inclusion "$\subset$" is clear. To show that $\mathcal{I}-\mathrm{LIM}\,(\mathcal{F})\subset\mathcal{I}-\mathrm{LIM}\,(\mathcal{F}_0)$, fix $f\in\mathcal{I}-\mathrm{LIM}\,(\mathcal{F})$. Then there is a sequence $\vec{f}=(f_n)_n\subset\mathcal{F}$ such that $\mathcal{I}-\lim_n f_n=f$. For every $n\in\mathbb{N}$, choose a function $g_n\in\mathcal{F}_0$ with $|f_n-g_n|<\frac{1}{n}$. Hence, $\mathcal{I}-\lim_n g_n=\mathcal{I}-\lim_n f_n=f$. □

From Lemma 5.5 and Theorem 5.4 we obtain:

Theorem 5.6 *Let X be a perfectly normal topological space, $1\leqslant\alpha<\omega_1$, and $\mathcal{F}$ be the class of all $\mathbf{\Pi}^0_{\alpha+1}(X)$-measurable functions. If $\mathcal{I}$ is an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets then $\mathcal{I}-\mathrm{LIM}\,(\mathcal{F})=\mathcal{B}_{\alpha+1}(X)$.*

Corollary 5.7 *Let X be a completely metrizable space. If $\mathcal{I}$ is an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets then $\mathcal{I}-\mathrm{LIM}\,(\mathcal{B}^*_1(X))=\mathcal{B}_2(X)$.*

In particular, for $\mathcal{I}=\mathrm{FIN}$ we obtain the following two corollaries.

Corollary 5.8 *If X is a perfectly normal topological space, $1\leqslant\alpha<\omega_1$, and $\mathcal{F}$ is the class of all $\mathbf{\Pi}^0_{\alpha+1}(X)$-measurable functions then $\mathrm{LIM}\,(\mathcal{F})=\mathcal{B}_{\alpha+1}(X)$.*

Corollary 5.9 *If X is a complete metric space then $\mathrm{LIM}\,(\mathcal{B}^*_1(X))=\mathcal{B}_2(X)$.*

Theorem 5.10 *Assume that $\mathcal{I}$ is an analytic ideal and X is an uncountable Polish space. Then $\mathcal{I}-\mathrm{LIM}\,(\mathcal{B}^*_1(X))=\mathcal{B}_2(X)$ iff $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets.*

Proof The implication "$\Leftarrow$" follows from Corollary 5.7. To show the implication "$\Rightarrow$", suppose that $\mathcal{I}^\star$ is not ω-diagonalizable by $\mathcal{I}^\star$ universal sets. Then $\mathcal{I}-\mathrm{LIM}\,(\mathcal{B}_1(X))\setminus\mathcal{B}_2(X)\neq\emptyset$; see [7, Theorem 6.1] (note that in [7] the authors assumed that $\mathcal{I}$ is a Borel ideal, but the same proof works also for all analytic ideals; see [6, Theorem 7.30]). By Lemma 5.5, $\mathcal{I}-\mathrm{LIM}\,(\mathcal{B}^*_1(X))=\mathcal{I}-\mathrm{LIM}\,(\mathcal{B}_1(X))$, so $\mathcal{I}-\mathrm{LIM}\,(\mathcal{B}^*_1(X))\neq\mathcal{B}_2(X)$. □

Corollary 5.11 *Assume that X is a Polish space, $\mathcal{I}$ is an ideal such that $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets, and α is a countable ordinal. Then*

$$\mathcal{I}-\mathrm{LIM}\,(\mathcal{S}_\alpha(X))=\mathcal{B}_{\alpha+2}(X).$$

6. Appendix: Darboux Baire α functions

In the last part of the paper we consider Darboux functions defined on the real line. Recall that $f\colon\mathbb{R}\to\mathbb{R}$ is a Darboux function if it has the intermediate value property. The class of all Darboux functions is denoted by $\mathcal{D}$. For a family $\mathcal{F}\subset\mathbb{R}^{\mathbb{R}}$, we denote by $\mathcal{DF}$ the class $\mathcal{D}\cap\mathcal{F}$.

Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Also let $\vec{f}=(f_n)_n$ and $\vec{g}=(g_n)_n$ be the sequences of functions defined on the real line. We say that:

- $\vec{f}$ and $\vec{g}$ are $\mathcal{I}$-equivalent if $\{n\in\mathbb{N}\colon f_n(x)\neq g_n(x)\}\in\mathcal{I}$ for each $x\in X$;
- $\vec{f}$ and $\vec{g}$ are equivalent if they are FIN-equivalent.

Proposition 6.1 *Let $\mathcal{I}$ be an ideal and $\mathcal{F} \subset \mathbb{R}^{\mathbb{R}}$ be a family of functions such that every sequence $\vec{f} \subset \mathcal{F}$ is $\mathcal{I}$-equivalent to some sequence $\vec{g} \subset \mathcal{DF}$. Then:*

1. $\Lambda^7_{\mathcal{I}}(\mathcal{DF}) = \Lambda^7_{\mathcal{I}}(\mathcal{F})$ *and*
2. $\mathcal{I} - \mathrm{LIM}\,(\mathcal{DF}) = \mathcal{I} - \mathrm{LIM}\,(\mathcal{F})$.

Proof (1). The inclusion "$\subset$" is clear. To see that the opposite inclusion is true, note that if $\vec{f} = (f_n)_n$ and $\vec{g} = (g_n)_n$ are $\mathcal{I}$-equivalent then the equalities $\mathcal{I}-\underline{\lim}_n f_n(x) = \mathcal{I}-\underline{\lim}_n g_n(x)$ and $\mathcal{I}-\overline{\lim}_n f_n(x) = \mathcal{I}-\overline{\lim}_n g_n(x)$ hold for each $x \in \mathbb{R}$. Thus, $E^i_{\mathcal{I}}(\vec{f}) = E^i_{\mathcal{I}}(\vec{g})$ for $i = 1, \ldots, 7$. The proof of (2) is similar. $\square$

Lemma 6.2 *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$ and let $\alpha \geqslant 1$ be a countable ordinal. Then every sequence $\vec{f} \subset \mathcal{B}_\alpha$ is equivalent to some sequence $\vec{g} \subset \mathcal{DB}_\alpha$.*

Proof Let $f \in \mathcal{B}_\alpha$ and let E be a meager subset of $\mathbb{R}$. Then there is $g \in \mathcal{DB}_\alpha$ such that $f = g$ except on a meager set $M \subset \mathbb{R}$, which is disjoint with E; see [2] and [19, Proposition 1] for the case of $\alpha = 1$. The case $\alpha > 1$ can be proved in a standard way; see, e.g., [20, Theorem II.1.2]. List all open intervals with rational end-points in a sequence $(I_n)_n$. For every $n, k \in \mathbb{N}$ choose a Cantor set $C_{n,k} \subset I_n \setminus E$ such that $C_{n,k} \cap C_{i,j} = \emptyset$ whenever $\langle n, k\rangle \neq \langle i, j\rangle$ and fix a continuous surjection $g_{n,k} \colon C_{n,k} \to [-k, k]$. Then $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} g_{n,k}(x) & \text{for } x \in C_{n,k},\ \langle n,k\rangle \in \mathbb{N}^2; \\ f(x) & \text{otherwise} \end{cases}$$

has all required properties.

Fix $\vec{f} = (f_n)_n \subset \mathcal{B}_\alpha$. Put $E_0 = \emptyset$ and for $n \geqslant 1$ choose inductively a function $g_n \in \mathcal{DB}_\alpha$ such that the set $E_n = \{x \in \mathbb{R} \colon f_n(x) \neq g_n(x)\}$ is meager and disjoint with the set $\bigcup_{i<n} E_i$. Let $\vec{g} = (g_n)_n$. Then $\vec{g} \subset \mathcal{DB}_\alpha$ and $\{n \in \mathbb{N} : f_n(x) \neq g_n(x)\}$ has at most one element. Thus, $\vec{f}$ and $\vec{g}$ are $\mathcal{I}$-equivalent. $\square$

Corollary 6.3 *Let $\mathcal{I}$ be an ideal and let $\alpha \geqslant 1$ be a countable ordinal. Then:*

1. $\Lambda^7_{\mathcal{I}}(\mathcal{DB}_\alpha) = \Lambda^7_{\mathcal{I}}(\mathcal{B}_\alpha)$ *and*
2. $\mathcal{I} - \mathrm{LIM}\,(\mathcal{DB}_\alpha) = \mathcal{I} - \mathrm{LIM}\,(\mathcal{B}_\alpha)$.

In particular, for $\mathcal{I} = \mathrm{FIN}$ we obtain:

Corollary 6.4 *If $\alpha \geqslant 1$ is a countable ordinal then*

1. $\Lambda^7(\mathcal{DB}_\alpha) = \Lambda^7(\mathcal{B}_\alpha)$ *and*
2. $\mathrm{LIM}\,(\mathcal{DB}_\alpha) = \mathrm{LIM}\,(\mathcal{B}_\alpha)$.

Finally, note that if $\alpha > 1$ and $\mathcal{F}$ is equal to the family of all $\mathbf{\Pi}^0_\alpha$-measurable functions then every sequence $\vec{f} \subset \mathcal{F}$ is $\mathcal{I}$-equivalent to some sequence $\vec{g} \subset \mathcal{DF}$. This fact can be proved exactly in the same way as Lemma 6.2. Thus, the following conditions hold:

1. $\Lambda^7_{\mathcal{I}}(\mathcal{DF}) = \Lambda^7_{\mathcal{I}}(\mathcal{F})$;

2. $\mathcal{I} - \mathrm{LIM}\,(\mathcal{DF}) = \mathcal{I} - \mathrm{LIM}\,(\mathcal{F})$.

In the case of $\alpha = 1$, i.e. for the class $\mathcal{DB}^*_1$, the situation is more complicated. In 1991 Grande [8, Theorem 1] proved that $\mathrm{LIM}\,(\mathcal{DB}^*_1) = \mathcal{B}_2 \cap \mathcal{PWD}$, where $\mathcal{PWD}$ is the class of all pointwise discontinuous functions, i.e. such mappings $f : \mathbb{R} \to \mathbb{R}$ for which the set $C(f)$ of all continuity points of f is dense in $\mathbb{R}$.

Theorem 6.5 *Let $\mathcal{I}$ be an ideal such that the filter $\mathcal{I}^\star$ is ω-+-diagonalizable. Then*

$$\mathcal{I} - \mathrm{LIM}\,(\mathcal{DB}^*_1) = \mathcal{B}_2 \cap \mathcal{PWD}.$$

Proof The inclusion "$\supset$" follows from Grande's theorem [8, Theorem 1] and Lemma 5.1. To prove the inclusion "$\subset$", fix $f = \mathcal{I} - \lim_n f_n$ for some sequence $(f_n)_n \subset \mathcal{DB}^*_1$. By Lemma 5.3, the filter $\mathcal{I}^\star$ is ω-diagonalizable by $\mathcal{I}^\star$-universal sets. Thus, Corollary 5.9 yields $f \in \mathcal{B}_2$.

Recall that if $g \in \mathcal{DB}^\star_1$ then $g|_{C(g)}$ is dense in g [26], and hence g is quasi-continuous in the sense of Kempisty (see, e.g., [9, Lemma 2]). Thus, $(f_n)_n$ is a sequence of quasi-continuous functions and $\mathcal{I}^\star$ is ω-+-diagonalizable, so $f = \mathcal{I} - \lim_n f_n$ is pointwise discontinuous; see [23, Proposition 3.1]. □

Problem 6.6 *Suppose $\mathcal{I}$ is an ideal such that $\mathcal{I} - \mathrm{LIM}\,(\mathcal{DB}^*_1) = \mathcal{B}_2 \cap \mathcal{PWD}$. Is the filter $\mathcal{I}^\star$ ω-+-diagonalizable?*

Problem 6.6 is open even under the assumption that $\mathcal{I}$ is an analytic ideal.

References

[1] Borzestowski D, Recław I. On Lunina's 7-tuples for ideal convergence. Real Analysis Exchange 2010; 35 (2): 479-485.

[2] Bruckner AM, Ceder JG, Keston R. Representations and approximations by Darboux functions in the first class of Baire. Revue Roumaine des Mathematiques Pures et Appliquees 1968; 13 (9): 1247-1254.

[3] Császár Á, Laczkovich M. Some remarks on discrete Baire classes. Acta Mathematica Academiae Scientiarum Hungaricae 1979; 33 (1-2): 51-70.

[4] Debs G, Saint Raymond J. Filter descriptive classes of Borel functions. Fundamenta Mathematicae 2009; 204 (3): 189-213.

[5] Dems K. On $\mathcal{J}$-Cauchy sequences. Real Analysis Exchange 2004; 30: 123-128.

[6] Filipów R, Natkaniec T, Szuca P. Ideal convergence. In: Filipczak M, Wagner-Bojakowska E, (editors). Traditional and Present-Day Topics in Real Analysis: Dedicated to Professor Jan Stanisław Lipiński. Łódź, Poland: Łódź University Press, 2013, pp. 69-91.

[7] Filipów R, Szuca P. Three kinds of convergence and the associated $\mathcal{I}$-Baire classes. Journal of Mathematical Analysis and Applications 2012; 391 (1): 1-9.

[8] Grande Z. Some remarks about Rosen's functions. Proceedings of the American Mathematical Society 1991; 113 (1): 117-122.

[9] Grande Z, Natkaniec T. Lattices generated by $\mathcal{T}$ - quasi-continuous functions. Bulletin of the Polish Academy of Sciences Mathematics 1986; 34: 525-530.

[10] Hahn H. Uber die Menge der Konvergenzpunkte einer Funktionfolge. Archiv der Mathematik und Physik 1919; 28: 34-45 (in German).

[11] Jayne JE, Rogers CA. First level Borel functions and isomorphisms. Journal de Mathématiques Pures et Appliquées 1982; 61 (2): 177-205.

[12] Kirchheim B. Baire one star functions. Real Analysis Exchange 1992; 18: 385-399.

[13] Kornfel'd IP. Sets of convergence and divergence of functional sequences. Izvestiya Vysshikh Uchebnykh Zavedenii Matematika 1963; 35 (4): 79-88 (in Russian).

[14] Kuratowski K. Topology, Volume I. New York, NY, USA: Academic Press, 1966.

[15] Laczkovich L, Recław I. Ideal limits of sequences of continuous functions. Fundamenta Mathematicae 2009; 203 (1): 39-46.

[16] Laflamme C. Filter games and combinatorial properties of strategies. In: Set Theory (Boise, ID, 1992-1994), Contemporary Math 192. Providence, RI, USA: AMS, 1996, pp. 51-67.

[17] Lipiński JS. Sets of points of convergence to infinity of a sequence of continuous functions. Fundamenta Mathematicae 1962/63; 51: 35-43 (in Russian).

[18] Lunina MA. Sets of convergence and divergence of sequences of real-valued continuous functions on a metric space. Matematicheskie Zametki 1975; 17: 205-217 (in Russian).

[19] Maliszewski A. On Darboux Baire one functions. Real Analysis Exchange 1994/95; 20 (1): 356-358.

[20] Maliszewski A. Darboux Property and Quasi-Continuity: A Uniform Approach. Słupsk, Poland: WSP Słupsk, 1996.

[21] Mazur K. F_σ-ideals and $\omega_1\omega_1^*$-gaps in the Boolean algebras $P(\omega)/I$. Fundamenta Mathematicae 1991; 138 (2): 103-111.

[22] Morayne M. Sierpiński's hierarchy and locally Lipschitz functions. Fundamenta Mathematicae 1995; 147 (1): 73-82.

[23] Natkaniec T, Szuca P. On the ideal convergence of sequences of quasi-continuous functions. Fundamenta Mathematicae 2016; 232 (3): 269-280.

[24] Natkaniec T, Wesołowska J. Sets of ideal convergence of sequences of quasi-continuous functions. Journal of Mathematical Analysis and Applications 2015; 423: 924-939.

[25] O'Malley RJ. Baire*1, Darboux functions. Proceedings of the American Mathematical Society 1976; 60: 187-192.

[26] Rosen H. Darboux Baire-.5 functions. Proceedings of the American Mathematical Society 1990; 110 (1): 285-286.

[27] Sierpiński W. Sur l'ensemble des points de convergence d'une suite de fonctions continues. Fundamenta Mathematicae 1921; 2: 41-49 (in French).

[28] Srivastava SM. A Course on Borel Sets. Graduate Texts in Mathematics, 180. New York, NY, USA: Springer, 1998.

[29] Wesołowska J. Sets of points of convergence of real functions. PhD, University of Gdańsk, Gdańsk, 2001 (in Polish).