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α-Associated metrics on rigged null hypersurfaces

Ferdinand NGAKEU[1], **Hans FOTSING TETSING**[2,*]
[1]Department of Mathematics and Computer Science, Faculty of Science, University of Douala, Douala, Cameroon
[2]African Institute for Mathematical Sciences, Limbe, Cameroon

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Abstract: Let $x : (M, g) \to (\overline{M}, \overline{g})$ be a null hypersurface isometrically immersed into a proper semi-Riemannian manifold $(\overline{M}, \overline{g})$. A rigging for M is a vector field ζ defined on some open subset of $\overline{M}$ containing M such that $\zeta_p \notin T_pM$ for every $p \in M$. Such a vector field induces an everywhere transversal null vector field N defined over M and which induces on M the same geometrical objects as ζ. Let $\bar{\eta}$ be the 1-form that is $\bar{g}$-metrically equivalent to N and let $\eta = x^\star\bar{\eta}$ be its pullback on M. For a given nowhere vanishing smooth function α on M, we have introduced and studied the so-called α-associated (semi-)Riemannian metric $g_\alpha = g + \alpha\eta \otimes \eta$. It turns out that this perturbation of the induced metric along a transversal null vector field is always nondegenerate, so we have established some relationships between geometrical objects of the (semi-)Riemannian manifold (M, g_α) and those of the lightlike hypersurface (M, g). For instance, in the case where N is closed, we give a constructive method to find a α-associated metric whose Levi-Civita connection coincides with the connection ∇ induced on M through the projection of the Levi-Civita connection $\overline{\nabla}$ of $\overline{M}$ along N. As an application, we show that given a null Monge hypersurface M in $\mathbb{R}^{n+1}_q$, there always exists a rigging and a α-associated metric whose Levi-Civita connection is the induced connection on M.

Key words: Null hypersurface, Monge hypersurface, screen distribution, rigging vector field, associated metric

1. Introduction

Let $(\overline{M}, \overline{g})$ be a proper semi-Riemannian manifold and $x : M \to \overline{M}$ an embedded hypersurface of $\overline{M}$. The pullback metric $g = x^\star\overline{g}$ can be either degenerate or nondegenerate on M. When g is nondegenerate, one says that (M, g) is a semi-Riemannian hypersurface of $(\overline{M}, \overline{g})$; otherwise, (M, g) is called a null (or degenerate, or lightlike) hypersurface of $(\overline{M}, \overline{g})$. Since any semi-Riemannian hypersurface has (locally) a canonical transversal vector field that is everywhere orthogonal to the hypersurface (the Gauss map), there is a standard way to study the extrinsic geometry of such a hypersurface. In fact, geometrical objects of the ambient manifold $\overline{M}$ are projected orthogonally on M and give new objects, which can be used to study the extrinsic geometry of the hypersurface.

For a null hypersurface, the tangent bundle contains the orthogonal bundle; hence, the orthogonal projection is no longer possible. One of the most used techniques to study a null hypersurface (M, g) immersed in a semi-Riemannian manifold $(\overline{M}, \overline{g})$ is to arbitrarily fix on it a screen distribution $S(TM)$ (a complementary to the normal bundle $TM^\perp$ in TM) and a (null) section $\xi \in \Gamma(TM^\perp)$. These choices locally fix a null

*Correspondence: hans.fotsing@aims-cameroon.org
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transversal vector field N, which is orthogonal to the screen distribution, transversal to M, and satisfies $\overline{g}(N,\xi)=1$. (See, for instance, [1, 5].) Instead of choosing a null section of the normal bundle and a screen distribution independently, we can make only one arbitrary choice of a transversal vector field ζ defined on an open neighborhood of M in $\overline{M}$. This vector field ζ is called rigging for M and induces a null section ξ (called the associated rigged vector field) of the normal bundle, a screen distribution, and a null transversal vector field N, all of them defined over M. This second technique (the rigging technique) was introduced in [8] and also used in other works such as [3, 11, 12].

A null rigging N for M induces a family (g_α) of nondegenerate metrics on M as follows. Let $\bar{\eta}$ be the 1-form that is $\bar{g}$-metrically equivalent to N (i.e. $\bar{\eta}=\bar{g}(N,.)$) and η the pullback of $\bar{\eta}$ on M via the immersion $x^\star$. For a given nowhere vanishing smooth function α on M we define the so-called α-associated (semi-)Riemannian metric $g_\alpha = g+\alpha\eta\otimes\eta$. When $\alpha=1$, the metric $g_1=g+\eta\otimes\eta$ is the so-called associated metric defined in [2]. Notice that for a function $\alpha>0$ the associated metric corresponding to the change of rigging $N'=\sqrt{\alpha}N$ is equal to the α-associated metric g_α corresponding to N. When the ambient manifold $(\overline{M},\bar{g})$ is a Lorentzian manifold, the associated metric is a Riemannian metric. This Riemannian metric was recently studied in [8].

When the null rigging N is defined over $\overline{M}$, it induces a perturbation $\bar{g}_\alpha=\bar{g}+\bar{\alpha}\bar{\eta}\otimes\bar{\eta}$ of the metric $\bar{g}$ whose restriction on M gives the associated metric. Such perturbations including those defined by spacelike or timelike vector fields at the place of null rigging have been considered in several works (see [9] for the α-associated type and [11] for canonical variation $g_t=g+t\eta\otimes\eta$ where t is constant). The Levi-Civita connection of the α-associated metric does not coincide in general with the connection ∇ induced on M from the Levi-Civita connection $\overline{\nabla}$ of $\bar{g}$ through the projection along N. A necessary and sufficient condition to have this coincidence for the case $\alpha=1$ was given in [2, 12].

In the present work, giving a null rigging N, we give a constructive method to obtain a nowhere vanishing smooth function α such that the Levi-Civita connection of the corresponding α-associated metric coincides with the induced connection. We also give relationships between curvatures of (M,∇) and those of (M,∇_α). We give some applications of our formalism on null Monge hypersurfaces in $\mathbb{R}^{n+1}_1$.

This paper is organized as follows: the first section is the Introduction. In Section 2 we present the twisted metric or a perturbation of a semi-Riemannian metric along a null vector field. Section 3 is devoted to the general setup on null hypersurfaces and new results on the α-associated metric. Theorem 3.2 gives a necessary and sufficient condition for the α-associated connection to coincide with the induced connection provided that α is constant along the leaves of the screen distribution. Section 4 is devoted to the computation of curvatures of the induced connection and the α-associated connection. Finally, in Section 5, we apply the formalism developed in the previous sections to null Monge hypersurfaces in $\mathbb{R}^{n+1}_1$ by showing that such hypersurfaces always admit suitable riggings and functions α such that $\nabla_\alpha=\nabla$.

2. Twisted metrics on a semi-Riemannian manifold

Throughout this work, $(\overline{M},\overline{g})$ is an $(n+1)$-dimensional semi-Riemannian manifold of index $q>0$, and $\overline{\nabla}$ and $\bar{R}$ will represent respectively the Levi-Civita connection and the Riemannian curvature of $\overline{g}$. (Objects from the metric $\overline{g}$ will be denoted with a line.) All manifolds are taken as smooth and connected. Let Σ be a d-dimensional manifold with $d\le n+1$. If there exists an immersion $x:\Sigma\to\overline{M}$, then $x(\Sigma)$ is called a

d-dimensional immersed submanifold of $\overline{M}$. If, moreover, x is injective, one says that $x(\Sigma)$ is a d-dimensional submanifold of $\overline{M}$. In addition, if the inverse map x^{-1} is a continuous map from $x(\Sigma)$ to Σ, $x(\Sigma)$ is a d-dimensional embedded submanifold of $\overline{M}$. When $x(\Sigma)$ is an embedded submanifold, one identifies Σ and $x(\Sigma)$. All submanifolds will be assumed embedded, and through the identification, stating that $x : M \to \overline{M}$ is a submanifold will mean that there is an embedding map $x : \Sigma \to \overline{M}$ such that $M = x(\Sigma)$. A hypersurface of $\overline{M}$ is a submanifold of $\overline{M}$ of dimension $d = n$. One says that $x : (M, g) \to (\overline{M}, \overline{g})$ is an isometrically immersed submanifold when, $x : M \to \overline{M}$ is a submanifold of $\overline{M}$ and $g = x^\star\overline{g}$. An isometrically immersed submanifold $x : (M, g) \to (\overline{M}, \overline{g})$ is called a nondegenerate submanifold if (M, g) is a semi-Riemannian manifold. Otherwise, one says that (M, g) is a degenerate or null or lightlike submanifold. The latter means that at any point $p \in M$ there exists a nonzero vector $u \in T_pM$ such that $g_p(u, v) = 0$ for every $v \in T_pM$.

Let N be a lightlike vector field defined over $\overline{M}$ and α be a nowhere vanishing smooth function on $\overline{M}$. We set $\overline{\eta}$ to be the 1-form $\overline{g}$-metrically equivalent to N. Using $\overline{g}$, we define the α-twisted metric on $\overline{M}$ as

$$\overline{g}_\alpha = \overline{g} + \alpha\overline{\eta} \otimes \overline{\eta}. \tag{1}$$

Lemma 2.1 *The pair $(\overline{M}, \overline{g}_\alpha)$ is a semi-Riemannian manifold.*

Proof Let $p \in \overline{M}$ and $u \in T_p\overline{M}$ such that $\overline{g}_\alpha(u, v) = 0$ for every $v \in T_p\overline{M}$. In particular, $\overline{g}_\alpha(u, N_p) = 0$, and hence $\overline{\eta}(u) = 0$, since N_p is a null vector. It follows that $\overline{g}(u, v) = 0$ for every $v \in T_p\overline{M}$, and then $u = 0$ since $\overline{g}$ is nondegenerate. This proves that $\overline{g}_\alpha$ is nondegenerate on $\overline{M}$. □

Let $\overline{\nabla}_\alpha$ be the Levi-Civita connection of $\overline{g}_\alpha$. The metrics $\overline{g}$ and $\overline{g}_\alpha$ are two semi-Riemannian metrics on $\overline{M}$. The following gives the relationship between their Levi-Civita connections.

Proposition 2.1 *The connections $\overline{\nabla}$ and $\overline{\nabla}_\alpha$ are related by*

$$\begin{aligned}\overline{\nabla}_{\alpha U} V = \overline{\nabla}_U V &+ \frac{1}{2}\left[\alpha\overline{\eta}(U)\,(i_V d\overline{\eta})^{\#\overline{g}_\alpha} + \alpha\overline{\eta}(V)\,(i_U d\overline{\eta})^{\#\overline{g}_\alpha} - \overline{\eta}(U)\overline{\eta}(V)d\alpha^{\#\overline{g}_\alpha}\right] \\ &+ \frac{1}{2}\left[\alpha\,(L_N\overline{g})\,(U, V) + d\alpha(U)\overline{\eta}(V) + d\alpha(V)\overline{\eta}(U)\right] N,\end{aligned} \tag{2}$$

where $d\alpha^{\#\overline{g}_\alpha}$ is the vector field $\overline{g}_\alpha$-metrically equivalent to the 1-form $d\alpha$, and $L_N\overline{g}$ is the Lie derivative of $\overline{g}$ along N.

Proof Let us start by recalling the Koszul equation defining $\overline{\nabla}_\alpha$. For all sections U, V, W of the tangent bundle $T\overline{M}$,

$$\begin{aligned}2\overline{g}_\alpha(\overline{\nabla}_{\alpha U} V, W) = &\, U \cdot \overline{g}_\alpha(V, W) + V \cdot \overline{g}_\alpha(W, U) - W \cdot \overline{g}_\alpha(U, V) \\ &+ \overline{g}_\alpha([U, V], W) - \overline{g}_\alpha([V, W], U) + \overline{g}_\alpha([W, U], V).\end{aligned}$$

Using (1) and the fact that $\overline{\nabla}$ is torsion-free and $\overline{g}$-metric, the latter equation leads to

$$\begin{aligned}2\overline{g}_\alpha(\overline{\nabla}^\alpha_U V, W) &= \overline{g}(\overline{\nabla}_U V, W) + \overline{g}(V, \overline{\nabla}_U W) + \alpha U\cdot(\overline{\eta}(V)\overline{\eta}(W)) + d\alpha(U)\overline{\eta}(V)\overline{\eta}(W)\\&+ \overline{g}(\overline{\nabla}_V W, U) + \overline{g}(W, \overline{\nabla}_V U) + \alpha V\cdot(\overline{\eta}(U)\overline{\eta}(W)) + d\alpha(V)\overline{\eta}(U)\overline{\eta}(W)\\&- \overline{g}(\overline{\nabla}_W U, V) - \overline{g}(U, \overline{\nabla}_W V) - \alpha W\cdot(\overline{\eta}(U)\overline{\eta}(V)) - d\alpha(W)\overline{\eta}(U)\overline{\eta}(V)\\&+ \overline{g}(\overline{\nabla}_U V - \overline{\nabla}_V U, W) + \alpha\overline{\eta}([U,V])\overline{\eta}(W) - \overline{g}(\overline{\nabla}_V W - \overline{\nabla}_W V, U)\\&- \alpha\overline{\eta}([V,W])\overline{\eta}(U) + \overline{g}(\overline{\nabla}_W U - \overline{\nabla}_U W, V) + \alpha\overline{\eta}([W,U])\overline{\eta}(V)\\&= 2\overline{g}(\overline{\nabla}_U V, W) + 2\alpha\overline{\eta}(\overline{\nabla}_U V)\overline{\eta}(W) + \alpha\left(L_N\overline{g}\right)(U,V)\overline{\eta}(W) + d\alpha(U)\overline{\eta}(V)\overline{\eta}(W)\\&+ \alpha\overline{\eta}(U)d\overline{\eta}(V,W) + \alpha\overline{\eta}(V)d\overline{\eta}(U,W) + d\alpha(V)\overline{\eta}(U)\overline{\eta}(W) - d\alpha(W)\overline{\eta}(U)\overline{\eta}(V)\\&= 2\overline{g}_\alpha(\overline{\nabla}_U V, W) + \alpha\left(L_N\overline{g}\right)(U,V)\overline{g}(N,W) + d\alpha(U)\overline{\eta}(V)\overline{g}(N,W)\\&+ \alpha\overline{\eta}(U)d\overline{\eta}(V,W) + \alpha\overline{\eta}(V)d\overline{\eta}(U,W) + d\alpha(V)\overline{\eta}(U)\overline{g}(N,W) - d\alpha(W)\overline{\eta}(U)\overline{\eta}(V),\end{aligned}$$

and (2) holds. □

3. Null hypersurfaces

3.1. Some preliminaries on null hypersurfaces

Let $x : (M, g) \to (\overline{M}, \overline{g})$ be a null hypersurface of $(\overline{M}, \overline{g})$. A rigging for M is a vector field ζ defined on an open subset containing M such that for any $p \in M$, $\zeta_p \notin T_pM$. We say that a rigging ζ is a null rigging for M when the restriction of ζ on M is lightlike. Therefore, if N is a null vector field on $\overline{M}$ anywhere transversal to M, then N is a null rigging for M. Determining conditions for the existence of a rigging for a given null hypersurface is still an open problem. However, it is clear that when the ambient manifold is a time-orientable Lorentzian manifold, any null hypersurface has a rigging, e.g., a timelike vector field globally defined.

Let ζ be a rigging for M and $\overline{\eta}$ be the 1-form $\overline{g}$-metrically equivalent to ζ (namely $\overline{\eta} = \overline{g}(\zeta, \cdot)$), and let $\eta = x^\star\overline{\eta}$ be the restriction of $\overline{\eta}$ on M. The associated metric $\widetilde{g}$ is given by

$$\widetilde{g} = g + \eta \otimes \eta. \tag{3}$$

The following is easy to prove.

Lemma 3.1 *[2] The associated metric $\widetilde{g}$ is nondegenerate.*

The associated rigged vector field is the vector field $\widetilde{g}$-metrically equivalent to the 1-form η and denoted ξ. As $\overline{g}$ is nondegenerate, it holds that

$$\eta(v) \neq 0, \qquad \forall v \in T_xM^\perp \setminus \{0\}. \tag{4}$$

Lemma 3.2 *The rigged field ξ is the unique section of $TM^\perp$ such that $\eta(\xi) = 1$.*

Proof Let $v \in T_xM$. By definition of ξ, $\eta(v) = \widetilde{g}(\xi, v) = g(\xi, v) + \eta(\xi)\eta(v)$, which implies that $g(\xi, v) = \eta(v)(1 - \eta(\xi))$. In particular, when $v \in T_xM^\perp \setminus \{0\}$, the latter gives $\eta(v)(1 - \eta(\xi)) = 0$. Hence, $\eta(\xi) = 1$, since from (4) $\eta(v) \neq 0$. It also follows that $g(\xi, v) = 0$ for every $v \in T_xM$. Hence, ξ is a section of $TM^\perp$ and the uniqueness follows from the fact that $TM^\perp$ is a rank 1 distribution. □

From now on, $\zeta = N$ is a null rigging and ξ is the associated rigged vector field. We will consider perturbations (1) of the ambient metric along this null rigging. Setting $\mathcal{S}(TM) = ker(\eta)$ and $tr(TM) = span(N)$, it is easy to prove that $S(TM)$ is a screen distribution and the following decompositions hold:

$$T\overline{M}_{|M} = TM \oplus tr(TM) = \mathcal{S}(TM) \oplus_{orth} \left(TM^\perp \oplus tr(TM)\right). \tag{5}$$

It also holds that

$$\overline{g}(\xi, N) = 1, \;\; \overline{g}(N, N) = \overline{g}(N, X) = 0, \;\; \forall X \in \Gamma(\mathcal{S}(TM)). \tag{6}$$

Let ∇ be the connection on M induced from $\overline{\nabla}$ through the projection along the transversal bundle $tr(TM) = span(N)$. To avoid confusion, we may denote the induced connection by ∇^N. For every section X of TM, one has $\overline{g}(\overline{\nabla}_X\xi, \xi) = 0$, which shows that $\overline{\nabla}_X\xi \in \Gamma(TM)$. The Weingarten map is the endomorphism field

$$\begin{array}{rccc} \chi: & \Gamma(TM) & \to & \Gamma(TM) \\ & X & \mapsto & \overline{\nabla}_X\xi \end{array}.$$

Gauss–Weingarten formulas of the immersion $x : (M, g) \to (\overline{M}, \overline{g})$ are given by

$$\begin{aligned} \overline{\nabla}_X Y &= \nabla_X Y + B(X, Y)N, & (7)\\ \nabla_X PY &= \overset{\star}{\nabla}_X PY + C(X, PY)\xi, & (8)\\ \overline{\nabla}_X N &= -A_N X + \tau(X)N, & (9)\\ \nabla_X \xi &= -\overset{\star}{A}_\xi X - \tau(X)\xi, & (10) \end{aligned}$$

for all $X, Y \in \Gamma(TM)$, where $\overset{\star}{\nabla}$ denotes the connection on the screen distribution $S(TM)$ induced by ∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma(\mathcal{S}(TM))$ along ξ. B and C are the local second fundamental forms of M and $\mathcal{S}(TM)$, respectively; A_N and $\overset{\star}{A}_\xi$ are the shape operators on TM and $\mathcal{S}(TM)$, respectively; and the rotation 1-form τ is given by $\tau(X) = \overline{g}(\overline{\nabla}_X N, \xi)$. It is easy to check that $\overset{\star}{A}_\xi$ and A_N are $S(TM)$-valued.

Shape operators and second fundamental forms are related by

$$B(X, Y) = g(\overset{\star}{A}_\xi X, Y), \tag{11}$$

$$C(X, PY) = g(A_N X, Y). \tag{12}$$

Using (6), (7), and (11), it is straightforward to show that $\overset{\star}{A}_\xi$ is g-symmetric and $\overset{\star}{A}_\xi(\xi) = 0$. On the contrary, A_N is not necessarily g-symmetric. However, A_N is g-symmetric on the screen distribution as a consequence of the following lemma.

Lemma 3.3 *For any sections X, Y of the tangent bundle TM, one has*

$$g(A_N X, Y) - g(X, A_N Y) = \tau(X)\eta(Y) - \tau(Y)\eta(X) - d\eta(X, Y).$$

Proof One just computes $d\eta(X,Y)$ by using the covariant derivative and Gauss–Weingarten equations. □

The mean curvatures of M and $S(TM)$ are respectively given by (cf. [4, 5])

$$\overset{\star}{H} = \sum_{i=2}^{n} \varepsilon_i B(E_i, E_i) \quad \text{and} \quad H = \sum_{i=2}^{n} \varepsilon_i C(E_i, E_i), \tag{13}$$

where $(E_2, \ldots, E_n)$ is an orthonormal basis of the screen distribution and $\varepsilon_i = g(E_i, E_i) = \pm 1$.

Proposition 3.1 *If the screen distribution $S(TM)$ is integrable and L is a leaf then $\vec{H} = H\xi + \overset{\star}{H} N$ is the mean curvature vector of the immersion $L \to (\overline{M}, \overline{g})$.*

Proof For every $x \in L$, one has $T_x L = span(N_x, \xi_x)$ and the Gauss formula of the immersion $L \to (\overline{M}, \overline{g})$ is given by

$$\overline{\nabla}_X Y = \overset{\star}{\nabla}_X Y + \overset{\star}{\nabla}{}^{\perp}_X Y = \overset{\star}{\nabla}_X Y + C(X,Y)\xi + B(X,Y)N, \tag{14}$$

for all $X, Y \in \Gamma(S(TM))$. Using (13), one has

$$\vec{H} = \sum_{i=2}^{n} \varepsilon_i \overset{\star}{\nabla}{}^{\perp}_{E_i} E_i = \sum_{i=2}^{n} \varepsilon_i C(E_i, E_i)\xi + \sum_{i=2}^{n} \varepsilon_i B(E_i, E_i) N = H\xi + \overset{\star}{H} N.$$

□

A null hypersurface M is said to be totally umbilical (resp. totally geodesic) if there exists a smooth function ρ on M such that at each point $x \in M$ and for all $X, Y \in T_x M$, $B_x(X,Y) = \rho(x) g_x(X,Y)$ (resp. B vanishes identically on M). It is equivalent to write $\overset{\star}{A}_\xi = \rho P$ and $\overset{\star}{A}_\xi = 0$, respectively. Notice that these are intrinsic notions on any null hypersurface in the sense that total umbilicity and total geodesibilicity of M do not depend on the chosen rigging. Also, the screen distribution $S(TM)$ is totally umbilical (resp. totally geodesic) if there exists a smooth function λ on M such that $C_x(X, PY) = \lambda(x) g_x(X,Y)$ for all $X, Y \in T_x M$ (resp. $C = 0$), which is equivalent to $A_N = \lambda P$ (resp. $A_N = 0$). We say that the rigged null hypersurface $x : (M, g, N) \to (\overline{M}, \overline{g})$ (or the rigging N) has a conformal screen distribution when there exists a nowhere vanishing smooth function φ on M such that

$$A_N = \varphi \overset{\star}{A}_\xi .$$

When the 1-form η is closed, we say that (M, g, N) is a null hypersurface with closed rigging. The following technical lemma* will be used in the sequel.

Lemma 3.4 *For any null hypersurface with closed rigging and conformal screen distribution, the rotation 1-form vanishes on the screen distribution.*

*Fotsing TH, Ngakeu F. Null hypersurfaces and trapping horizons, 12 June 2017, arxiv: 1706.03861v1 [math.DG].

For all sections X, Y, Z, T of TM, the so-called Gauss–Codazzi equations of (M, g, N) are given by

$$\begin{aligned}\overline{g}(\overline{R}(X,Y)Z,PT) &= g(R(X,Y)Z,PT)\\ &\quad + B(X,Z)C(Y,PT) - B(Y,Z)C(X,PT), \qquad (15)\\ \overline{g}(\overline{R}(X,Y)Z,N) &= \overline{g}(R(X,Y)Z,N), \qquad (16)\\ \overline{g}(\overline{R}(X,Y)PZ,N) &= (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ)\\ &\quad + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X), \qquad (17)\\ \overline{g}(\overline{R}(X,Y)Z,\xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z)\\ &\quad + B(Y,Z)\tau(X) - B(X,Z)\tau(Y), \qquad (18)\\ \overline{g}(\overline{R}(X,Y)\xi,N) &= C(Y,\overset{\star}{A}_\xi X) - C(X,\overset{\star}{A}_\xi Y) - d\tau(X,Y), \qquad (19)\end{aligned}$$

where $\nabla_X C$ is defined by $(\nabla_X C)(Y,PZ) = X \cdot C(Y,PZ) - C(\nabla_X Y, PZ) - C(Y, \overset{\star}{\nabla}_X PZ)$.

3.2. α-Associated metric and α-twisted metric

For $\alpha \in \mathcal{C}^\infty(\overline{M})^\star$ a nowhere vanishing smooth function, the restriction on M of the α-twisted metric (1) is given by

$$g_\alpha = g + \alpha\eta \otimes \eta, \qquad (20)$$

where we have denoted again by α the restriction of α on M. We call g_α the α-associated metric of (M, g, N). One can observe that the 1-associated metric g_1 is just the associated metric $\widetilde{g}$.

Lemma 3.5 *The pair (M, g_α) is a semi-Riemannian manifold of index*

$$\nu_\alpha = q - \frac{1}{2}(1 + sign(\alpha)) = \begin{cases} q-1 & \text{if } \alpha > 0\\ q & \text{if } \alpha < 0.\end{cases}$$

Proof Let $x \in M$ and $u \in T_xM$ such that $g_\alpha(u,v) = 0$ for all $v \in T_xM$. In particular, $0 = g_\alpha(u, \xi_x) = \alpha(x)\eta_x(u) \Rightarrow \eta_x(u) = 0$ since $\alpha(x) \neq 0$. Thus, $u \in S(T_xM)$. One then has $g(u,v) = 0$ for all $v \in S(T_xM)$, and hence $u = 0$ since the restriction of g on the screen distribution is nondegenerate. Thus, (M, g_α) is a semi-Riemannian manifold. For the index, one just remarks that g is of index $q-1$ on $S(TM)$ and $g_\alpha(\xi,\xi) = \alpha$.
□

We now know that $x_\alpha : (M, g_\alpha) \to (\overline{M}, \overline{g}_\alpha)$ is a nondegenerate hypersurface of the semi-Riemannian manifold $(\overline{M}, \overline{g}_\alpha)$. The Gauss map of the isometric immersion x_α is given by

$$\delta_\alpha = \sqrt{|\alpha|}N - \frac{sign(\alpha)}{\sqrt{|\alpha|}}\xi. \qquad (21)$$

In fact, $g(X,\xi) = 0 \implies \overline{g}_\alpha(X, \delta_\alpha) = 0$, $\forall X \in \Gamma(TM)$, and also $\overline{g}_\alpha(\delta_\alpha, \delta_\alpha) = -sign(\alpha)$. It follows that $(\overline{M}, \overline{g}_\alpha)$ is a semi-Riemannian manifold of index q, since (M, g) is of index $\nu_\alpha = q - \frac{1}{2}(1 + sign(\alpha))$. For the rest of this subsection, we assume that the rigging N is closed. This means that its equivalent 1-form $\overline{\eta}$ is closed. It is easy to check that this is equivalent to

$$\overline{g}(\overline{\nabla}_U N, V) = \overline{g}(U, \overline{\nabla}_V N), \quad \forall U, V \in \Gamma(T\overline{M}). \tag{22}$$

Using (2), one has

$$\overline{\nabla}_{\alpha X}\delta_\alpha = \overline{\nabla}_X \delta_\alpha + \frac{1}{2}[\alpha\,(L_N\overline{g})\,(X,\delta_\alpha) + d\alpha(X)\overline{\eta}(\delta_\alpha) + d\alpha(\delta_\alpha)\eta(X)]N - \frac{1}{2}\eta(X)\overline{\eta}(\delta_\alpha)d\alpha^{\#\overline{g}_\alpha}.$$

Using (7)–(22) and by direct calculations, one gets

$$\overline{\nabla}_X\delta_\alpha = \frac{sign(\alpha)}{\sqrt{|\alpha|}}\left[-\alpha A_N X + \overset{\star}{A}_\xi X + \tau(X)\xi\right] + (X\cdot\sqrt{|\alpha|} + \sqrt{|\alpha|}\tau(X))N + \frac{d\alpha(X)}{2\alpha}\xi,$$

$$(L_N\overline{g})(X,N) = 0, \qquad d\alpha^{\#\overline{g}_\alpha} = d\alpha^{\#g_\alpha} - sign(\alpha)d\alpha(\delta_\alpha)\delta_\alpha,$$

$$(L_N\overline{g})(X,\xi) = 2\tau(X), \qquad \overline{\eta}(\delta_\alpha) = -\frac{sign(\alpha)}{\sqrt{|\alpha|}}.$$

Thus,

$$\overline{\nabla}_{\alpha X}\delta_\alpha = \frac{sign(\alpha)}{\sqrt{|\alpha|}}\left[-\alpha A_N X + \overset{\star}{A}_\xi X + \tau(X)\xi + \frac{d\alpha(X)}{2\alpha}\xi + \frac{\eta(X)}{2\sqrt{|\alpha|}}\left(\sqrt{|\alpha|}d\alpha^{\#g_\alpha} + d\alpha(\delta_\alpha)\xi\right)\right].$$

The shape operator of the immersion x_α is then given by

$$A_{\delta_\alpha}(X) = \frac{sign(\alpha)}{\sqrt{|\alpha|}}\left[\alpha A_N X - \overset{\star}{A}_\xi X - \tau(X)\xi - \frac{d\alpha(X)}{2\alpha}\xi - \frac{\eta(X)}{2\sqrt{|\alpha|}}\left(\sqrt{|\alpha|}d\alpha^{\#g_\alpha} + d\alpha(\delta_\alpha)\xi\right)\right].$$

If α is constant on each leaf of the screen distribution and the screen distribution is conformal with conformal factor $1/\alpha$, then the shape operator of the isometric immersion x_α is given by

$$A_{\delta_\alpha}(X) = -\frac{sign(\alpha)}{2\sqrt{|\alpha|}}\eta(X)\left[2\tau(\xi) + \eta(d\alpha^{\#g_\alpha}) + d\alpha(N)\right]\xi.$$

We then have the following result.

Theorem 3.1 *Let $x : (M, g, N) \to (\overline{M}^{n+1}, \overline{g})$ be a null hypersurface with closed rigging and conformal screen distribution with conformal factor $1/\alpha$ constant on leaves of the screen distribution. Then the isometric immersion $x_\alpha : (M, g_\alpha) \to (\overline{M}, \overline{g}_\alpha)$ ($\overline{g}_\alpha$ being defined by (20)) is a nondegenerate hypersurface with at most two principal curvatures: 0 with multiplicity $n-1$ and eigenvectors the sections of $S(TM)$, and $-\frac{sign(\alpha)}{2\sqrt{|\alpha|}}\left[2\tau(\xi) + \eta(d\alpha^{\#g_\alpha}) + d\alpha(N)\right]$ with multiplicity 1 and eigenvectors the sections of $Rad(TM)$.*

3.3. Induced metric and α-associated metric

In this subsection, we will relate some geometrical objects of the α-associated metric g_α with those of the induced metric g. From here on, N is strictly a null rigging for M, meaning that we do not require N to be lightlike globally on $\overline{M}$, but on M. Recall that ∇_α is the Levi-Civita connection of the α-associated semi-Riemannian manifold (M, g_α) and ∇ is the connection on the rigged null hypersurface $x : (M, g, N) \to (\overline{M}, \overline{g})$ induced from $\overline{\nabla}$ through the projection along N.

Proposition 3.2 *The connections* ∇_{α} *and* ∇ *are related by*

$$\nabla_{\alpha X}Y = \nabla_X Y - \frac{1}{2}\eta(X)\eta(Y)d\alpha^{\#g_\alpha} + \frac{\alpha}{2}\left[\eta(X)(i_Y d\eta)^{\#g_\alpha} + \eta(Y)(i_X d\eta)^{\#g_\alpha}\right]$$
$$+ \frac{1}{2\alpha}\left[\alpha\left(L_N\overline{g}\right)(X,Y) + 2B(X,Y) + d\alpha(X)\eta(Y) + d\alpha(Y)\eta(X)\right]\xi. \qquad (23)$$

Proof Reasoning as in the proof of (2), one has

$$\begin{aligned}
2g_\alpha(\nabla_{\alpha X}Y, Z) &= \overline{g}(\overline{\nabla}_X Y, Z) + \overline{g}(Y, \overline{\nabla}_X Z) + \alpha X\cdot(\eta(Y)\eta(Z)) + d\alpha(X)\eta(Y)\eta(Z)\\
&+ \overline{g}(\overline{\nabla}_Y Z, X) + \overline{g}(Z, \overline{\nabla}_Y X) + \alpha Y\cdot(\eta(X)\eta(Z)) + d\alpha(Y)\eta(X)\eta(Z)\\
&- \overline{g}(\overline{\nabla}_Z X, Y) - \overline{g}(X, \overline{\nabla}_Z Y) - \alpha Z\cdot(\eta(X)\eta(Y)) - d\alpha(Z)\overline{\eta}(X)\overline{\eta}(Y)\\
&+ \overline{g}(\overline{\nabla}_X Y - \overline{\nabla}_Y X, Z) + \alpha\eta(\overline{\nabla}_X Y - \overline{\nabla}_Y X)\eta(Z) - \overline{g}(\overline{\nabla}_Y Z - \overline{\nabla}_Z Y, X)\\
&- \alpha\eta([Y,Z])\eta(X) + \overline{g}(\overline{\nabla}_Z X - \overline{\nabla}_X Z, Y) + \alpha\eta([Z,X])\eta(Y)\\
&= 2g_\alpha(\nabla_X Y, Z) + 2B(X,Y)\eta(Z) + \alpha\left(L_N\overline{g}\right)(X,Y)\eta(Z) + d\alpha(X)\eta(Y)\eta(Z)\\
&+ \alpha\eta(X)d\eta(Y,Z) + \alpha\eta(Y)d\eta(X,Z) + d\alpha(Y)\eta(X)\eta(Z) - d\alpha(Z)\eta(X)\eta(Y).
\end{aligned}$$

From here, using the fact that

$$\alpha\eta(X) = g_\alpha(X,\xi) \quad \forall X \in \Gamma(TM), \qquad (24)$$

one obtains (23). □

From here on, we assume that the rigging N is closed. Then using (22), (9), and (12), one has

$$(L_N\overline{g})(X,Y) = 2\tau(X)\eta(Y) - 2C(X,PY),$$

and equation (23) becomes

$$\nabla_{\alpha X}Y = \nabla_X Y - \frac{1}{2}\eta(X)\eta(Y)d\alpha^{\#g_\alpha}$$
$$+ \frac{1}{2\alpha}\left[2B(X,Y) - 2\alpha C(X,PY) + 2\alpha\tau(X)\eta(Y) + d\alpha(X)\eta(Y) + d\alpha(Y)\eta(X)\right]\xi. \qquad (25)$$

From now on, we use the following range of indexes:

$$i,j = 0,1,\ldots,n; \qquad a,b = 1,\ldots,n \qquad k,l = 2,\ldots,n,$$

for summations (often with Einstein summation convention). For free indexes, we shall use

$$\beta,\gamma = 1,\ldots,n.$$

Let $\left(E_1 = \frac{1}{\sqrt{|\alpha|}}\xi, E_2,\ldots,E_n\right)$ be a g_α-orthonormal frame field of TM such that $(E_2,\ldots,E_n)$ is a frame field of $S(TM)$. The matrix of g_α in this frame is given by

$$g_\alpha = (g_\alpha(E_a,E_b)),$$

and we set (g_α^{ab}) to be the inverse matrix. Note that $g_\alpha^{ab} = \varepsilon^a\delta^{ab}$, with $\varepsilon^a := \pm 1$.

Proposition 3.3 *One has:*

① *for all* X, Y *sections of* TM, $(L_\xi g_\alpha)(X,Y) = -2B(X,Y) + \eta(X)\eta(Y)d\alpha(\xi)$;

② *in particular,* $div^{g_\alpha}(\xi) = \frac{1}{2|\alpha|}d\alpha(\xi) - \overset{\star}{H}$;

③ *if* ξ *is* g_α*-Killing conformal (or* g_α*-Killing) with conformal factor* φ*, then* (M, g, N) *is totally umbilical (or geodesic) with umbilical factor* $\rho = -\frac{1}{2}\varphi$.

Proof Since $\overset{\alpha}{\nabla}$ is the Levi-Civita connection of g_α, one has

$$(L_\xi g_\alpha)(X,Y) = g_\alpha(\overset{\alpha}{\nabla}_X \xi, Y) + g_\alpha(X, \overset{\alpha}{\nabla}_Y \xi). \tag{26}$$

Using (25), the latter becomes

$$(L_\xi g_\alpha)(X,Y) = g_\alpha(\nabla_X \xi, Y) + g_\alpha(X, \nabla_Y \xi) + \eta(X)\eta(Y)d\alpha(\xi) + \alpha\left[\eta(X)\tau(Y) + \eta(Y)\tau(X)\right].$$

From here, using (20) and Gauss–Weingarten formulas, the first item holds. By definition and using (26), one has

$$div^{g_\alpha}(\xi) = tr\left(\overset{\alpha}{\nabla}\xi\right) = \varepsilon^k g_\alpha\left(\overset{\alpha}{\nabla}_{E_k}\xi, E_k\right) = \frac{1}{2}\varepsilon^k (L_\xi g_\alpha)(E_k, E_k).$$

From here, using the first item, we obtain the second item. For the last item, let us assume that ξ is g_α-conformal Killing with conformal factor φ. Then the first item says that for all X, Y sections of the tangent bundle TM,

$$-2B(X,Y) + \eta(X)\eta(Y)d\alpha(\xi) = \varphi g(X,Y) + \alpha\varphi\eta(X)\eta(Y). \tag{27}$$

Setting $X = Y = \xi$, one finds $d\alpha(\xi) = \alpha\varphi$, and (27) becomes

$$-2B(X,Y) = \varphi g(X,Y),$$

which completes the proof. □

With the above proof we see that when ξ is g_α-Killing, α is necessarily constant along integral lines of ξ. We have two connections on M, namely the induced connection ∇ and the α-associated connection $\overset{\alpha}{\nabla}$. A natural question is to ask if both connections can coincide. The following result gives a necessary and sufficient condition to have an affirmative answer.

Theorem 3.2 *Let* $x : (M, g, N) \to (\overline{M}, \overline{g})$ *be a null hypersurface with closed rigging.*

① *Let* α *be a nowhere vanishing function constant on each leaf of the screen distribution. Then the induced connection is the Levi-Civita connection of the* α*-associated metric if and only if*

$$\overset{\star}{A}_\xi = \alpha A_N \quad \text{and} \quad 2\alpha\tau(\xi) + d\alpha(\xi) = 0. \tag{28}$$

② *Let* α *be a nonzero real number. Then the induced connection is the Levi-Civita connection of the* α*-associated metric if and only if*

$$\overset{\star}{A}_\xi = \alpha A_N \quad \text{and} \quad \tau \equiv 0. \tag{29}$$

Proof If α is constant along the leaves of the screen distribution, then

$$d\alpha(X) = \eta(X)d\alpha(\xi) \text{ and } \alpha d\alpha^{\#g_\alpha} = d\alpha(\xi)\xi,$$

and equation (25) becomes

$$\nabla_{\alpha X}Y = \nabla_X Y + \frac{1}{2\alpha}\left[2B(X,Y) - 2\alpha C(X,PY) + 2\alpha\tau(X)\eta(Y) + \eta(X)\eta(Y)d\alpha(\xi)\right]\xi. \tag{30}$$

Now $\nabla_\alpha = \nabla$ if and only if

$$2B(X,Y) - 2\alpha C(X,PY) + 2\alpha\tau(X)\eta(Y) + \eta(X)\eta(Y)d\alpha(\xi) = 0. \tag{31}$$

Replacing X and Y by ξ in the latter, one obtains $d\alpha(\xi) + 2\alpha\tau(\xi) = 0$. The latter together with (31) allows us to conclude that if α is constant along the leaves of the screen distribution then (28) holds. Now if α is constant on M then the screen distribution is conformal and $\tau(\xi) = 0$, which by the Lemma 3.4 implies that τ identically vanishes. The converse is straightforward by using (30). □

Thus, given a null rigging N, to find an α-associated perturbation of g for which the coincidence of connections happens, we have to solve equation (28) with α as unknown. By using Theorem 4.1 in [2], the proof of the following result is a straightforward computation.

Proposition 3.4 *Let $(M,g,N) \to (\overline{M},\overline{g})$ be a rigged null hypersurface. If α is a function such that (28) holds, then the same equations hold for any change of rigging $\widetilde{N} = \phi N$, with ϕ constant on each leaf of the screen distribution, and for $\tilde{\alpha} = \frac{\alpha}{\phi^2}$.*

We notice that for another nowhere vanishing function ϕ on M, the α-associated metric along N coincides with the $\frac{\alpha}{\phi^2}$-associated metric along $\widetilde{N} = \phi N$. Therefore, if $\nabla_\alpha^N = \nabla$, then we also have $\nabla_{\frac{\alpha}{\phi^2}}^{\widetilde{N}} = \nabla$ without the condition that ϕ is constant along leaves of the screen distribution or the condition that $\widetilde{N}$ is closed.

4. Curvature of the α-associated metric

In this section, $x : (M,g,N) \to (\overline{M},\overline{g})$ is a rigged null hypersurface with closed null rigging N of a semi-Riemannian manifold, and α is a nowhere vanishing smooth function on M constant on each leaf of the screen distribution. Let X,Y,Z be sections of TM. We recall that the Riemannian curvature R_α of the α-associated metric g_α is given by

$$R_\alpha(X,Y)Z = \nabla_{\alpha X}\nabla_{\alpha Y}Z - \nabla_{\alpha Y}\nabla_{\alpha X}Z - \nabla_{\alpha [X,Y]}Z. \tag{32}$$

It is straightforward to relate each of the three terms of the right-hand side of the above relation with tools of the lightlike metric. Using equation (25) and Gauss–Weingarten equations, one finds

$$\begin{aligned}\nabla_{\alpha X}\nabla_{\alpha Y}Z &= \nabla_X\nabla_Y Z - \left[\frac{1}{\alpha}B(Y,Z) - C(Y,PZ) + \tau(Y)\eta(Z) + \frac{1}{2\alpha}\eta(Y)\eta(Z)d\alpha(\xi)\right]\overset{\star}{A}_\xi X\\ &+ \left\{\frac{1}{\alpha}B(\nabla_Y X,Z) - C(X,P\nabla_Y Z) + \tau(X)\eta(\nabla_Y Z) + \frac{1}{2\alpha}\eta(X)\eta(\nabla_Y Z)d\alpha(\xi)\right.\\ &- \left[\frac{1}{\alpha}B(Y,Z) - C(Y,PZ) + \tau(Y)\eta(Z) + \frac{1}{2\alpha}\eta(Y)\eta(Z)d\alpha(\xi)\right]\tau(X)\\ &- \frac{d\alpha(X)}{\alpha^2}\eta(Y)\eta(Z)d\alpha(\xi) + \frac{1}{2\alpha}X\cdot(\eta(Y)\eta(Z))d\alpha(\xi)\\ &\left.-\frac{d\alpha(X)}{2\alpha^2}B(Y,Z) + \frac{1}{\alpha}X\cdot B(Y,Z) - X\cdot C(Y,PZ) + X\cdot(\tau(Y)\eta(Z))\right\}\xi.\end{aligned}$$

Similarly, we express the two other terms of (32) to obtain the following:

Proposition 4.1 *Riemannian curvatures of the connections* ∇_α *and* ∇ *are related by*

$$\begin{aligned}R_\alpha(X,Y)Z &= R(X,Y)Z - \left[\frac{1}{\alpha}B(Y,Z) - C(Y,PZ) + \tau(Y)\eta(Z) + \frac{1}{2\alpha}\eta(Y)\eta(Z)d\alpha(\xi)\right]\overset{\star}{A}_\xi X\\ &+ \left[\frac{1}{\alpha}B(X,Z) - C(X,PZ) + \tau(X)\eta(Z) + \frac{1}{2\alpha}\eta(X)\eta(Z)d\alpha(\xi)\right]\overset{\star}{A}_\xi Y + d\tau(X,Y)\eta(Z)\\ &+ \left\{\frac{1}{\alpha}(\nabla_X B)(Y,Z) - \frac{1}{\alpha}(\nabla_Y B)(X,Z) + (\nabla_Y C)(X,PZ) - (\nabla_X C)(Y,PZ)\right.\\ &- \left[\frac{1}{\alpha}B(Y,Z) - 2C(Y,PZ)\right]\tau(X) + \left[\frac{1}{\alpha}B(X,Z) - 2C(X,PZ)\right]\tau(Y)\\ &\left.+\frac{d\alpha(\xi)}{2\alpha^2}\big[\eta(Y)(2B(X,Z) - \alpha C(X,PZ)) - \eta(X)(2B(Y,Z) - \alpha C(Y,PZ))\big]\right\}\xi.\end{aligned}$$

Let $\mathcal{X},\mathcal{Y},\mathcal{Z},\mathcal{T}$ be sections of the screen distribution. Using the above proposition, one finds

$$\begin{aligned}g_\alpha(R_\alpha(X,Y)Z,\mathcal{X}) &= g(R(X,Y)Z,\mathcal{X}) + \left[\frac{1}{\alpha}B(X,Z) - C(X,PZ) + \tau(X)\eta(Z) + \frac{1}{2\alpha}\eta(X)\eta(Z)d\alpha(\xi)\right]B(Y,\mathcal{X})\\ &- \left[\frac{1}{\alpha}B(Y,Z) - C(Y,PZ) + \tau(Y)\eta(Z) + \frac{1}{2\alpha}\eta(Y)\eta(Z)d\alpha(\xi)\right]B(X,\mathcal{X}).\end{aligned}$$

Now using equation (15), this becomes

$$\begin{aligned}g_\alpha(R_\alpha(X,Y)Z,\mathcal{X}) &= \overline{g}(\overline{R}(X,Y)Z,\mathcal{X}) - B(X,Z)C(Y,\mathcal{X}) + B(Y,Z)C(X,\mathcal{X})\\ &+ \left[\frac{1}{\alpha}B(X,Z) - C(X,PZ) + \tau(X)\eta(Z) + \frac{1}{2\alpha}\eta(X)\eta(Z)d\alpha(\xi)\right]B(Y,\mathcal{X})\\ &- \left[\frac{1}{\alpha}B(Y,Z) - C(Y,PZ) + \tau(Y)\eta(Z) + \frac{1}{2\alpha}\eta(Y)\eta(Z)d\alpha(\xi)\right]B(X,\mathcal{X}).\end{aligned} \tag{33}$$

Using equations (16)–(17) and the above proposition, we obtain

$$\overline{g}(R_\alpha(\xi,\mathcal{X})\mathcal{Y},N) = \frac{1}{\alpha}\left(\nabla_\xi B\right)(\mathcal{X},\mathcal{Y}) - \frac{1}{\alpha}\left(\nabla_{\mathcal{X}} B\right)(\xi,\mathcal{Y}) - \left[\frac{1}{\alpha}B(\mathcal{X},\mathcal{Y}) - C(\mathcal{X},\mathcal{Y})\right]\tau(\xi)$$
$$- \frac{d\alpha(\xi)}{2\alpha^2}\left[2B(\mathcal{X},\mathcal{Y}) + \alpha C(\mathcal{X},\mathcal{Y})\right] - C(\xi,\mathcal{Y})\tau(\mathcal{X}). \tag{34}$$

Equation (34) together with Gauss–Codazzi equation (18) gives

$$\overline{g}(R_\alpha(\xi,\mathcal{X})\mathcal{Y},N) = \frac{1}{\alpha}\overline{g}(\overline{R}(\xi,\mathcal{X})\mathcal{Y},\xi) - \left[\frac{2}{\alpha}B(\mathcal{X},\mathcal{Y}) - C(\mathcal{X},\mathcal{Y})\right]\tau(\xi)$$
$$- \frac{d\alpha(\xi)}{2\alpha^2}\left[2B(\mathcal{X},\mathcal{Y}) + \alpha C(\mathcal{X},\mathcal{Y})\right] - C(\xi,\mathcal{Y})\tau(\mathcal{X}). \tag{35}$$

In Proposition 4.1, we have given relationships between Riemannian curvatures of the connections ∇_α and ∇. Since ∇ is not a g-metric connection, the $(1,3)$-tensor R does not have all Riemannian curvature symmetries and does not allow us to define the classical Ricci tensor. However, if one defines a Ricci tensor as $Ric(X,Y) = tr(Z \mapsto R(Z,X)Y)$, this gives a nonnecessarily symmetric tensor and the definition of the scalar curvature becomes ambiguous. For this reason, we will relate the Ricci tensor of ∇_α with the one of $\overline{\nabla}$ for sections of TM. In [8], such a relationship was found for $\alpha = 1$ and by assuming that M is totally geodesic. We are going to relate this Ricci tensor for a function α constant on the leaves of the screen distribution and without the total geodesic condition. Let us start with sections of the screen distribution.

Proposition 4.2 *For all $\mathcal{X},\mathcal{Y}$ sections of the screen distribution, one has*

$$Ric_\alpha(\mathcal{X},\mathcal{Y}) = \overline{Ric}(\mathcal{X},\mathcal{Y}) - \overline{g}(\overline{R}(\xi,\mathcal{X})\mathcal{Y},N) - \overline{g}(\overline{R}(\xi,\mathcal{Y})\mathcal{X},N) + \frac{1}{\alpha}\overline{g}(\overline{R}(\xi,\mathcal{X})\mathcal{Y},\xi) - C(\xi,\mathcal{Y})\tau(\mathcal{X})$$
$$+ \frac{1}{\alpha}g(\overset{\star}{A}_\xi\mathcal{X},\overset{\star}{A}_\xi\mathcal{Y}) - g(\overset{\star}{A}_\xi X, A_N\mathcal{Y}) - g(A_N\mathcal{X},\overset{\star}{A}_\xi\mathcal{Y}) + B(\mathcal{X},\mathcal{Y})\left(H - \frac{1}{\alpha}\overset{\star}{H}\right) \tag{36}$$
$$+ C(\mathcal{X},\mathcal{Y})\overset{\star}{H} - \left[\frac{2}{\alpha}B(\mathcal{X},\mathcal{Y}) - C(\mathcal{X},\mathcal{Y})\right]\tau(\xi) - \frac{d\alpha(\xi)}{2\alpha^2}\left[2B(\mathcal{X},\mathcal{Y}) + \alpha C(\mathcal{X},\mathcal{Y})\right].$$

Proof By definition,

$$Ric_\alpha(\mathcal{X},\mathcal{Y}) = tr(\mathcal{Z} \mapsto R_\alpha(\mathcal{Z},\mathcal{X})\mathcal{Y}) = \sum_{k=2}^{n}\varepsilon^k g(R_\alpha(E_k,\mathcal{X})\mathcal{Y},E_k) + \overline{g}(R_\alpha(\xi,\mathcal{X})\mathcal{Y},N).$$

Let us compute each term of the latter. Using (33), one has

$$\varepsilon^k g(R_\alpha(E_k,\mathcal{X})\mathcal{Y},E_k) = \varepsilon^k\overline{g}(\overline{R}(E_k,\mathcal{X})\mathcal{Y},E_k) - B(A_N\mathcal{X},\mathcal{Y}) + B(\mathcal{X},\mathcal{Y})H$$
$$+ \frac{1}{\alpha}B(\overset{\star}{A}_\xi\mathcal{X},\mathcal{Y}) - B(\mathcal{X},A_N\mathcal{Y}) - \left[\frac{1}{\alpha}B(\mathcal{X},\mathcal{Y}) - C(\mathcal{X},\mathcal{Y})\right]\overset{\star}{H}.$$

Again by definition,

$$\overline{Ric}(\mathcal{X},\mathcal{Y}) = \varepsilon^k g(\overline{R}(E_k,\mathcal{X})\mathcal{Y},E_k) + \overline{g}(\overline{R}(\xi,\mathcal{X})\mathcal{Y},N) + \overline{g}(\overline{R}(\xi,\mathcal{Y})\mathcal{X},N),$$

where we have used the quasi-orthonormal basis $(N,\xi,E_2,\ldots,E_n)$. Hence,

$$\begin{aligned}\varepsilon^k g(R_\alpha(E_k,\mathcal{X})\mathcal{Y},E_k) &= \overline{Ric}(\mathcal{X},\mathcal{Y}) - \overline{g}(\overline{R}(\xi,\mathcal{X})\mathcal{Y},N) - \overline{g}(\overline{R}(\xi,\mathcal{Y})\mathcal{X},N)\\ &\quad - B(A_N\mathcal{X},\mathcal{Y}) + B(\mathcal{X},\mathcal{Y})H\\ &\quad + \frac{1}{\alpha}B(\overset{\star}{A}_\xi\mathcal{X},\mathcal{Y}) - B(\mathcal{X},A_N\mathcal{Y}) - \left[\frac{1}{\alpha}B(\mathcal{X},\mathcal{Y}) - C(\mathcal{X},\mathcal{Y})\right]\overset{\star}{H}.\end{aligned}$$

Then one obtains (36) by summing the latter with (35). □

To complete the computation of the Ricci of all two sections of TM, it remains to compute $Ric_\alpha(\xi,\xi)$ and $Ric_\alpha(\xi,\mathcal{X})$.

Proposition 4.3 *For any function α constant on each leaf of the screen distribution of the rigged null hypersurface $(M,g,N)\to(\overline{M},\overline{g})$ with closed null rigging N, the following hold:*

① $Ric_\alpha(\xi,\xi) = \overline{Ric}(\xi,\xi) - \left[\tau(\xi) + \frac{1}{2\alpha}d\alpha(\xi)\right]\overset{\star}{H}$.

② *For any section $\mathcal{X}$ of $S(TM)$,*

$$Ric_\alpha(\xi,\mathcal{X}) = \overline{Ric}(\xi,\mathcal{X}) + d\tau(\xi,\mathcal{X}) + g(A_N\xi,\mathcal{X})\overset{\star}{H}.$$

Proof By definition, $Ric_\alpha(\xi,X) = \varepsilon^k g_\alpha(R_\alpha(E_k,\xi)X,E_k)$. Equation (33) gives

$$\begin{aligned}\varepsilon^k g_\alpha(R_\alpha(E_k,\xi)X,E_k) &= \varepsilon^k\overline{g}(\overline{R}(E_k,\xi)X,E_k) - \varepsilon^k B(E_k,X)C(\xi,E_k)\\ &\quad + \varepsilon^k\left[C(\xi,PX) - \tau(\xi)\eta(X) - \frac{1}{2\alpha}\eta(X)d\alpha(\xi)\right]B(E_k,E_k). \qquad (37)\end{aligned}$$

Replacing X by ξ and summing over k, one finds

$$Ric_\alpha(\xi,\xi) = \sum\varepsilon_k\overline{g}(\overline{R}(E_k,\xi)\xi,E_k) - \left[\tau(\xi) + \frac{1}{2\alpha}d\alpha(\xi)\right]\overset{\star}{H}.$$

Since $\overline{Ric}(\xi,\xi) = \sum\varepsilon_k g(\overline{R}(E_k,\xi)\xi,E_k)$, the first item holds. Now replacing X by $\mathcal{X}$ in (37) and summing, one finds

$$Ric_\alpha(\xi,\mathcal{X}) = \overline{Ric}(\xi,\mathcal{X}) - \overline{g}(\overline{R}(\xi,\mathcal{X})\xi,N) + g(A_N\xi,\overset{\star}{A}_\xi\mathcal{X}) + g(A_N\xi,\mathcal{X})\overset{\star}{H},$$

since $\overline{Ric}(\xi,\mathcal{X}) = \varepsilon^k g(\overline{R}(E_k,\xi)\mathcal{X},E_k) + \overline{g}(\overline{R}(N,\xi)\mathcal{X},\xi) = \varepsilon^k g(\overline{R}(E_k,\xi)\mathcal{X},E_k) + \overline{g}(\overline{R}(\xi,\mathcal{X})\xi,N)$. Then using Gauss–Codazzi equation (19), the second item follows. □

The following relates sectional curvatures of ∇_α and $\overline{\nabla}$. Recall that the sectional curvature of a plane $\Pi = span(X,Y)$ is given by

$$K_\alpha(\Pi) = \frac{g_\alpha(R_\alpha(X,Y)X,Y)}{g_\alpha(X,X)g_\alpha(Y,Y) - g_\alpha(X,Y)^2}.$$

By using equation (33), the proof of the following proposition is a straightforward calculation.

Proposition 4.4 *Let $\mathcal{X}$ and $\mathcal{Y}$ be two orthogonal sections of the screen structure. Let us consider the planes $\Pi_0 = span(\xi, \mathcal{X})$ and $\Pi = span(\mathcal{X}, \mathcal{Y})$. Then:*

① $K_\alpha(\Pi_0) = \frac{1}{\alpha g(\mathcal{X},\mathcal{X})}\left[\overline{K}_\xi(\Pi_0) + \left[\tau(\xi) + \frac{1}{2\alpha}d\alpha(\xi)\right]B(\mathcal{X},\mathcal{X})\right]$;

②

$$K_\alpha(\Pi) = \overline{K}(\Pi) + \frac{B(\mathcal{X},\mathcal{X})B(\mathcal{Y},\mathcal{Y}) - B(\mathcal{X},\mathcal{Y})^2}{\alpha g(\mathcal{X},\mathcal{X})g(\mathcal{Y},\mathcal{Y})} + \frac{2B(\mathcal{X},\mathcal{Y})C(\mathcal{X},\mathcal{Y}) - B(\mathcal{X},\mathcal{X})C(Y,\mathcal{Y}) - B(\mathcal{Y},\mathcal{Y})C(\mathcal{X},\mathcal{X})}{g(\mathcal{X},\mathcal{X})g(\mathcal{Y},\mathcal{Y})}.$$

Let us now relate scalar curvatures of (M, g_α) and $(\overline{M}, \overline{g})$.

Theorem 4.1 *Let $(M, g, N) \to (\overline{M}, \overline{g})$ be a null hypersurface with closed rigging of a semi-Riemannian manifold and g_α the semi-Riemannian metric on M defined as in (20). The scalar curvatures s_α and $\overline{s}$ of (M, g_α) and $(\overline{M}, \overline{g})$, respectively, are related (on M) by*

$$s_\alpha = \overline{s} - 4\overline{Ric}(\xi, N) + 2\overline{K}(\xi, N) + \frac{2}{\alpha}\overline{Ric}(\xi,\xi) - 2tr\left(\overset{\star}{A}_\xi \circ A_N\right) + \frac{1}{\alpha}tr\left(\overset{\star}{A}_\xi^{\,2}\right) + \left(2H - \frac{1}{\alpha}\overset{\star}{H}\right)\overset{\star}{H} - \tau(A_N\xi) + \left(H - \frac{3}{\alpha}\overset{\star}{H}\right)\tau(\xi) - \frac{d\alpha(\xi)}{2\alpha^2}\left(H + 3\overset{\star}{H}\right).$$

Proof By definition,

$$s_\alpha = g_\alpha^{aa}Ric_\alpha(E_a, E_a) = \varepsilon^k Ric_\alpha(E_k, E_k) + \frac{1}{\alpha}Ric_\alpha(\xi,\xi).$$

Let us compute each term of the latter. Replacing $\mathcal{X}$ and $\mathcal{Y}$ by E_k in equation (36) and summing over k, one obtains

$$\varepsilon^k Ric_\alpha(E_k, E_k) = \overline{s} - 4\overline{Ric}(\xi, N) + 2\overline{K}(\xi, N) + \frac{1}{\alpha}\overline{Ric}(\xi,\xi) - 2tr\left(\overset{\star}{A}_\xi \circ A_N\right) + \frac{1}{\alpha}tr\left(\overset{\star}{A}_\xi^{\,2}\right) + \left(2H - \frac{1}{\alpha}\overset{\star}{H}\right)\overset{\star}{H} - \tau(A_N\xi) + \left(H - \frac{2}{\alpha}\overset{\star}{H}\right)\tau(\xi) - \frac{d\alpha(\xi)}{2\alpha^2}\left(H + 2\overset{\star}{H}\right). \tag{38}$$

The first item of Proposition 4.3 gives:

$$\frac{1}{\alpha}Ric_\alpha(\xi,\xi) = \frac{1}{\alpha}\overline{Ric}(\xi,\xi) - \frac{1}{\alpha}\left[\tau(\xi) + \frac{1}{2\alpha}d\alpha(\xi)\right]\overset{\star}{H}. \tag{39}$$

One obtains the announced result by summing (38) and (39). □

5. Application on Monge null hypersurfaces of $\mathbb{R}^{n+1}_q$

Let us now set $(\overline{M}, \overline{g}) = \mathbb{R}^{n+1}_q$, the real standard semi-Euclidean space with its canonical metric

$$\overline{g} = \varepsilon_i(dx^i)^2,$$

with Einstein's summation and where $(x^0,\dots,x^n)$ is the rectangular coordinate of $\mathbb{R}^{n+1}$, and we have set

$$\varepsilon^i=\varepsilon_i:=\begin{cases}-1 & \text{if } 0\leq i\leq q-1\\ +1 & \text{if } q\leq i\leq n\end{cases}.$$

Let $\mathcal{D}$ be an open subset of $\mathbb{R}^n_{q-1}$ and let $F:\mathcal{D}\to\mathbb{R}$ be a nowhere vanishing smooth function. Let us consider the immersion

$$x:\begin{array}{ccc}\mathcal{D} & \longrightarrow & \mathbb{R}^{n+1}_q\\ p=(u^1,\dots,u^n) & \mapsto & x(p)=(x^0=F(p),x^1=u^1,\dots,x^n=u^n).\end{array} \tag{40}$$

Then $M=x(\mathcal{D})$ is called a Monge hypersurface. It is easy to check that a vector field $X=X^i\frac{\partial}{\partial x^i}$ (Einstein's summation) on $\mathbb{R}^{n+1}_q$ is tangent to M if and only if $X^0=X^aF'_{u^a}$, so $\mathbf{n}=\dfrac{\partial}{\partial x^0}+\varepsilon^aF'_{u^a}\frac{\partial}{\partial x^a}$ is normal to M. The Monge hypersurface M is a null hypersurface if and only if $\mathbf{n}$ is a null vector field. It is equivalent to write

$$\varepsilon^a\left(F'_{u^a}\right)^2=||\nabla F||^2=1, \tag{41}$$

where ∇F is the gradient of F in the semi-Euclidean space $\mathbb{R}^n_{q-1}$. Then taking the partial derivative of (41) with respect to x^β leads to

$$\varepsilon^aF'_{u^a}F''_{u^au^\beta}=0. \tag{42}$$

5.1. Generic UCC-normalization on a Monge null hypersurface

Let us endow the Monge null hypersurface $x:M\to\mathbb{R}^{n+1}_q$ with the (physically and geometrically) relevant rigging

$$\mathscr{N}_F=\frac{1}{\sqrt{2}}\Big[-\frac{\partial}{\partial x^0}+\varepsilon^aF'_{u^a}\frac{\partial}{\partial x^a}\Big]. \tag{43}$$

The corresponding rigged vector field is given by

$$\xi_F=\frac{1}{\sqrt{2}}\mathbf{n}=\frac{1}{\sqrt{2}}\Big[\frac{\partial}{\partial x^0}+\varepsilon^aF'_{u^a}\frac{\partial}{\partial x^a}\Big]. \tag{44}$$

We show below that this is a closed normalization with vanishing rotation 1-form τ and conformal screen distribution with unit conformal factor $\varphi=1$. Let us consider the natural (global) parametrization of M given by

$$\begin{cases}x^0=F(u^1,...,u^n)\\ x^\alpha=u^\alpha\\ \alpha=1,...,n\end{cases}\qquad (u^1,...,u^n)\in\mathcal{D}\,. \tag{45}$$

Then $\Gamma(TM)$ is spanned by $\{\frac{\partial}{\partial u^\beta}\}_\beta$ with

$$\frac{\partial}{\partial u^\beta}=F'_{u^\beta}\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^\beta}. \tag{46}$$

Now taking the covariant derivative of $\mathbf{n}$ by the flat connection $\overline{\nabla}$ and using (42), we obtain

$$\begin{aligned}\overline{\nabla}_{\frac{\partial}{\partial u^\beta}}\mathbf{n} &= \varepsilon^a F''_{u^\beta u^a}\frac{\partial}{\partial x^a} \\ &= \varepsilon^a F''_{u^\beta u^a}\left(F'_{u^a}\frac{\partial}{\partial x^0}+\frac{\partial}{\partial x^a}\right), \\ \overline{\nabla}_{\frac{\partial}{\partial u^\alpha}}\mathbf{n} &= \varepsilon^a F''_{u^\beta u^a}\frac{\partial}{\partial u^a}. \end{aligned} \tag{47}$$

Using (42) again we have

$$\overline{g}(\overline{\nabla}_{\frac{\partial}{\partial u^\beta}}\mathbf{n}, \mathscr{N}_F) = \varepsilon^a F'_{x^\beta}F''_{x^\beta x^a} = 0.$$

Hence, $\overline{\nabla}_{\frac{\partial}{\partial u^\beta}}\mathbf{n}$ is a section of the screen distribution.

Proposition 5.1 *Let $x:(M,g,\mathscr{N}_F)\to\mathbb{R}^{n+1}_q$ be a Monge null hypersurface graph of a function F endowed with the rigging $\mathscr{N}_F$ as in (43). Then the following hold:*

1. *The rigging $\mathscr{N}_F$ is closed and the corresponding rotation 1-form $\tau^{\mathscr{N}_F}$ vanishes identically.*
2. *The screen distribution is conformal with $\varphi = 1$ as conformal factor.*
3. *The screen distribution is integrable with leaves the level sets of the function F.*
4. *The induced connection ∇ coincides with the Levi-Civita connection of the associated metric g_1, i.e.*

$$\nabla_1 = \nabla.$$

5. *In the natural basis $\{\frac{\partial}{\partial u^a}\}_a$, the divergence (with respect to the induced connection) of a vector field $X = X^a\frac{\partial}{\partial u^a}$ takes the form*

$$divX = \frac{\partial X^a}{\partial u^a}$$

(as in the usual Euclidean case).

Proof Since $\overline{\nabla}$ is a flat connection and the difference between both of the vectors $\mathscr{N}_F$, ξ_F and $\frac{1}{\sqrt{2}}\mathbf{n}$ is a constant vector, then

$$\overline{\nabla}.\mathscr{N}_F = \overline{\nabla}.\xi_F = \frac{1}{\sqrt{2}}\overline{\nabla}.\mathbf{n}.$$

Then by using (47) and (10), $\tau^{\mathscr{N}_F}$ identically vanishes and

$$A_{\mathscr{N}_F}\left(\frac{\partial}{\partial u^\alpha}\right) = \overset{\star}{A}_{\xi_F}\left(\frac{\partial}{\partial u^\beta}\right) = -\frac{1}{\sqrt{2}}\varepsilon^a F''_{u^\beta u^a}\frac{\partial}{\partial u^a}. \tag{48}$$

Hence, $\overset{\star}{A}_{\xi_F} = A_{\mathscr{N}_F}$, which shows that the screen distribution is conformal with conformal factor $\varphi = 1$. The 1-form η is given by

$$\eta = \sqrt{2}F'_{u^a}du^a.$$

Using the Gauss lemma it follows that

$$d\eta = \sqrt{2}F''_{u^a u^b} du^b \wedge du^a = \sqrt{2}\sum_{a\neq b}(F''_{u^a u^b} - F''_{u^b u^a})\, du^b \otimes du^a = 0,$$

which shows that the rigging $\mathscr{N}_F$ is closed. Then the screen distribution is integrable. Let us now show that the leaves of the screen distribution are really the level sets of F. Let $c \in Im(F)$ be a regular value of F and $M_c = F^{-1}(c)$ the c-level set of F in $\mathbb{R}^n_{q-1}$. Then $\psi_c : M_c \to \mathbb{R}^n_{q-1}$ is a semi-Riemannian hypersurface of the semi-Euclidean space $\mathbb{R}^n_{q-1}$ and the Gauss map is the gradient ∇F of F. We take ψ_c to be the inclusion map and M_c is a subset of $\mathcal{D}$. We then have the following diagram:

$$\begin{aligned} M_c \overset{\psi_c}{\hookrightarrow} \mathcal{D} \quad &\overset{\psi}{\longrightarrow} \quad M \overset{i}{\hookrightarrow} \mathbb{R}^{n+1}_q, \\ p(u^1,...,u^n) \quad &\mapsto \quad x(x^0 = F(u^1,...,u^n), x^1 = u^1,...,x^n = u^n). \end{aligned} \tag{49}$$

We denote by $\overset{\circ}{\nabla}$ and ∇_c the Levi-Civita connections of $\mathbb{R}^n_{q-1}$ and M_c, respectively. Taking the Jacobian matrix of ψ, it is easy to check that for any $X \in \Gamma(TM_c)$, $\psi_\star(\psi_{c\star}X) = \psi_\star(X) = (\langle X, \nabla F\rangle, X) = (0, X)$ and

$$\begin{aligned} \langle \psi_\star(X), \xi_F\rangle &= (1/\sqrt{2})\langle (0,X), (1,\nabla F)\rangle = (1/\sqrt{2})\,(-0 + \langle X, \nabla F\rangle) = 0, \\ \langle \psi_\star(X), \mathscr{N}_F\rangle &= (1/\sqrt{2})\langle (0,X), (-1,\nabla F)\rangle = (1/\sqrt{2})\,(0 + \langle X, \nabla F\rangle) = 0. \end{aligned}$$

Thus, the level sets $\psi(M_c)$ are the leaves of the screen distribution $\mathscr{S}(\mathscr{N}_F)$ of M (endowed with the normalization (43)). Conversely, let $L \to (M,g) \to \mathbb{R}^{n+1}_q$ be a (connected) leaf of the screen distribution. We need to show that L is a level set of F. For every $x = (F(p), p) \in L$, for every $X = X^i \frac{\partial}{\partial x^i} \in T_x\mathbb{R}^{n+1}$,

$$X \in T_x L \iff \langle X, \xi_F\rangle = 0 = \langle X, \mathscr{N}_F\rangle \iff X^0 = 0.$$

Hence, for every $\beta = 1,\dots,n$, $\frac{\partial}{\partial x^\beta} \in T_x L$. In addition, since $\langle \frac{\partial}{\partial x^\beta}, \xi_F\rangle = \frac{1}{\sqrt{2}}F'_{u^\beta}$, it follows that $F'_{u^\beta}(p) = 0, \quad \forall x = (F(p),p) \in L, \forall \beta = 1,\dots,n$, so there exists $c \in \mathbb{R}$ such that

$$F(p) = c, \qquad \forall x = (F(p), p) \in L.$$

Thus, the leaf L is defined on M by $F = c$.

Since $\tau^{\mathscr{N}_F}$ identically vanishes and $\overset{\star}{A}_{\xi_F} = A_{\mathscr{N}_F}$, ∇ is the Levi-Civita connection of the (semi-Riemannian) associated metric g_1 (see Theorem 4.1 in [2]). Let $X = X^a \frac{\partial}{\partial u^a}$ be a section of TM:

$$X = X^a \frac{\partial}{\partial u^a} = X^0 \frac{\partial}{\partial x^1} + X^a \frac{\partial}{\partial x^a},$$

with $X^0 = F'_{u^a} X^a$. We have

$$\overline{\nabla}_{\partial_{u^b}} X = \partial_{u^b}(X^0)\partial x^0 + \partial_{u^b}(X^a)\partial_{x^a}.$$

By using (7) and (43) the left-hand side of the above equation gives

$$\begin{aligned}\overline{\nabla}_{\partial_{u^b}} X &= \nabla_{\partial_{u^b}} X + B(\partial_{u^b}, X)\mathcal{N}_F \\ &= f^a \partial_{u^a} + B(\partial_{u^b}, X)\mathcal{N}_F \\ &= \left(F'_{u^a} f^a - \frac{1}{\sqrt{2}} B(\partial_{u^b}, X) \right) \partial_{x^0} \\ &+ \sum_{a=1}^{q-1} \left(f^a - F'_{u^a} \frac{1}{\sqrt{2}} B(\partial_{u^b}, X) \right) \partial_{x^a} + \sum_{a=q}^{n} \left(f^a + F'_{u} \frac{1}{\sqrt{2}} B(\partial_{u^b}, X) \right) \partial_{x^a}.\end{aligned}$$

After identification, one gets

$$f^a = \begin{cases} \partial_{u^b}(X^a) + \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^b}, X) & \text{if } 1 \leq a \leq q-1 \\ \partial_{u^b}(X^a) - \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^b}, X) & \text{if } q \leq a \leq n \end{cases}.$$

Hence,

$$\nabla_{\partial_{u^b}} X = \sum_{a=1}^{q-1} \left(\partial_{u^b}(X^a) + \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^b}, X) \right) \partial_{u^a} + \sum_{a=q}^{n} \left(\partial_{u^b}(X^a) - \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^b}, X) \right) \partial_{u^a}.$$

The above relation together with equation (41) leads to

$$\begin{aligned} divX &= tr(\nabla X) \\ &= \sum_{a=1}^{q-1} \left(\partial_{u^a}(X^a) + \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^a}, X) \right) + \sum_{a=q}^{n} \left(\partial_{u^a}(X^a) - \frac{1}{\sqrt{2}} F'_{u^a} B(\partial_{u^a}, X) \right) \\ &= \partial_{u^a}(X^a) + \frac{1}{\sqrt{2}} \sum_{a=1}^{q-1} F'_{u^a} B(F'_{u^a} \partial_{x^0} + \partial_{x^a}, X) - \frac{1}{\sqrt{2}} \sum_{a=q}^{n} F'_{u^a} B(F'_{u^a} \partial_{x^0} + \partial_{x^a}, X) \\ &= \partial_{u^a}(X^a) - B(\xi_F, X) \\ &= \partial_{u^a}(X^a). \end{aligned}$$

□

Hence, on any Monge null hypersurface, our rigging $\mathcal{N}_F$ has many good properties: the screen distribution is integrable, the 1-form τ identically vanishes, and

$$A_{\mathcal{N}_F} = \overset{\star}{A}_{\xi_F}. \tag{50}$$

On a Monge null hypersurface, the rigging (43) is called generic unitary conformally closed (UCC)-rigging, since it is closed and has a conformal screen with conformal factor $\varphi = 1$. Recall that a hypersurface of a semi-Riemannian manifold is said to be totally geodesic when its shape operator identically vanishes. The above proposition together with Theorem 3.1 gives the following result.

Theorem 5.1 *For any Monge null hypersurface $(M, g, \mathcal{N}_F) \to \mathbb{R}^{n+1}_q$ endowed with its generic UCC-rigging (43), the isometric immersion $x_1 : (M, g_1) \to (\mathbb{R}^{n+1}, \overline{g}_1)$ into the twisted semi-Riemannian manifold $(\mathbb{R}^{n+1}, \overline{g}_1)$ with the metric (20) is a totally geodesic semi-Riemannian hypersurface.*

5.2. A special rigging on a Monge null hypersurface of $\mathbb{R}_q^{n+1}$

Let us now consider for $x : M \to \mathbb{R}_q^{n+1}$, as in (40), the rigging

$$\mathcal{N}_F = \frac{1}{2x^0}\Big[\frac{\partial}{\partial x^0} - \varepsilon^a F'_{u^a}\frac{\partial}{\partial x^a}\Big] \tag{51}$$

with corresponding rigged vector field

$$\xi_F = -x^0\mathbf{n} = x^0\Big[-\frac{\partial}{\partial x^0} - \varepsilon^a F'_{u^a}\frac{\partial}{\partial x^a}\Big]. \tag{52}$$

These two vector fields are defined on $\mathbb{R}^\star \times \mathcal{D}$, which is an open subset containing our Monge null hypersurface M. However, they are lightlike only along M. Since $\mathcal{N}_F$ is pointwise conformal to the generic UCC-rigging, the rigging $\mathcal{N}_F$ also has integrable screen distribution and corresponding leaves are the level sets of the function F. Furthermore, for this rigging,

$$\overline{\eta} = -\frac{1}{2x^0}\Big[dx^0 + F'_{u^a}dx^a\Big] \tag{53}$$

and

$$\eta = -\frac{1}{x^0}F'_{u^a}du^a \tag{54}$$

since $dx^0 = F'_{u^a}du^a$ along M. Let us set for this subsection $\alpha = 2(x^0)^2$, which is constant along the leaves of the screen distribution. After a direct calculation we obtain

$$g_\alpha = \big[\varepsilon_a + (F'_{u^a})^2\big]\,(du^a)^2 + 2\sum_{a<b} F'_{u^a}F'_{u^b}du^a du^b, \tag{55}$$

where $2dx^i dx^j = dx^i \otimes dx^j + dx^j \otimes dx^i$. Notice that g_α is a semi-Riemannian metric of index $q-1$ on M, but since $\mathcal{N}_F$ is lightlike only along M, the metric $\overline{g}_\alpha$ is not necessary nondegenerate. The problem is to find integers n and q for which this metric is nondegenerate, in order to apply results of Section 3 to the Monge null hypersurface M endowed with this rigging. For example, by a calculation of determinant, one shows that for $n = 3$ and $q = 2$, this metric $\overline{g}_\alpha$ is nondegenerate for any F.

Using (46) one has

$$\overline{\nabla}_{\frac{\partial}{\partial u^a}}\xi_F = -x^0\overline{\nabla}_{\frac{\partial}{\partial u^a}}\mathbf{n} - \frac{\partial x^0}{\partial u^a}\mathbf{n} = -x^0\overline{\nabla}_{\frac{\partial}{\partial u^a}}\mathbf{n} + \frac{F'_{u^a}}{x^0}\xi_F.$$

The latter together with (10) and (54) gives

$$\overset{\star}{A}_\xi = x^0\overline{\nabla}_{.}\mathbf{n} \qquad \text{and} \qquad \tau = \eta. \tag{56}$$

We also have

$$\overline{\nabla}_{\frac{\partial}{\partial u^\beta}}\mathcal{N}_F = -\frac{1}{2x^0}\overline{\nabla}_{\frac{\partial}{\partial u^\beta}}\mathbf{n} - (2x^0)\frac{\partial\, 1/(2x^0)}{\partial u^\beta}\mathcal{N}_F = -\frac{1}{2x^0}\overline{\nabla}_{\frac{\partial}{\partial u^\beta}}\mathbf{n} - \frac{F'_{u^\beta}}{x^0}\mathcal{N}_F,$$

which allows us to find

$$A_N = \frac{1}{2x^0}\overline{\nabla}_{.}\mathbf{n}. \tag{57}$$

Then the screen distribution is conformal with

$$\overset{\star}{A}_{\xi} = 2(x^0)^2 A_N \quad \text{and} \quad \tau = \eta. \tag{58}$$

From here, it is easy to check that (28) holds. By Theorem 3.2, the induced connection is the Levi-Civita connection of the α-associated metric g_α. Thus, $\nabla = \nabla\!\!\!\!\diagup_{\alpha}$, where $\alpha = 2(x^0)^2$.

Remark 5.1 *It is noteworthy that for all changes of rigging $\widetilde{N} = \phi \mathcal{N}_F$, the Levi-Civita connection of the $\frac{\alpha}{\phi^2}$-associated metric coincides with the induced connection $\nabla^{\widetilde{N}}$.*

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