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On oscillatory and nonoscillatory behavior of solutions for a class of fractional order differential equations

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Abstract: This work aims to develop oscillation criterion and asymptotic behavior of solutions for a class of fractional order differential equation:

$$D_0^{\alpha} u(t) + \lambda u(t) = f(t, u(t)), \quad t > 0,$$

$$D_0^{\alpha-1} u(t)|_{t=0} = u_0, \quad \lim_{t \to 0} J_0^{2-\alpha} u(t) = u_1,$$

where D_0^{α} denotes the Riemann–Liouville differential operator of order α with $1 < \alpha \leq 2$ and $\lambda \in [1, \infty)$. Properties of the Mittag–Leffler function are utilized to establish our main results.

Key words: Fractional differential equations, oscillation, asymptotic behavior, the Riemann–Liouville differential operator, the Mittag–Leffler function

1. Introduction

The study of fractional derivatives and integrals is a branch of mathematical analysis, known as fractional calculus. Fractional differential equations, as the generalization of classical integer order differential equations, have gained great interest because of their considerable applications in the fields of science and engineering that are yet increasing. In the beginning, there were almost no well-known functional uses of this field and many researchers considered it a theocratical territory consisting of just mathematical manipulations of a few or no practical utilization. Almost three decades back, this view changed worldwide and gradually, the fractional calculus penetrated in applied mathematics. In the most recent decade, fractional calculus has been practically connected to almost every field of science and engineering. It should be noted that the majority of research work is devoted to the solvability of fractional differential equations and in this respect fractional differential equations have gained significant developments [1, 22].

Recently, the exploration on the oscillation theory of fractional differential equations has been exceptionally productive and developed rapidly and has drawn the attention of many analysts [4, 5, 8, 13, 19, 21].

In [5, 19, 21], oscillation criteria are established for nonlinear fractional differential equations by generalized Riccati transformation technique and by using certain parameter functions. In [13], the generalized Riccati transformation technique and integral averaging method has been used to establish sufficient conditions for oscillation of solutions of time fractional partial fractional differential equations.

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In [4], some oscillation results for fractional order delay differential equations are given by using the Laplace transformation formulations of fractional order derivatives.

In [8], oscillation criteria for a class of nonlinear fractional differential equations is established by obtaining an equivalent volterra integral equation. Some interesting results are established by considering different conditions.

We likewise allude [2, 3, 9–12, 15, 18, 20] to see some exploration of very late days on oscillation of a variety of fractional differential equations.

In this manuscript, we established the oscillation criteria and asymptotic behavior of solutions for a class of fractional differential equations by considering equations of the form

$$D_0^\alpha u(t) + \lambda u(t) = f(t, u(t)), \quad t > 0, \tag{1.1}$$

$$D_0^{\alpha-1} u(t)|_{t=0} = u_0, \quad \lim_{t\to 0} J_0^{2-\alpha} u(t)|_{t=0} = u_1, \tag{1.2}$$

where D_0^α denotes the Riemann-Liouville(RL) differential operator of order α with $1 < \alpha \leq 2,\ \lambda \in [1, \infty)$ and $u_0, u_1 \in \mathbb{R}^+ = [0, \infty)$. We let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ be a continuous function.

However in any case, to the best of our insight nothing is discussed about the oscillation of the solutions of the fractional differential equations (1.1) so far. The motivation for the present work has been inspired basically by the paper of N. Parhi and N. Misra [14], and the cited papers in the references. In this paper, we used the Laplace transformation to the differential equation to get an equivalent integral equation. Some properties of Mittag-Leffler function are used to obtain oscillation criteria.

The remainder of this paper is organized as follows: In the next section, we give some basic definitions and properties. Some related properties of the Mittag–Leffler function are established in Section 3. Section 4 presents main results about oscillatory and nonoscillatory solutions, and the last section discusses the asymptotic behavior of oscillatory and non-oscillatory solutions.

2. Preliminaries

In this section, we recall fundamentals of fractional calculus needed for the further developments of this paper. We define the Riemann–Liouville fractional integral and derivative, and also give some of their properties that will be used in this article.

The Riemann–Liouville integral is named after Bernhard Riemann and Joseph Liouville, and is defined as follows:

Definition 2.1 *Let $\alpha > 0$ and $u \in L^1[a, b]$, for $a \leq t \leq b$. Then the Riemann–Liouville fractional integral I_a^α is defined as*

$$I_a^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} u(s) ds,$$

where $\Gamma(.)$ is the Gamma function and a is the fixed initial point.

Next we present semigroup property and linearity property of the Riemann–Liouville fractional integral. These properties are used in the proofs of next section.

1. If $\alpha, \beta \geq 0$ and $u \in L^1[a, b]$, then

   $$I_a^\alpha I_a^\beta u = I_a^\beta I_a^\alpha u = I_a^{\alpha+\beta} u$$

   holds almost everywhere on $[a, b]$.

2. Let u and v be two functions defined on $[a,b]$ such that $I_a^\alpha u$ and $I_a^\alpha v$ exist almost everywhere, then

$$I_a^\alpha(c_1 u + c_2 v) = c_1 I_a^\alpha u + c_2 I_a^\alpha v$$

almost everywhere on $[a,b]$.

Next we give the definition of fractional-order derivative.

Definition 2.2 *[7] Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $u \in AC^n[a,b]$, for $a \leq t \leq b$. Then the Riemann–Liouville fractional derivative D_a^α is defined as*

$$D_a^\alpha u = D^n I_a^{n-\alpha} u = D^n \left(\frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-\alpha-1} u(s) ds \right), \quad n-1 < \alpha \leq n.$$

In particular, for $\alpha = n$, $D_a^\alpha u = D^n u$.

For some $\beta > -1$ and $\alpha > 0$,

$$D_a^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}. \tag{2.1}$$

The Riemann–Liouville fractional derivative and integral also satisfy the following properties:

(a) For $\alpha, \beta \geq 0$ and $u \in L^1[a,b]$, $D_a^\alpha I_a^\alpha u = u$.

(b) For $\alpha, \beta > 0$, $\beta > \alpha$ and $u \in L^1[a,b]$, $D_a^\alpha I_a^\beta u = I_a^{\beta-\alpha} u$ holds almost everywhere on $[a,b]$.

3. The Mittag–Leffler function and related properties

In this section, we define a two-parameter Mittag–Leffler function and prove some related properties that play an important role in the upcoming progress of the paper.

Definition 3.1 *The two-parameter Mittag–Leffler function $E_{\alpha,\beta}$ is defined by the power series*

$$E_{\alpha,\beta}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\alpha+\beta)},$$

where $\alpha, \beta > 0$ are the parameters.

On the basis of numerical evidences [17], for $\alpha \in (1,2]$, the estimate

$$-1 < E_{\alpha,\alpha}(-t^\alpha) \leq \frac{1}{\Gamma(\alpha)}$$

holds for $t \geq 0$.

It is evident from Figure 1 that for $t \geq 0$ and $1 < \alpha \leq 2$, the function $t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha)$ is a bounded function, that is $|t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha)| \leq 1.$, and the function $t^{\alpha-2}E_{\alpha,\alpha-1}(-t^\alpha)$ is bounded below by -1. Furthermore, $\lim_{t\to\infty} t^{\alpha-2}E_{\alpha,\alpha-1}(-t^\alpha) \leq 1$. Also, note that at $t=0$, the graph of $t^{\alpha-2}E_{\alpha,\alpha-1}(-t^\alpha)$ blows up.

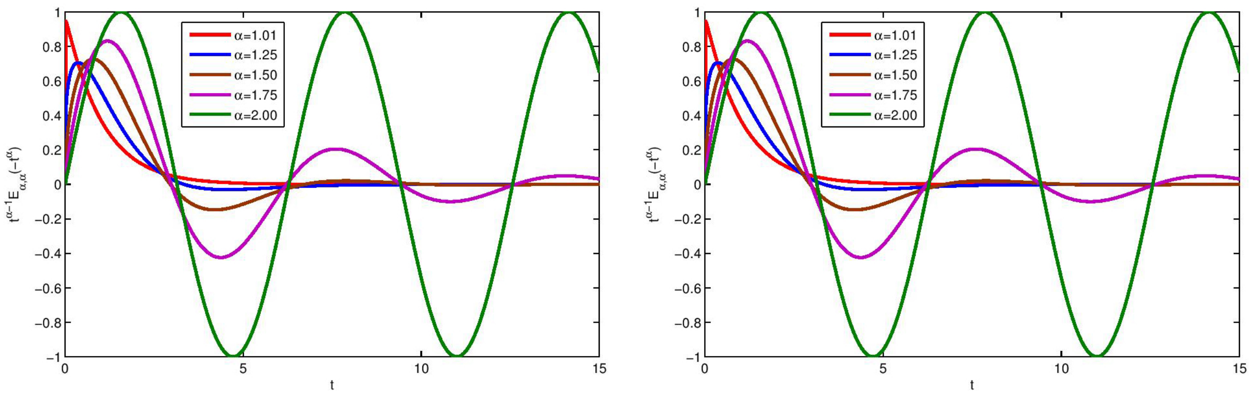

Figure 1. Graphs of $t^{\alpha-1}E_{\alpha,\alpha}(-t^{\alpha})$ and $t^{\alpha-2}E_{\alpha,\alpha-1}(-t^{\alpha})$ for $\alpha = 1.01, 1.25, 1.50, 1.75, 2.00$.

For simplicity, we introduce the notation

$$J_0^{\alpha} f(t, u(t)) := \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha}) f(s, u(s)) ds.$$

Now we prove the following properties that are important tools in the proof of Theorem 3.2.

For $1 < \alpha \leq 2$,

(P_1) $D_0^{\alpha} J_0^{\alpha} f(t, u(t)) = f(t, u(t)) - \lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^{\alpha}) f(s, u(s)) ds.$

(P_2) $\lim_{t\to 0} \; D_0^{\alpha-1} J_0^{\alpha} f(t, u(t)) = 0.$

(P_3) $D_0^{\alpha} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha})$ and, $D_0^{\alpha} t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^{\alpha}) = -\lambda t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^{\alpha})$.

Proof

(P_1) By definition 3.1, we get $D_0^{\alpha} J_0^{\alpha} f(t, u(t)) = D_0^{\alpha} \int_0^t \sum_{k=0}^{\infty} \frac{(-\lambda)^k (t-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} f(s, u(s)) ds.$

As the power series defining $E_{\alpha,\beta}(t)$ is convergent for all real t, we can interchange summation and integration, and by definition 2.1, we have

$$D_0^{\alpha} J_0^{\alpha} f(t, u(t)) = D_0^{\alpha} \sum_{k=0}^{\infty} (-\lambda)^k I_0^{\alpha k+\alpha} f(t, u(t))$$

$$= D_0^{\alpha} I_0^{\alpha} f(t, u(t)) + D_0^{\alpha} \sum_{k=1}^{\infty} (-\lambda)^k I_0^{\alpha k+\alpha} f(t, u(t)).$$

Using property (a) of RL-derivative and integral, and replacing $k-1$ by k in the sum, we obtain

$$D_0^{\alpha} J_0^{\alpha} f(t, u(t)) = f(t, u(t)) + D_0^{\alpha} \sum_{k=0}^{\infty} (-\lambda)^{k+1} I_0^{\alpha k+\alpha+\alpha} f(t, u(t)).$$

In view of power series convergence, interchanging summation and derivative in the second term of the right hand side, we get

$$D_0^{\alpha} J_0^{\alpha} f(t, u(t)) = f(t, u(t)) + \sum_{k=0}^{\infty} (-\lambda)^{k+1} D_0^{\alpha} I_0^{\alpha k+\alpha+\alpha} f(t, u(t)).$$

Using property (b) of RL-derivative and integral, we obtain

$$D_0^\alpha J_0^\alpha f(t,u(t)) = f(t,u(t)) + \sum_{k=0}^{\infty}(-\lambda)^{k+1} I_0^{\alpha k+\alpha} f(t,u(t)).$$

Applying definition 2.1, and then interchanging summation and integral, we get

$$D_0^\alpha J_0^\alpha f(t,u(t)) = f(t,u(t)) + \int_0^t \sum_{k=0}^{\infty} \frac{(-\lambda)^{k+1}(t-s)^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} f(s,u(s))ds.$$

By definition 3.1, we have

$$D_0^\alpha J_0^\alpha f(t,u(t)) = f(t,u(t)) - \lambda \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s,u(s))ds.$$

(P_2) Using the similar arguments as above in the proof of (P_2), we get $D_0^{\alpha-1} J_0^\alpha f(t,u(t))$

$= \int_0^t E_{\alpha,1}(-\lambda(t-s)^\alpha) f(s,u(s))ds.$

Now, since $\lim_{t\to 0} \int_0^t E_{\alpha,1}(-\lambda(t-s)^\alpha) f(s,u(s))ds = 0$. Thus, we get $\lim_{t\to 0} D_0^{\alpha-1} J_0^\alpha f(t,u(t)) = 0$.

(P_3) $D_0^\alpha t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) = D_0^\alpha \sum_{k=0}^\infty \frac{(-\lambda)^k t^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} = \sum_{k=1}^\infty \frac{(-\lambda)^k t^{\alpha k-1}}{\Gamma(\alpha k)} = \sum_{k=0}^\infty \frac{(-\lambda)^{k+1} t^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)}$

$= -\lambda t^{\alpha-1} \sum_{k=0}^\infty \frac{(-\lambda)^k t^{\alpha k}}{\Gamma(\alpha k+\alpha)} = -\lambda t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha)$, and $D_0^\alpha t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha)$

$= D_0^\alpha \sum_{k=0}^\infty \frac{(-\lambda)^k t^{\alpha k+\alpha-2}}{\Gamma(\alpha k+\alpha-1)} = \sum_{k=1}^\infty \frac{(-\lambda)^k t^{\alpha k-2}}{\Gamma(\alpha k-1)} = -\lambda t^{\alpha-2} \sum_{k=0}^\infty \frac{(-\lambda)^k t^{\alpha k}}{\Gamma(\alpha k+\alpha-1)}$

$= -\lambda t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha).$

□

Theorem 3.2 *Let* $1 < \alpha \le 2$, $\lambda \in [1,\infty)$ *and* $u_0, u_1 \in \mathbb{R}^+$. *Let* $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ *is continuous. Then* u *is solution of* (1.1), (1.2) *if and only if* u *satisfies the integral equation*

$$\begin{aligned} u(t) =& u_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + u_1 t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t^\alpha) \\ &+ \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s,u(s))ds. \end{aligned} \tag{3.1}$$

Proof Assume that u is the the solution of Equation (1.1), (1.2), then u satisfies the Volterra integral equation (3.1)(see [6]).

Conversely, we assume that u is the solution of (3.1), then we show that it solves (1.1), (1.2).

From (P_1) and (P_3), we have

$$\begin{aligned}D_0^\alpha u(t) =& D_0^\alpha[u_0 t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + u_1 t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha)\\ &+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)f(s,u(s))ds]\\ =& -u_0\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) - u_1\lambda t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha) + f(t,u(t))\\ &-\lambda\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)f(s,u(s))ds\\ =& -\lambda u(t) + f(t,u(t)).\end{aligned}$$

Consequently, $D_0^\alpha u(t) + \lambda u(t) = f(t,u(t))$. That is, the equation (1.1) is satisfied.

Next we prove that the initial conditions are also satisfied.

As $D_0^{\alpha-1}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) = \sum_{k=0}^\infty \frac{(-\lambda)^k D_0^{\alpha-1}t^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} = \sum_{k=0}^\infty \frac{(-\lambda)^k t^{\alpha k}}{\Gamma(\alpha k+1)} = E_{\alpha,1}(-\lambda t^\alpha)$. And $J_0^{2-\alpha}t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha) = \sum_{k=0}^\infty \frac{(-\lambda)^k J_0^{2-\alpha}t^{\alpha k+\alpha-2}}{\Gamma(\alpha k+\alpha-1)} = \sum_{k=0}^\infty \frac{(-\lambda)^k t^{\alpha k}}{\Gamma(\alpha k+1)} = E_{\alpha,1}(-\lambda t^\alpha)$.

Now, since $E_{\alpha,1}(-\lambda t^\alpha) = \sum_{k=0}^\infty \frac{(-\lambda)^k t^{\alpha k}}{\Gamma(\alpha k+1)} = 1 + \sum_{k=1}^\infty \frac{(-\lambda)^k t^{\alpha k}}{\Gamma(\alpha k+1)}$. Thus, we obtain

$$\lim_{t\to 0} D_0^{\alpha-1}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) = \lim_{t\to 0} J_0^{2-\alpha}t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha) = 1. \tag{3.2}$$

Also $J_0^{2-\alpha}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) = \sum_{k=0}^\infty \frac{(-\lambda)^k J_0^{2-\alpha}t^{\alpha k+\alpha-1}}{\Gamma(\alpha k+\alpha)} = \sum_{k=0}^\infty \frac{(-\lambda)^k t^{\alpha k+1}}{\Gamma(\alpha k+2)} = tE_{\alpha,2}(-\lambda t^\alpha)$, and $D_0^{\alpha-1}t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha) = \sum_{k=0}^\infty \frac{(-\lambda)^k D_0^{\alpha-1}t^{\alpha k+\alpha-2}}{\Gamma(\alpha k+\alpha-1)} = \sum_{k=1}^\infty \frac{(-\lambda)^k t^{\alpha k-1}}{\Gamma(\alpha k)} = -\lambda t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)$.

Thus, we have

$$\lim_{t\to 0} J_0^{2-\alpha}t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) = \lim_{t\to 0} D_0^{\alpha-1}t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t^\alpha) = 0. \tag{3.3}$$

Thus from (P_2), (3.2) and (3.3) the initial conditions are also satisfied.

Hence, every solution of (3.1) is also a solution of (1.1), (1.2) and vice versa. □

We consider only those real solutions $u:\mathbb{R}^+\to\mathbb{R}$ of (1.1) that are continuous and exist on the half line $[0,\infty)$ and are nontrivial in any neighborhood of infinity.

A solution $u(t)$ of (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative. Otherwise, it is called nonoscillatory.

In Sections 4 and 5, we provide sufficient conditions for the oscillatory and nonoscillatory solutions of (1.1), (1.2).

4. Oscillatory and nonoscillatory solutions

Theorem 4.1 *For $1<\alpha<2$, $u_1 = 0$. Assume that $f:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ is a continuous function, and let there exists a constant $M>0$ such that*

$$|f(t,u)| \le \frac{M}{\Gamma(1-\alpha)(t-a)^\alpha} \text{ for some } a>0 \text{ and } t>a.$$

Then all unbounded solutions of (1.1), (1.2) are oscillatory.

Proof Let $u(t)$ be an unbounded solution of (1.1), (1.2) on $[0,\infty)$ such that it is not oscillatory. Then there exists a $t_0 \geq 0$ such that $u(t) > 0$ for $t \geq t_0$. Since we have $-1 \leq t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) \leq 1$, and also we know $-1 < E_{\alpha,\alpha}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\alpha)}$. Thus, we get

$$\begin{aligned}
0 < u(t) =& u_0 t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)f(s,u(s))ds\\
=& u_0 t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^{t_1} (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)f(s,u(s))ds\\
&+ \int_{t_1}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)f(s,u(s))ds\\
\leq& u_0 + \int_0^{t_1} f(s,u(s))ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^t (t-s)^{\alpha-1}f(s,u(s))ds\\
\leq& u_0 + \int_0^{t_1} f(s,u(s))ds + \frac{M}{\Gamma(\alpha)}\int_a^t \frac{(t-s)^{\alpha-1}}{\Gamma(1-\alpha)(s-a)^\alpha}ds\\
\leq& u_0 + Lt_1 + M,
\end{aligned}$$

where $L = \sup_{t\in[0,t_1]} |f(t,u(t))|$ and $t_1 = \max\{a, t_0\}$. Hence, $u(t)$ is bounded, a contradiction.

For the other case, let $u(t) < 0$ for $t \geq t_0$. We have

$$\begin{aligned}
0 > u(t) =& u_0 t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)f(s,u(s))ds\\
=& u_0 t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^{t_1} (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)f(s,u(s))ds\\
&+ \int_{t_1}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)f(s,u(s))ds\\
>& -u_0 - \int_0^{t_1} f(s,u(s))ds - \int_{t_1}^t (t-s)^{\alpha-1}f(s,u(s))ds\\
>& -u_0 - \int_0^{t_1} f(s,u(s))ds - M\int_{t_1}^t \frac{(t-s)^{\alpha-1}}{\Gamma(1-\alpha)(s-t_1)^\alpha}ds\\
>& -u_0 - Lt_1 - \frac{M}{\Gamma(\alpha)},
\end{aligned}$$

where $L = \sup_{t\in[0,t_1]} |f(t,u(t))|$ and $t_1 = \max\{a, t_0\}$. Hence, $u(t)$ is bounded, a contradiction. □

Theorem 4.2 *Let* $u_1 = 0$, *and* $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ *satisfy* $f(t,-u) = -f(t,u)$ *and* $u_2 \leq u_3$ *implies* $f(t,u_2) \geq f(t,u_3)$ *for each fixed* t. *Let*

$$\lim_{t\to\infty} \int_\rho^t f(s,K)ds = +\infty, \quad t \geq \rho$$

for some $\rho > 0$ *and* $K > 0$, *then all bounded solutions of* (1.1), (1.2) *are oscillatory.*

Proof Let $u(t)$ be a bounded solution of (1.1), (1.2) that is not an oscillatory solution on $[0,\infty)$, so there exists a constant M and $t_0 \geq 0$ such that $|u(t)| \leq M$ for $t \geq 0$. Let $u(t) < 0$ for $t \geq t_0$. We have

$$\begin{aligned} u(t) &= u_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s,u(s))ds \\ &\geq -u_0 - \int_0^{t_0} f(s,u(s))ds - \int_{t_0}^t f(s,u(s))ds \\ &\geq -u_0 - Lt_0 + \int_{t_0}^t f(s,M)ds, \end{aligned}$$

where $L = \sup_{t\in[0,t_0]} |f(t,u(t))|$. This leads to $u(t) > 0$ for large t, a contradiction.

For the other case, let $u(t) > 0$ for $t \geq t_0$. We get

$$\begin{aligned} u(t) &= u_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t-s)^\alpha) f(s,u(s))ds \\ &\leq u_0 + \int_0^{t_0} f(s,u(s))ds + \int_{t_0}^t f(s,u(s))ds \\ &\leq u_0 + Lt_0 - \int_{t_0}^t f(s,M)ds, \end{aligned}$$

where $L = \sup_{t\in[0,t_0]} |f(t,u(t))|$. We obtain $u(t) < 0$ for large t, a contradiction. □

Theorem 4.3 *Let $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ that satisfy $f(t,-u) = -f(t,u)$. Let $f(t,u)$ be monotonically increasing in u for each fixed t. If*

$$\lim_{t\to\infty} \inf \int_0^t f(s,K)ds = -\infty$$

for some $K > 0$, then all bounded solutions of (1.1), (1.2) are nonoscillatory.

Proof Let $u(t)$ be a bounded solution of (1.1), (1.2) on $[0,\infty)$ such that $|u(t)| \leq M$ for $t \geq 0$. Let $u(t)$ be an oscillatory solution of (1.1), (1.2). Thus, there exists a sequence (t_n) such that $u(t_n) = 0$ and $t_n \to \infty$ as $n \to \infty$. We have

$$\begin{aligned} u_0 t_n^{\alpha-1} E_{\alpha,\alpha}(-\lambda t_n^\alpha) + u_1 t_n^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t_n^\alpha) &= -\int_0^{t^n} (t_n-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda(t_n-s)^\alpha) f(s,u(s))ds \\ &\geq -\int_0^{t_n} f(s,u(s))ds \\ &\geq -\int_0^{t_n} f(s,M)ds. \end{aligned}$$

We obtain $\limsup_{n\to\infty}(u_0 t_n^{\alpha-1} E_{\alpha,\alpha}(-\lambda t_n^\alpha) + u_1 t_n^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda t_n^\alpha)) = \infty$, a contradiction. □

5. Asymptotic behavior of oscillatory and nonoscillatory solutions

Theorem 5.1 *Let $f:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}$ be monotonically increasing in u for each fixed t and it satisfy $f(t,-u)=-f(t,u)$ and $uf(t,u)<0$ if $u\neq 0$.*

Let $\lim_{t\to\infty}\int_\rho^t f(s,K)ds=+\infty,\ t\geq\rho$ for some $\rho>0$ and $K>0$.

If $u(t)$ are oscillatory solutions of (1.1), (1.2) such that $\lim_{t\to\infty}u(t)$ exists, then $\lim_{t\to\infty}u(t)=0$.

Proof Let $\lim_{t\to\infty}u(t)=r\neq 0$. Let $r<0$. Choose $0<\epsilon<-r$, so there exists a $t_0>0$ such that $t\geq t_0$ implies $u(t)<r+\epsilon<0$. Thus,

$$f(t,u(t))<f(t,r+\epsilon)=-f(t,-(r+\epsilon)).$$

Since $u(t)$ is oscillatory, there exists a sequence (t_n) such that $u(t_n)=0$ and $t_n\to\infty$ as $n\to\infty$. Now, choosing $t_n\geq t_0$, we get

$$\begin{aligned}u_0t_n^{\alpha-1}E_{\alpha,\alpha}(-\lambda t_n^\alpha)+u_1t_n^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t_n^\alpha)=&-\int_0^{t_0}(t_n-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t_n-s)^\alpha)f(s,u(s))ds\\&-\int_{t_0}^{t_n}(t_n-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t_n-s)^\alpha)f(s,u(s))ds\\ \leq&\int_0^{t_0}f(s,u(s))ds+\int_{t_0}^{t_n}f(s,u(s))ds\\ \leq& Lt_0-\int_{t_0}^{t_n}f(s,-(r+\epsilon))ds,\end{aligned}$$

where $L=\sup_{t\in[0,t_0]}|f(t,u(t))|$. Thus, $\lim_{n\to\infty}(u_0t_n^{\alpha-1}E_{\alpha,\alpha}(-t_n^\alpha)+u_1t_n^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t_n^\alpha))=-\infty$, a contradiction.

For the other case, let $r>0$. Then for $0<\epsilon<r$, there exists a $t_0>0$ such that $u(t)>r-\epsilon$ for $t\geq t_0$. Choosing $t_n\geq t_0$, we have

$$\begin{aligned}u_0t_n^{\alpha-1}E_{\alpha,\alpha}(-\lambda t_n^\alpha)+u_1t_n^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t_n^\alpha)=&-\int_0^{t_0}(t_n-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t_n-s)^\alpha)f(s,u(s))ds\\&-\int_{t_0}^{t_n}(t_n-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t_n-s)^\alpha)f(s,u(s))ds\\ \geq&\int_0^{t_0}f(s,u(s))ds+\int_{t_0}^{t_n}f(s,u(s))ds\\ \geq&-Lt_0+\int_{t_0}^{t_n}f(s,r-\epsilon)ds,\end{aligned}$$

where $L=\sup_{t\in[0,t_0]}|f(t,u(t))|$. Thus, $\lim_{n\to\infty}(u_0t_n^{\alpha-1}E_{\alpha,\alpha}(-t_n^\alpha)+u_1t_n^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda t_n^\alpha))=\infty$, a contradiction. □

Theorem 5.2 *Let $u_1=0$, and $f:\mathbb{R}^+\times\mathbb{R}\to\mathbb{R}^+$ be monotonic decreasing in u for each fixed t and let it satisfy $f(t,-u)=-f(t,u)$. If*

$$\lim_{t\to\infty}\int_0^t f(s,K)ds=+\infty \text{ for some } K>0,$$

then all bounded solutions of (1.1), (1.2) are eventually negative.

Proof Let $u(t)$ be a bounded solution of (1.1), (1.2) such that $|u(t)| \leq M$ for $t \geq 0$. It follows from Theorem 4.3 that $u(t)$ is eventually positive or eventually negative. Thus, we let $u(t)$ be eventually positive. Then there exists a $t_0 > 0$ such that $u(t) > 0$ for $t \geq t_0$. We have

$$\begin{aligned} u(t) &= u_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^\alpha) f(s,u(s))ds \\ &= u_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^{t_0} (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^\alpha) f(s,u(s))ds \\ &\leq u_0 + \int_0^{t_0} f(s,u(s))ds + \int_{t_0}^t f(s,u(s))ds \\ &\leq u_0 + Lt_0 - \int_{t_0}^t f(s,M)ds, \end{aligned}$$

where $L = \sup_{t \in [0,t_0]} |f(t,u(t))|$. Applying lim as $t \to \infty$, we get a contradiction. □

Theorem 5.3 *Let $u_1 = 0$, and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ that satisfy $f(t,-u) = -f(t,u)$ and $uf(t,u) < 0$ if $u \neq 0$. Also, $u_2 \leq u_3$ implies $f(t,u_2) \leq f(t,u_3)$ for each fixed t. If*

$$\lim_{t\to\infty} \int_\rho^t f(s,K)ds = +\infty, \ \ t \geq \rho$$

for some $\rho > 0$ and $K > 0$, then no nonoscillatory solution of (1.1), (1.2) is bounded away from zero as $t \to \infty$.

Proof Assume $u(t)$ a nonoscillatory solution of (1.1), (1.2). Furthermore, assume that as $t \to \infty$ it is bounded away from zero. Then there exist $t_0 > 0$, $\epsilon > 0$ such that $|u(t)| \geq \epsilon$, for $t \geq t_0$. Let us assume $u(t)$ be eventually negative. Then there exists a $t_1 > t_0$ such that $u(t) < 0$ for $t \geq t_1$. So $-u(t) \geq \epsilon$ for $t \geq t_1$. We get

$$\begin{aligned} u(t) &\geq -u_0 - \int_0^{t_1} f(s,u(s))ds - \int_{t_1}^t f(s,u(s))ds \\ &\geq -u_0 - Lt_1 + \int_{t_1}^t f(s,\epsilon)ds, \end{aligned}$$

where $L = \sup_{t \in [0,t_0]} |f(t,u(t))|$. So

$$\lim_{t\to\infty} u(t) \geq -u_0 - Lt_1 + \lim_{t\to\infty} \int_{t_1}^t f(s,\epsilon)ds > 0,$$

a contradiction.

For the other case, let $u(t) > 0$ for $t \geq t_1$. So, $u(t) > \epsilon$ for $t \geq t_1$. We get

$$\begin{aligned} u(t) &\leq u_0 - \int_0^{t_1} f(s,u(s))ds - \int_{t_1}^t f(s,u(s))ds \\ &\leq u_0 + Lt_1 - \int_{t_1}^t f(s,\epsilon)ds, \end{aligned}$$

implies $\lim_{t\to\infty} u(t) < 0$, a contradiction. □

Theorem 5.4 *Let $u_1 = 0$, and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ satisfying $f(t,-u) = -f(t,u)$ and $u_2 \leq u_3$ implies $f(t,u_2) \geq f(t,u_3)$ for each fixed t. If*

$$\lim_{t\to\infty} \int_{\rho}^{t} f(s,K)ds = +\infty, \;\; t \geq \rho$$

for some $\rho > 0$ and $K > 0$, then no nonoscillatory solution of (1.1), (1.2) goes to zero as $t \to \infty$.

Proof Let $u(t)$ be a nonoscillatory solution of (1.1), (1.2). Suppose that $\lim_{t\to\infty} u(t) = 0$. Then for every $\epsilon > 0$ there exists a $T > 0$ such that $|u(t)| < \epsilon$ for $t \geq T$. Now we assume that $u(t)$ is eventually negative. Then there exists a $t_0 > T$ such that $u(t) < 0$ for $t \geq t_0$. Thus, $0 < -u(t) < \epsilon$ for $t \geq t_0$. We have

$$\begin{aligned} u(t) &\geq -u_0 - \int_{0}^{t_0} f(s,u(s))ds - \int_{t_0}^{t} f(s,u(s))ds \\ &\geq -u_0 - Lt_0 + \int_{t_0}^{t} f(s,\epsilon)ds, \end{aligned}$$

where $L = \sup_{t\in[0,t_0]} |f(t,u(t))|$. Hence $\lim_{t\to\infty} u(t) > 0$, a contradiction.

For the other case, let $u(t)$ be eventually positive. Thus, there exists a $t_0 > T$ such that $u(t) > 0$ for $t \geq t_0$. Then $0 < u(t) < \epsilon$ for $t \geq t_0$. We get

$$\begin{aligned} u(t) &\leq u_0 + \int_{0}^{t_0} f(s,u(s))ds + \int_{t_0}^{t} f(s,u(s))ds \\ &= u_0 + \int_{0}^{t_0} f(s,u(s))ds - \int_{t_0}^{t} f(s,-u(s))ds \\ &\leq u_0 + Lt_0 - \int_{t_0}^{t} f(s,\epsilon)ds, \end{aligned}$$

implies $\lim_{t\to\infty} u(t) < 0$, a contradiction. □

Theorem 5.5 *Let $u_1 = 0$, and $f : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$ that satisfy $f(t,-u) = -f(t,u)$ and $uf(t,u) < 0$ if $u \neq 0$. Also, $u_2 \leq u_3$ implies $f(t,u_2) \leq f(t,u_3)$ for each fixed t. If*

$$\lim_{t\to\infty} \int_{\rho}^{t} f(s,K)ds = +\infty, \;\; t \geq \rho$$

for some $\rho > 0$ and $K > 0$. If $u(t)$ is a nonoscillatory solution of (1.1), (1.2) such that $\lim_{t\to\infty} u(t)$ exists, then $\lim_{t\to\infty} u(t) = 0$.

Proof Suppose $\lim_{t\to\infty} u(t) \neq 0$. Furthermore, assume that $u(t)$ is eventually negative. Then there exists a $t_0 > 0$ such that $u(t) < 0$ for $t \geq t_0$. Let $\lim_{t\to\infty} u(t) = r < 0$. For $0 < \epsilon < -r$, there exists a $t_1 > t_0$ such that $u(t) < r + \epsilon < 0$, for $t \geq t_1$. We have

$$\begin{aligned} u(t) &\leq u_0 + \int_{0}^{t_1} f(s,u(s))ds + \int_{t_1}^{t} f(s,u(s))ds \\ &< u_0 + Lt_1 - \int_{t_1}^{t} f(s,-(r+\epsilon))ds, \end{aligned}$$

where $L = \sup_{t\in[0,t_1]} |f(t,u(t))|$. Hence, $\lim_{t\to\infty} u(t)$ not exists, a contradiction.

For the other case, let $r > 0$. For $0 < \epsilon < r$, there exists a $t_1 \geq t_0$ such that $u(t) > r - \epsilon > 0$ for $t \geq t_1$. Thus, for $t \geq t_1$, we get

$$\begin{aligned} u(t) &\geq -u_0 + \int_0^{t_1} f(s, u(s))ds + \int_{t_1}^{t} f(s, u(s))ds \\ &> -u_0 - Lt_1 + \int_{t_1}^{t} f(s, r-\epsilon)ds, \end{aligned}$$

which implies that $\lim_{t\to\infty} u(t)$ does not exist, a contradiction. □

We hope that oscillation theory for Caputo fractional differential equations can also be developed similarly.

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