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Existence results for some neutral functional integrodifferential equations with bounded delay

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Abstract: In this paper, we study a class of neutral functional integrodifferential equations with finite delay in Banach spaces. We are interested in the global existence, uniqueness of mild solutions with values in the Banach space and in its subspace $D(A)$. The results are based on Banach's and Schauder's fixed point theorems and on the technique of equivalent norms. As an application, we consider a diffusion neutral functional integrodifferential equation.

Key words: Mild solution, functional integrodifferential equation, neutral equation, semigroup of bounded linear operators, infinitesimal generator, finite delay

1. Introduction

Neutral differential equations arise in many areas of applied mathematics and for this reason these equations have received much attention in the last few decades. In the following, we provide several examples studied in the literature resulting from various physical systems. In [30–32] the authors studied a model for a continuous circular array of resistively coupled transmission lines with mixed initial boundary conditions, given by the partial functional differential equations

$$\frac{\partial}{\partial t}\left[u(t,\xi)-q(t-r,\xi)\right]=k\Delta\left[u(t,\xi)-q(t-r,\xi)\right]+f(u_t(\cdot,\xi)) \quad \text{for} \quad t\geq 0,$$

where ξ is in the unit circle S^1, $u_t(s,\xi)=u(t+s,\xi)$, $-r\leq s\leq 0$, $t\geq 0$, k is a positive constant, f is a continuous function, and $0<q<1$. The phase space $C\left([-r,0],H^1(S^1)\right)$ is the space of continuous functions provided with the uniform norm topology. Motivated by the above model, Hale in [17–19] considered a more general class of partial neutral functional differential equations of the form

$$\begin{cases} \frac{\partial}{\partial t}Du_t=k\Delta Du_t+f(u_t) \quad \text{for} \quad t\geq 0 \\ u_0=\phi, \end{cases}$$

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with k as a positive constant, $\phi \in \mathcal{C} = C\left([-r, 0], C\left(S^1, \mathbb{R}\right)\right)$,

$$Dv = v(0) - \int_{-r}^{0} v(s)\, d\eta(s) \quad \text{for} \quad v \in \mathcal{C}, \tag{1.1}$$

where the function η is of bounded variation and nonatomic at 0, that is, there exists a continuous nondecreasing function $\mu : [0, r] \to \mathbb{R}_+$ such that $\mu(0) = 0$ and

$$\left| \int_{-\epsilon}^{0} v(s)\, d\eta(s) \right| \leq \mu(\epsilon) \sup_{-\epsilon \leq s \leq 0} |v(s)| \quad \text{for} \quad v \in \mathcal{C}, \ \epsilon \in [0, r],$$

with $|\cdot|$ a norm in $C(S^1, \mathbb{R})$. The Laplace operator $A = k\Delta$ with domain $C^2(S^1, \mathbb{R})$ in an infinitesimal generator of a strongly continuous semigroup of bounded linear operators on $C(S^1, \mathbb{R})$. Adimy and Ezzinbi [1] considered the problem

$$\begin{cases} \frac{d}{dt} Du_t = ADu_t + f(u_t) \quad \text{for} \quad t \geq 0 \\ u_0 = \phi, \end{cases}$$

where $\phi \in \mathcal{C} = C([-r, 0], X)$, A is a nondensely defined linear operator that satisfies the Hille–Yosida condition on the Banach space X, $D : \mathcal{C} \to X$ is the bounded linear operator given by (1.1), and f is a continuous function from of $\mathcal{C}$ into X. The model of rigid heat conduction with finite wave speeds, studied in [8], also can be expressed as an integrodifferential equation of neutral type with infinite delay

$$\frac{d}{dt}\left[u(t) + \int_{-\infty}^{t} K(t-s)u(s)ds\right] = A\left[u(t) + \int_{-\infty}^{t} K(t-s)u(s)ds\right] + f(t, u_t) \quad \text{for} \quad t \geq 0,$$

where the operator A is a generator of a strongly continuous semigroup on a Banach space. Some other models are discussed by Hernandez and Henriquez in [9, 10], namely

$$\begin{cases} \frac{d}{dt}[u(t) - F(t, u_t)] = Au(t) + G(t, u_t) \quad \text{for} \quad t \geq 0 \\ u_0 = \phi, \end{cases}$$

where A generates an analytic semigroup on a Banach space X, $\mathcal{B}$ is the phase space of functions mapping $(-\infty, 0]$ into X, $\phi \in \mathcal{B}$, and G, F are continuous functions from $\mathbb{R}_+ \times \mathcal{B}$ into X. For more on this topic and related applications we refer the reader to [17] and [32], which contain a comprehensive description of those equations.

It has been noted that partial functional differential equations with delay have attracted widespread attention in the literature, see for example [2, 3, 7, 11–14, 20] and the references therein.

Our work is mainly motivated by [29], where the author considered the initial value problem for an abstract integrodifferential equation

$$\begin{cases} u'(t) = Au(t) + \int_0^t g(t-s, u(s))\, ds + f(t) \quad \text{for} \quad t \geq 0 \\ u(0) = x \in X. \end{cases}$$

In this paper, we consider more generally the initial value problem for neutral integrodifferential equations with a finite delay, more exactly the problem

$$\begin{cases} \frac{d}{dt}[u(t) - G(t, u_t)] = A[u(t) - G(t, u_t)] + \int_0^t B(t-s, u_s)ds + F(t, u_t), \\ u_0 = \phi, \end{cases} \tag{1.2}$$

where A is a linear operator with domain $D(A)$ which generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X, $\phi \in C([-r,0],X)$, and B,F,G are nonlinear operators with values in X. Recall that the notation u_t stands for the history function defined by

$$u_t(s) = u(t+s) \quad \text{for} \quad s \in [-r,0].$$

We assume that

$$G(0,\phi) = 0$$

and we are interested in mild solutions of problem (1.2), first in the spaces $C([0,t_1],X)$ $(0 < t_1 < \infty)$ and $C(\mathbb{R}_+,X)$, and next in the spaces $C([0,t_1],Y)$ and $C(\mathbb{R}_+,Y)$, where Y is the space $D(A)$ endowed with the graph norm. The results are based on Banach's and Schauder's fixed point theorems and on the technique of equivalent norms. As an application, we consider a diffusion partial functional integrodifferential equation.

The paper is organized as follows. In Section 2, we recall some preliminary notions and results. In Section 3, we state and prove our main results on the existence and uniqueness, or only on the existence of mild solutions with values in the Banach space X or in its subspace $D(A)$, on a finite time interval $[0,t_1]$ and on $[0,\infty)$. We use the theory of c_0-semigroups, Banach's and Schauder's fixed point theorems and the technique of equivalent norms. Finally, in Section 4, as an application, we consider a diffusive neutral partial functional integrodifferential equation.

2. Preliminaries

In this section, we recall some notions and results that we need in the following. Throughout the paper, X is a Banach space, $A : D(A) \subset X \to X$ is closed linear operator which generates a c_0-semigroup $(T(t))_{t\geq 0}$ on X. For more details, refer to [24] and [28].

Recall that for such a semigroup, there exists $M > 0$ and $\omega \in \mathbb{R}$ such that

$$|T(t)| \leq Me^{\omega t}, \quad t \geq 0, \tag{2.1}$$

where $|T(t)|$ is the norm of the bounded linear operator $T(t)$.

We denote by Y the space $D(A)$ equipped with the graph norm defined by

$$|y|_{D(A)} = |y|_X + |Ay|_X. \tag{2.2}$$

It is well known that $D(A)$ equipped with norm $|\cdot|_{D(A)}$ is a Banach space.

We recall the following result from [21, p. 486].

Lemma 2.1 *Let $k : [0,t_1] \to X$ be continuously differentiable, and q be defined by*

$$q(t) = \int_0^t T(t-s)k(s)ds \quad \textit{for} \quad t \in [0,t_1].$$

Then $q(t) \in D(A)$, for every $t \in [0,t_1]$, q is continuously differentiable, and

$$Aq(t) = q'(t) - k(t) = \int_0^t T(t-s)k'(s)ds + T(t)k(0) - k(t).$$

In regards with the abstract initial value problem

$$\begin{aligned} u'(t) &= Au(t) + h(t), \quad t \in [0, t_1] \\ u(0) &= v, \end{aligned}$$

where $v \in X$ and $h \in C([0, t_1], X)$, we have the following definitions: by the *mild solution* of the problem, one means the function $u \in C([0, t_1], X)$ given by

$$u(t) = T(t)v + \int_0^t T(t-s)h(s)\,ds, \quad t \in [0, t_1]. \tag{2.3}$$

If $v \in Y$ and either $h \in C^1([0, t_1], X)$ or $h(t) \in Y$ for all $t \in [0, t_1]$ and $h(t)$, $Ah(t)$ are continuous in t on $[0, t_1]$, then the function $u(t)$ given by (2.3) is a *strong solution* of the problem, i.e. $u \in C^1([0, t_1], X) \cap C([0, t_1], Y)$, $u(0) = u_0$ and u satisfies pointwise the differential equation.

In what follows, we shall work in the Banach space $C([0, t_1], X)$ endowed with the max norm $|u|_\infty = |u|_{C([0,t_1],X)} = \max_{t\in[0,t_1]} |u(t)|_X$, or as in [5], with a suitable equivalent norm of the form

$$|u|_\theta = \max_{t\in[r,t_1]} \left(|u_t|_{C([-r,0],X)}\, e^{-\theta t} \right),$$

for some $\theta > 0$, where it is assumed that $0 < r < t_1$, and for each $t \in [r, t_1]$, u_t is the function in $C([-r, 0], X)$ defined by

$$u_t(s) = u(t+s), \quad s \in [-r, 0].$$

It is easy to check that $|\cdot|_\theta$ is a norm on $C([0, t_1], X)$ and that

$$e^{-\theta t_1} |u|_\infty \le |u|_\theta \le |u|_\infty,$$

which proves the equivalence of the norms $|\cdot|_\infty$ and $|\cdot|_\theta$ on $C([0, t_1], X)$.

Similarly, we shall consider the Banach space $C([0, t_1], Y)$ endowed with the corresponding max norm $|u|_\infty = |u|_{C([0,t_1],Y)} = \max_{t\in[0,t_1]} |u(t)|_{D(A)}$ or an equivalent one of the form

$$|u|_\theta = \max_{t\in[r,t_1]} \left(|u_t|_{C([-r,0],Y)}\, e^{-\theta t} \right).$$

As it will follow from the next section, the use of an equivalent norm $|\cdot|_\theta$ with a suitable large enough $\theta > 0$ is extremely convenient when dealing with Volterra-type integral operators (see also [5, 6, 26, 27]).

3. Main results

3.1. Mild solutions with values in X

First we consider problem (1.2) in a compact interval $[0, t_1]$. In view of (2.3), by a mild solution of (1.2) on the interval $[0, t_1]$ we mean a function $u \in C([0, t_1], X)$ with $u(0) = \phi(0)$ such that for each $t \in [0, t_1]$,

$$u(t) = T(t)\phi(0) + G(t, \widetilde{u}_t) + \int_0^t T(t-s) \int_0^s B(s-\tau, \widetilde{u}_\tau)\,d\tau ds + \int_0^t T(t-s) F(s, \widetilde{u}_s)\,ds,$$

where

$$\widetilde{u}(t) = \begin{cases} \phi(t), & t \in [-r, 0] \\ u(t), & t \in [0, t_1]. \end{cases}$$

Our first result is about the existence and uniqueness of the mild solution of (1.2) in the set

$$K(t_1, X) = \{u \in C([0, t_1], X) : u(0) = \phi(0)\}.$$

Clearly $K(t_1, X)$ is a closed subset of the space $C([0, t_1], X)$. The result is obtained via Banach's contraction principle with respect to the norm $|\cdot|_\theta$ on $C([0, t_1], X)$, with a suitable large enough number $\theta > 0$.

Here are the hypotheses:

(H) $B, F, G : [0, t_1] \times C([-r, 0], X) \to X$ are continuous and Lipschitzian with respect to the second argument, that is there are constants $L_B, L_F, L_G \geq 0$ with $L_G < 1$ such that

$$|B(t, u) - B(t, v)|_X \leq L_B |u - v|_{C([-r,0],X)}$$

$$|F(t, u) - F(t, v)|_X \leq L_F |u - v|_{C([-r,0],X)}$$

$$|G(t, u) - G(t, v)|_X \leq L_G |u - v|_{C([-r,0],X)}$$

for all $u, v \in C([-r, 0], X)$ and $t \in [0, t_1]$.

Theorem 3.1 *Under the assumption (H), problem (1.2) has a unique mild solution $u \in C([0, t_1], X)$.*

Proof A mild solution of (1.2) is a fixed point in $K(t_1, X)$ of the operator

$$(Pu)(t) = T(t)\phi(0) + G(t, \widetilde{u}_t) + \int_0^t T(t-s) \int_0^s B(s-\tau, \widetilde{u}_\tau) d\tau ds + \int_0^t T(t-s) F(s, \widetilde{u}_s) ds.$$

Since $G(0, u_0) = G(0, \phi) = 0$, one has $P(K(t_1, X)) \subset K(t_1, X)$. Hence, in order to apply Banach's contraction principle, it remains to prove that P is a contraction on $K(t_1, X)$ with respect to a suitable norm $|\cdot|_\theta$ on $C([0, t_1], X)$. To show this, consider two arbitrary functions $u, v \in K(t_1, X)$ and any $t \in [0, t_1]$. Using (2.1) and (H) we have

$$\begin{aligned} |(Pu)(t) - (Pv)(t)|_X &\leq L_G |\widetilde{u}_t - \widetilde{v}_t|_{C([-r,0],X)} + M L_B e^{\omega t_1} \int_0^t \int_0^s |\widetilde{u}_\tau - \widetilde{v}_\tau|_{C([-r,0],X)} d\tau ds \\ &\quad + M L_F e^{\omega t_1} \int_0^t |\widetilde{u}_s - \widetilde{v}_s|_{C([-r,0],X)} ds \\ &= L_G |\widetilde{u}_t - \widetilde{v}_t|_{C([-r,0],X)} e^{-\theta t} e^{\theta t} \\ &\quad + M L_B e^{\omega t_1} \int_0^t \int_0^s |\widetilde{u}_\tau - \widetilde{v}_\tau|_{C([-r,0],X)} e^{-\theta \tau} e^{\theta \tau} d\tau ds \\ &\quad + M L_F e^{\omega t_1} \int_0^t |\widetilde{u}_s - \widetilde{v}_s|_{C([-r,0],X)} e^{-\theta s} e^{\theta s} ds. \end{aligned}$$

It follows that

$$|(Pu)(t)-(Pv)(t)|_X \leq \left(L_G e^{\theta t} + M L_B e^{\omega t_1}\int_0^t\int_0^s e^{\theta\tau}d\tau ds + M L_F e^{\omega t_1}\int_0^t e^{\theta s}ds\right)|u-v|_\theta .$$

Since

$$\int_0^t e^{\theta s}ds = \frac{1}{\theta}\left(e^{\theta t}-1\right) \leq \frac{1}{\theta}e^{\theta t}, \quad \int_0^t\int_0^s e^{\theta\tau}d\tau ds \leq \frac{1}{\theta^2}e^{\theta t},$$

we deduce that

$$|(Pu)(t)-(Pv)(t)|_X \leq \left[L_G + M e^{\omega t_1}\left(\frac{L_B}{\theta^2}+\frac{L_F}{\theta}\right)\right]e^{\theta t}|u-v|_\theta .$$

For $t\in[r,t_1]$ and $s\in[-r,0]$, this inequality yields

$$\begin{aligned}|(Pu)(t+s)-(Pv)(t+s)|_X &\leq \left[L_G + M e^{\omega t_1}\left(\frac{L_B}{\theta^2}+\frac{L_F}{\theta}\right)\right]e^{\theta(t+s)}|u-v|_\theta \\ &\leq \left[L_G + M e^{\omega t_1}\left(\frac{L_B}{\theta^2}+\frac{L_F}{\theta}\right)\right]e^{\theta t}|u-v|_\theta .\end{aligned}$$

Taking the maximum for $s\in[-r,0]$ gives

$$|(Pu)_t-(Pv)_t|_{C([-r,0],X)} \leq \left[L_G + M e^{\omega t_1}\left(\frac{L_B}{\theta^2}+\frac{L_F}{\theta}\right)\right]e^{\theta t}|u-v|_\theta ,$$

for every $t\in[r,t_1]$. Now we divide by $e^{\theta t}$ and take the maximum for $t\in[r,t_1]$ to give

$$|Pu-Pv|_\theta \leq \left[L_G + M e^{\omega t_1}\left(\frac{L_B}{\theta^2}+\frac{L_F}{\theta}\right)\right]|u-v|_\theta .$$

Therefore, in view of the assumption $L_G<1$, for $\theta>0$ large enough that

$$L_G + M e^{\omega t_1}\left(\frac{L_B}{\theta^2}+\frac{L_F}{\theta}\right) < 1,$$

the operator P is a contraction on $K(t_1,X)$, and according to Banach's fixed point theorem it has in $K(t_1,X)$ a unique fixed point. □

Under a weaker condition than (H), where the Lipschitz continuity is replaced by the condition of at most linear growth, and a stronger assumption on G and on the semigroup $(T(t))_{t\geq 0}$, we can still prove the existence of a mild solution in $C([0,t_1],X)$, but not the uniqueness.

($\mathbf{H_w}$) (a) $B,F,G:[0,t_1]\times C([-r,0],X)\to X$ are continuous and have a growth at most linear with respect to the second argument, that is there are constants $L_B,L_F,L_G\geq 0$ with $L_G<1$, and $C_B,C_F,C_G\geq 0$ such that

$$|B(t,u)|_X \leq L_B|u|_{C([-r,0],X)} + C_B$$

$$|F(t,u)|_X \leq L_F|u|_{C([-r,0],X)} + C_F$$

$$|G(t,u)|_X \leq L_G|u|_{C([-r,0],X)} + C_G$$

for all $u\in C([-r,0],X)$ and $t\in[0,t_1]$;

(b) G maps bounded subsets of $[0,t_1]\times C([-r,0],X)$ into relatively compact sets of X; and the semigroup $(T(t))_{t\geq 0}$ is compact.

Theorem 3.2 *Under the assumption (H_w), problem (1.2) has at least one mild solution $u \in C([0,t_1],X)$.*

Proof From ($\mathrm{H_w}$)(b), the operator P from $C([0,t_1],X)$ to its self is completely continuous. By similar estimations as in the proof of Theorem 3.1, using the growth conditions on B,F,G, and the technique based on equivalent norms, we obtain

$$|Pu|_\theta \leq \left[L_G + Me^{\omega t_1}\left(\frac{L_B}{\theta^2}+\frac{L_F}{\theta}\right)\right]|u|_\theta + c,$$

for every $u \in K(t_1,X)$, where

$$c = C_G + Me^{\omega t_1}t_1 C_F + Me^{\omega t_1}t_1^2 C_B.$$

Since $L_G < 1$, we may choose $\theta > 0$ large enough that

$$L_G + Me^{\omega t_1}\left(\frac{L_B}{\theta^2}+\frac{L_F}{\theta}\right) < 1,$$

and then a number $R > 0$ such that

$$\left[L_G + Me^{\omega t_1}\left(\frac{L_B}{\theta^2}+\frac{L_F}{\theta}\right)\right]R + c \leq R.$$

For example we can take

$$R = \left(1 - L_G - Me^{\omega t_1}\left(\frac{L_B}{\theta^2}+\frac{L_F}{\theta}\right)\right)^{-1}c.$$

Thus, the operator P maps the closed bounded convex set $K_R = \{u \in K(t_1,X) : \ |u|_\theta \leq R\}$ into itself. Consequently, from Schauder's fixed point theorem, there exists at least one $u \in K_R$ with $Pu = u$. Clearly u is a mild solution of problem (1.2). □

Next we consider problem (1.2) on the semiline $[0,\infty)$ and we look for a mild solution on $[0,\infty)$, that is a function $u \in C([0,\infty),X)$ which is a mild solution on any finite interval $[0,t_1]$. To guarantee the existence and uniqueness of such a solution we require the following conditions:

(H*) $B,F,G : [0,\infty)\times C([-r,0],X) \to X$ are continuous and Lipschitzian with respect to the second argument, that is there are continuous functions $l_B, l_F, l_G : [0,\infty) \to [0,\infty)$ with $l_G(t) < 1$ for all $t \in [0,\infty)$ such that

$$|B(t,u) - B(t,v)|_X \leq l_B(t)\,|u-v|_{C([-r,0],X)}$$

$$|F(t,u) - F(t,v)|_X \leq l_F(t)\,|u-v|_{C([-r,0],X)}$$

$$|G(t,u) - G(t,v)|_X \leq l_G(t)\,|u-v|_{C([-r,0],X)}$$

for all $u,v \in C([-r,0],X)$ and $t \in [0,\infty)$.

Theorem 3.3 *Under the assumption (H^*), problem (1.2) has a unique mild solution* $u \in C([0,\infty), X)$.

Proof It suffices to apply Theorem 3.1 to any finite interval $[0,n]$, for any integer $n > r$, with

$$L_B = \max_{t\in[0,n]} l_B(t), \quad L_F = \max_{t\in[0,n]} l_F(t), \quad L_G = \max_{t\in[0,n]} l_G(t).$$

Clearly, since $l_G(t) < 1$ for all $t \geq 0$, one has $L_G < 1$. Notice that the Lipschitz constants being dependent on n, the corresponding numbers θ and their associated norms $|\cdot|_\theta$ may differ as well from one interval $[0,n]$ to the other.

Thus, for each such interval $[0,n]$, problem (1.2) has a unique mild solution on $[0,n]$, let it be denoted by u^n. The uniqueness property implies

$$u^{n+1}(t) = u^n(t) \quad \text{for} \quad t \in [0,n].$$

Based on this, the following definition of a function $u \in C([0,\infty), X)$,

$$u(t) = u^n(t), \quad t \in [0,n], \; n \in \mathbb{N}, \; n > r.$$

Obviously, this function u is the unique mild solution in $C([0,\infty), X)$ of (1.2). □

An analogue existence but not uniqueness result on semiline can be established relaxing the Lipschitz continuity conditions and reinforcing the semigroup assumption.

($\mathbf{H_w^*}$) **(a)** $B, F, G : [0,\infty) \times C([-r,0], X) \to X$ are continuous and there are continuous functions $l_B, l_F, l_G : \mathbb{R}_+ \to \mathbb{R}_+$ with $l_G(t) < 1$ for all $t \geq 0$, and $c_B, c_F, c_G : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|B(t,u)|_X \leq l_B(t)\, |u|_{C([-r,0],X)} + c_B(t)$$

$$|F(t,u)|_X \leq l_F(t)\, |u|_{C([-r,0],X)} + c_F(t)$$

$$|G(t,u)|_X \leq l_G(t)\, |u|_{C([-r,0],X)} + c_G(t)$$

for all $u \in C([-r,0], X)$ and $t \in [0,\infty)$;

(b) G maps bounded subsets of $[0,\infty) \times C([-r,0], X)$ into relatively compact sets of X; and the semigroup $(T(t))_{t\geq 0}$ is compact.

Theorem 3.4 *Under the assumption (H_w^*), problem (1.2) has at least one mild solution* $u \in C([0,\infty), X)$.

Proof One solution $u^1 \in C([-r,1], X)$ can be guaranteed using Theorem 3.2, where $t_1 = 1$, $L_B = \max_{t\in[0,1]} l_B(t)$, $C_B = \max_{t\in[0,1]} c_B(t)$ and L_F, L_G, C_F, C_G are defined in a similar way. The solution u^1 is then continued on the interval $[1,2]$ by considering the operator

$$\begin{aligned}(Pu)(t) \quad &= \quad T(t-1)\, u^1(1) + G(t, \widetilde{u}_t) + \int_1^t T(t-s) \int_0^s B(s-\tau, \widetilde{u}_\tau)\, d\tau ds \\ &\quad + \int_1^t T(t-s)\, F(s, \widetilde{u}_s)\, ds,\end{aligned}$$

where

$$\widetilde{u}(t)=\begin{cases} u^1(t), & t\in[-r,1] \\ u(t), & t\in[1,2]. \end{cases}$$

A fixed point of P in $C([1,2],X)$ is obtained via Schauder's fixed point theorem using a similar method. This extends u^1 to a solution $u^2\in C([-r,2],X)$, which in its turn is continued on the interval $[2,3]$, and so on. □

3.2. Mild solutions with values in $D(A)$

In applications, when X is a space of functions of space variables, the appurtenance of an element to the subspace $D(A)$ of X means some regularity with respect to the space variables. Therefore, it is of interest to guarantee that the mild solution takes values in $D(A)$. Recalling that Y, the space $D(A)$ equipped with the norm (2.2), is a Banach space, we are able to adapt the previous results to this aim assuming that $\phi\in C([-r,0],Y)$. Our assumptions are now:

(A1) $F,G:[0,t_1]\times C([-r,0],Y)\to Y$ are continuous and Lipschitzian with respect to the second argument, that is there are constants $L_F,L_G\geq 0$ with $L_G<1$ such that

$$|F(t,u)-F(t,v)|_{D(A)}\leq L_F\,|u-v|_{C([-r,0],Y)}$$

$$|G(t,u)-G(t,v)|_{D(A)}\leq L_G\,|u-v|_{C([-r,0],Y)}$$

for all $u,v\in C([-r,0],Y)$ and $t\in[0,t_1]$.

(A2) $B:[0,t_1]\times C([-r,0],Y)\to X$, the derivative $\frac{\partial B}{\partial t}(t,u)$ exists and is continuous from $[0,t_1]\times Y$ to X, there exist constants $L_B,L_B^1\geq 0$ such that:

$$|B(t,u)-B(t,v)|_X\leq L_B\,|u-v|_{C([-r,0],Y)}$$

and

$$\left|\frac{\partial B}{\partial t}(t,u)-\frac{\partial B}{\partial t}(t,v)\right|_X\leq L_B^1\,|u-v|_{C([-r,0],Y)}$$

for all $u,v\in C([-r,0],Y)$ and $t\in[0,t_1]$.

Theorem 3.5 *Under the assumption (A1) and (A2), problem (1.2) has a unique mild solution $u\in C([0,t_1],Y)$.*

Proof Denote

$$\begin{aligned}(P_1u)(t) &= T(t)\phi(0)+G(t,\widetilde{u}_t)+\int_0^t T(t-s)F(s,\widetilde{u}_s)\,ds,\\ (P_2u)(t) &= \int_0^t T(t-s)\int_0^s B(s-\tau,\widetilde{u}_\tau)\,d\tau ds.\end{aligned}$$

As in the proof of Theorem 3.1, we obtain, this time for the norm $|\cdot|_\theta$ in $C([0,t_1],Y)$, for every $u,v\in K(t_1,Y):=\{u\in C([0,t_1],Y):u(0)=\phi(0)\}$,

$$|P_1u-P_1v|_\theta\leq\left(L_G+Me^{\omega t_1}\frac{L_F}{\theta}\right)|u-v|_\theta\,. \tag{3.1}$$

Also,

$$\left|\left(P_2 u\right)(t)-\left(P_2 v\right)(t)\right|_X \leq M e^{\omega t_1} \frac{L_B}{\theta^2} e^{\theta t}|u-v|_\theta . \tag{3.2}$$

Next, for an estimate of $\left|\left(A P_2 u\right)(t)-\left(A P_2 v\right)(t)\right|_X$, we apply Lemma 2.1 to

$$k(s)=\int_0^s\left[B\left(s-\tau, u_\tau\right)-B\left(s-\tau, v_\tau\right)\right] d \tau,$$

for which

$$k^{\prime}(s)=B\left(0, u_s\right)-B\left(0, v_s\right)+\int_0^s\left[\frac{\partial B}{\partial t}\left(s-\tau, u_\tau\right)-\frac{\partial B}{\partial t}\left(s-\tau, v_\tau\right)\right] d \tau.$$

In this case,

$$q(t)=\left(P_2 u\right)(t)-\left(P_2 v\right)(t)$$

and since

$$A q(t)=\int_0^t T(t-s) k^{\prime}(s) d s-k(t)$$

Then

$$\begin{aligned}
\left|\left(A P_2 u\right)(t)-\left(A P_2 v\right)(t)\right|_X & \leq\left|\int_0^t T(t-s) k^{\prime}(s) d s\right|_X+|k(t)|_X \\
& \leq M e^{\omega t_1} \int_0^t\left|k^{\prime}(s)\right|_X d s+|k(t)|_X .
\end{aligned}$$

Next by similar estimations,

$$|k(t)|_X \leq \frac{L_B}{\theta} e^{\theta t}|u-v|_\theta$$

and

$$\int_0^t\left|k^{\prime}(s)\right|_X d s \leq\left(\frac{L_B}{\theta}+\frac{L_B^1}{\theta^2}\right) e^{\theta t}|u-v|_\theta .$$

Hence,

$$\left|\left(A P_2 u\right)(t)-\left(A P_2 v\right)(t)\right|_X \leq\left[M e^{\omega t_1}\left(\frac{L_B}{\theta}+\frac{L_B^1}{\theta^2}\right)+\frac{L_B}{\theta}\right] e^{\theta t}|u-v|_\theta . \tag{3.3}$$

Now (3.2) and (3.3) yield

$$\left|\left(P_2 u\right)(t)-\left(P_2 v\right)(t)\right|_{D(A)} \leq\left[M e^{\omega t_1}\left(\frac{2 L_B}{\theta}+\frac{L_B^1}{\theta^2}\right)+\frac{L_B}{\theta}\right] e^{\theta t}|u-v|_\theta$$

This gives

$$\left|\left(P_2 u\right)_t-\left(P_2 v\right)_t\right|_{C([-r, 0], Y)} \leq\left[M e^{\omega t_1}\left(\frac{2 L_B}{\theta}+\frac{L_B^1}{\theta^2}\right)+\frac{L_B}{\theta}\right] e^{\theta t}|u-v|_\theta$$

and after dividing by $e^{\theta t}$ and taking the maximum over t, becomes

$$\left|P_2 u-P_2 v\right|_\theta \leq\left[M e^{\omega t_1}\left(\frac{2 L_B}{\theta}+\frac{L_B^1}{\theta^2}\right)+\frac{L_B}{\theta}\right]|u-v|_\theta . \tag{3.4}$$

Finally, (3.1) and (3.4) give

$$|Pu - Pv|_\theta \leq \left[L_G + Me^{\omega t_1}\left(\frac{2L_B}{\theta} + \frac{L_B^1}{\theta^2} + \frac{L_F}{\theta}\right) + \frac{L_B}{\theta}\right]|u - v|_\theta \,.$$

Thus, choosing $\theta > 0$ sufficiently large, the operator P is a contraction on $K(t_1, Y)$ with respect to the norm $|\cdot|_\theta$ on $C([0, t_1], Y)$. □

Next we consider problem (1.2) on the semiline $[0, \infty)$ and we look for a mild solution with values in $D(A)$, that is a function $u \in C([0, \infty), Y)$ which is a mild solution on any finite interval $[0, t_1]$. The hypotheses are:

(A1*) $F, G : [0, \infty) \times C([-r, 0], Y) \to Y$ are continuous and lipschitzian with respect to the second argument, that is there are continuous functions $l_B, l_F, l_G : [0, \infty) \to [0, \infty)$ with $l_G(t) < 1$ for all $t \in [0, \infty)$ such that

$$|F(t, u) - F(t, v)|_{D(A)} \leq l_F(t)\,|u - v|_{C([-r,0],Y)}$$

$$|G(t, u) - G(t, v)|_{D(A)} \leq l_G(t)\,|u - v|_{C([-r,0],Y)}$$

for all $u, v \in C([-r, 0], Y)$ and $t \in [0, \infty)$.

(A2*) $B : [0, \infty) \times C([-r, 0], Y) \to X$, the derivative $\frac{\partial B}{\partial t}(t, u)$ exists and is continuous from $[0, \infty) \times Y$ to X, there exist continuous functions $l_B, l_B^1 : [0, \infty) \to [0, \infty)$ such that:

$$|B(t, u) - B(t, v)|_X \leq l_B(t)\,|u - v|_{C([-r,0],Y)}$$

and

$$\left|\frac{\partial B}{\partial t}(t, u) - \frac{\partial B}{\partial t}(t, v)\right|_X \leq l_B^1(t)\,|u - v|_{C([-r,0],Y)}$$

for all $u, v \in C([-r, 0], Y)$ and $t \in [0, \infty)$.

Theorem 3.6 *Under the assumptions (A1*) and (A2*), problem (1.2) has a unique mild solution* $u \in C([0, \infty), Y)$.

Proof The proof is similar to that of Theorem 3.3. □

4. Application

To illustrate the previous results, we consider the following diffusion model given by a neutral integrodifferential equation with delay

$$\begin{cases} \frac{d}{dt}\left[u(t,\xi) - \int_{-r}^{0}\gamma(s)u(t+s,\xi))ds\right] = \frac{d^2}{d\xi^2}\left[u(t,\xi) - \int_{-r}^{0}\gamma(s)u(t+s,\xi))ds\right] \\ + \int_0^t \beta(t-s, u(s,\xi))\,ds + \int_{-r}^{0}\rho(s)\,u(t+s,\xi))ds, \quad t \geq 0, \quad \xi \in [0,a] \\ u(t,0) = u(t,a) = 0, \quad t \geq 0 \\ u(s,\xi) = \phi(s,\xi), \quad s \in [-r,0], \quad \xi \in [0,a]. \end{cases} \tag{4.1}$$

With the notations

$$\begin{aligned} u(t)(\xi) &= u(t,\xi) \quad \text{for } \ t \geq 0, \ \ \xi \in [0,a], \\ \phi(s)(\xi) &= \phi(s,\xi) \quad \text{for } \ s \in [-r,0], \ \ \xi \in [0,a], \end{aligned}$$

problem (4.1) can be rewritten in the form

$$\begin{cases} \frac{d}{dt}\left[u(t) - \int_{-r}^{0} \gamma(s)\, u_t(s))ds\right] = \frac{d^2}{d\xi^2}\left[u(t) - \int_{-r}^{0} \gamma(s)\, u_t(s))ds\right] \\ + \int_0^t \beta\left(t-s, \frac{d^2}{d\xi^2} u(s)\right) ds + \int_{-r}^{0} \rho(s)\, u_t(s))ds, \quad t \geq 0, \quad \xi \in [0,a] \\ u(t)(0) = u(t)(\pi) = 0, \quad t \geq 0 \\ u_0(s) = \phi(s), \quad s \in [-r,0], \quad \xi \in [0,a]. \end{cases} \tag{4.2}$$

Then problem (4.2) is of type (1.2), if we let $X = L^2(0,a)$, $A : D(A) \subset X \to X$ be defined by

$$D(A) = H^2(0,a) \cap H_0^1(0,a), \qquad Au = u'',$$

and for $t \geq 0$, $v \in C([-r,0],Y)$,

$$G(t,v) = \int_{-r}^{0} \gamma(s)\, v(s)\, ds, \quad B(t,v) = \beta(t,v''), \quad F(t,v) = \int_{-r}^{0} \rho(s)\, v(s))ds.$$

Notice that the operator A defined as above is the infinitesimal generator of c_0-semigroup on X (see [28, p. 64]).

Assume that

(i) $\phi \in C([-r,0],Y)$; $\gamma, \rho \in L^1([-r,0], \mathbb{R}_+)$, $|\gamma|_{L^1(-r,0)} < 1$ and $\int_{-r}^{0} \gamma(s)\, \phi(s)\, ds = 0$.

(ii) $\beta : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and continuously differentiable in its first variable; $\beta(t,x)$ and $\beta_t(t,x)$ are Lipschitz continuous in x uniformly in t.

Notice that here G and F do not depend on t, they are bounded linear operators from $C([-r,0],Y)$ to Y, and

$$\begin{aligned} |G(t,v)|_Y &\leq |\gamma|_{L^1(-r,0)}\, |v|_{C([-r,0],Y)} \\ |F(t,v)|_Y &\leq |\rho|_{L^1(-r,0)}\, |v|_{C([-r,0],Y)}. \end{aligned}$$

Hence the hypothesis (A1*) on F and G is fulfilled with $l_F(t) \equiv |\rho|_{L^1(-r,0)}$ and $l_G(t) \equiv |\gamma|_{L^1(-r,0)}$. As concerns hypothesis (A2*), we refer to paper [29].

Consequently, the conditions of Theorem 3.6 are fulfilled and we have the following result.

Theorem 4.1 *Under the assumptions (i) and (ii), problem (4.1) has a unique mild solution* $u \in C([0,\infty),Y)$.

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