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Some results related to the Berezin number inequalities

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Abstract: In this paper, we prove reverse inequalities for the so-called Berezin number of some operators. Also, by using the classical Jensen and Young inequalities, we obtain upper bounds for Berezin number of $A^{\alpha}XB^{\alpha}$ and $A^{\alpha}XB^{1-\alpha}$ for the case when $0 \leq \alpha \leq 1$.

Key words: Berezin number, positive operator, reproducing kernel Hilbert space, Berezin symbol, McCarthy inequality

1. Introduction

Denote by $\mathcal{F}(\Omega)$ the set of all complex valued functions on some set Ω. A reproducing kernel Hilbert space (RKHS for short) on the set Ω is a Hilbert space $\mathcal{H} \subset \mathcal{F}(\Omega)$ with a function $k_\lambda : \Omega \times \Omega \to \mathcal{H}$, which is called the reproducing kernel enjoying the reproducing property $k_\lambda := k(., \lambda) \in \mathcal{H}$ for all $\lambda \in \Omega$, and

$$f(\lambda) = \langle f, k_\lambda \rangle_{\mathcal{H}}$$

holds for all $\lambda \in \Omega$ and all $f \in \mathcal{H}$ (see [2, 28]).

Let $\widehat{k}_\lambda = \frac{k_\lambda}{\|k_\lambda\|}$ be the normalized reproducing kernel of the space $\mathcal{H}$. For any bounded linear operator A on $\mathcal{H}$, the Berezin symbol of A is the function $\widetilde{A}$ defined by (see [5])

$$\widetilde{A}(\lambda) := \left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle_{\mathcal{H}} \ (\lambda \in \Omega).$$

The Berezin symbol is a very useful tool in studying operators on the RKHS, including Hardy, Bergman, and Fock spaces. For example, boundedness, invertibility, compactness, and positivity of some operators are characterized or related to their Berezin symbols (see [8, 10, 22, 26, 31]).

Following Coburn [9], note that since the Berezin map $A \overset{B}{\to} \widetilde{A}$ is linear and in most familar RKHSs it is one-to-one, it "encodes" operator-theoretic information into function theory in a striking but somewhat impenetrable way. In fact, since $\widehat{k}_\lambda \to 0$ weakly as $\lambda \to \partial\Omega$ (of course, if the space $\mathcal{H}(\Omega)$ is standard in the sense of Nordgren and Rosenthal [27]), it is clear that B maps compact operators on these spaces into functions that vanish at the boundary $\partial\Omega$. Because of these properties, the mapping B has found useful applications in dealing with operators "of function-theoretic significance" such as Toeplitz and Hankel operators on the Hardy,

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Bergman, and Fock spaces (for more information, see, for instance, Coburn [9], Berger and Coburn [6], and Engliš [15, 16]).

Recall that the Berezin set and the Berezin number for an operator $A \in \mathcal{B}(\mathcal{H}(\Omega))$ were introduced by the second author in [22, 23] as follows:

$$Ber(A) := Range(\widetilde{A}) = \left\{ \widetilde{A}(\lambda) : \lambda \in \Omega \right\} \quad \text{(Berezin set)},$$

$$ber(A) := \sup \left\{ \left| \widetilde{A}(\lambda) \right| : \lambda \in \Omega \right\} \quad \text{(Berezin number)}.$$

Clearly, $Ber(A) \subset W(A) := \{\langle Ax, x\rangle : \|x\|_{\mathcal{H}} = 1\}$ (numerical range) and $ber(A) \leq w(A) := \sup\{|\langle Ax, x\rangle| : \|x\|_{\mathcal{H}} = 1\}$ (numerical radius). More information about numerical range and numerical radius can be found in [1, 4, 14, 19, 21, 24, 25]. Recently, some results about the Berezin number were obtained in [3, 18, 20, 29, 30].

In the present paper, by using some ideas from [12, 13], we will prove reverse inequalities for the so-called Berezin number of some operators acting in the reproducing kernel Hilbert space. Also, we obtain upper bounds for the Berezin number of $A^{\alpha}XB^{\alpha}$ and $A^{\alpha}XB^{1-\alpha}$ for the case when $0 \leq \alpha \leq 1$.

2. Relations between numerical radius and Berezin number

Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS of complex-valued functions on a set Ω. A subset $\mathcal{M}(\Omega)$ in $\mathcal{H}(\Omega)$ is called the multiplier for the space $\mathcal{M}(\Omega)$ if $\mathcal{M}(\Omega)\mathcal{H}(\Omega) \subset \mathcal{H}(\Omega)$, i.e. $fg \in \mathcal{H}(\Omega)$ for any $f \in \mathcal{M}(\Omega)$ and $g \in \mathcal{H}(\Omega)$.

The following two lemmas are well known (and very easy to verify).

Lemma 1 *If f is a multiplier of $\mathcal{H}(\Omega)$, then $\widetilde{M_f}(\lambda) = f(\lambda)$ for all $\lambda \in \Omega$.*

Proof Indeed, if f is a multiplier, then we have

$$\begin{aligned} \widetilde{M_f}(\lambda) &= \left\langle M_f \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle = \left\langle f \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \\ &= \frac{1}{\|k_\lambda\|_{\mathcal{H}}^2} f(\lambda) k_\lambda(\lambda) = f(\lambda) \end{aligned}$$

for all $\lambda \in \Omega$, as desired. □

Lemma 2 *If f is a multiplier of $\mathcal{H}(\Omega)$, then f is bounded.*

Proof In fact, since $f \in \mathcal{H}(\Omega)$ and $fg \in \mathcal{H}(\Omega)$ for all $g \in \mathcal{H}(\Omega)$, by using Lemma 1, we have:

$$|f(\lambda)| = \left| \widetilde{M_f}(\lambda) \right| = \left| \left\langle M_f \widehat{k}_\lambda, \widehat{k}_\lambda \right\rangle \right| \leq \|M_f\|.$$

Since M_f is a closed operator defined in hull space $\mathcal{H}(\Omega)$, by the closed graph theorem M_f is bounded (see, for instance, Aronzajn [2]). The last inequality shows that f is bounded. □

We set by $\mathcal{H}_1$ the unit sphere of $\mathcal{H}$, $\mathcal{H}_1 = \{f \in \mathcal{H} : \|f\|_{\mathcal{H}} = 1\}$, and also we set $\mathcal{J} := \{V \in \mathcal{B}(\mathcal{H}) : V \text{ is isometry}\}$, where $\mathcal{B}(\mathcal{H})$ is the Banach algebra of all bounded linear operators on $\mathcal{H}$.

In this short section, we prove the relations between the numerical radius and Berezin number of reproducing kernel Hilbert space operators, which improves some results in [17, 23].

Theorem 1 *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS of complex-valued functions on Ω with reproducing kernel k_λ such that it has a dense multiplier $\mathcal{M}(\Omega)$ and $k_0 = \mathbf{1}$.*
Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator (i.e. $A \in \mathcal{B}(\mathcal{H})$). Then

$$\sup_{V \in \mathcal{J}} ber\left(V^* A V\right) \leq w(A) \leq \|\mathbf{1}\|_{\mathcal{H}}^2 \sup_{f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1} ber\left(M_f^* A M_f\right).$$

Proof By assumption $\mathcal{M}(\Omega)$ is dense in $\mathcal{H}$. Then it is standard to show that

$$\sup\{|\langle Af, f\rangle| : f \in \mathcal{H}_1\} = \sup\{|\langle Af, f\rangle| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1\}.$$

According to Lemma 2, $\mathcal{M}(\Omega)$ consists of bounded functions of the space $\mathcal{H}(\Omega)$. Then we have:

$$\begin{aligned}
w(A) &= \sup\{|\langle Af, f\rangle| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1\} \\
&= \sup\{|\langle AM_f\mathbf{1}, M_f\mathbf{1}\rangle| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1\} \\
&= \sup\left\{\left|\left\langle M_f^* A M_f \mathbf{1}, \mathbf{1}\right\rangle\right| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1\right\} \\
&= \|k_0\|_{\mathcal{H}}^2 \sup\left\{\left|\left\langle M_f^* A M_f \frac{k_0}{\|k_0\|}, \frac{k_0}{\|k_0\|}\right\rangle\right| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1\right\} \\
&= \|\mathbf{1}\|_{\mathcal{H}}^2 \sup\left\{\left|\left\langle M_f^* A M_f \widehat{k}_0, \widehat{k}_0\right\rangle\right| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1\right\} \\
&= \|\mathbf{1}\|_{\mathcal{H}}^2 \sup\left\{\left|\widetilde{M_f^* A M_f}(0)\right| : f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1\right\} \\
&\leq \|\mathbf{1}\|_{\mathcal{H}}^2 \sup_{f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1} \sup\left\{\left|\widetilde{M_f^* A M_f}(\lambda)\right| : \lambda \in \Omega\right\} \\
&= \|\mathbf{1}\|_{\mathcal{H}}^2 \sup_{f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1} ber\left(M_f^* A M_f\right),
\end{aligned}$$

and hence

$$w(A) \leq \|\mathbf{1}\|_{\mathcal{H}}^2 \sup_{f \in \mathcal{M}(\Omega) \cap \mathcal{H}_1} ber\left(M_f^* A M_f\right). \tag{2.1}$$

On the other hand, for any $V \in \mathcal{J}$ and $g \in \mathcal{H}$, we have:

$$\begin{aligned}
ber\left(V^* A V\right) &= \sup_{\lambda \in \Omega}\left|\widetilde{V^* A V}(\lambda)\right| = \sup_{\lambda \in \Omega}\left|\left\langle V^* A V \widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right| \\
&= \sup_{\lambda \in \Omega}\left|\left\langle AV\widehat{k}_\lambda, V\widehat{k}_\lambda\right\rangle\right|.
\end{aligned}$$

Since V is isometry, $V\widehat{k}_\lambda \in \mathcal{H}_1$ for all $\lambda \in \Omega$. Then

$$ber\left(V^* A V\right) \leq \sup_{h \in \mathcal{H}_1} |\langle Ah, h\rangle| = w(A).$$

Thus,

$$\sup_{V \in \mathcal{J}} ber\left(V^* A V\right) \leq w(A). \tag{2.2}$$

It remains only to combine inequalities (2.1) and (2.2) to get the required result. The theorem is proven. □

3. Reverse inequalities for the Berezin numbers of operators

Our next results in this section are mainly motivated with Dragomir's survey paper [13], where he proved relations only between the norm and numerical radius of operators. Here we investigate similar questions also for Berezin numbers of operators A and $|A|^2 := A^*A$; here, $|A| := (A^*A)^{1/2}$ is a so-called module of operator A.

Theorem 2 *Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a RKHS on Ω and $A \in \mathcal{B}(\mathcal{H})$ be an operator. If $\mu \in \mathbb{C}\backslash\{0\}$ and $r > 0$ are such that*

$$\sqrt{ber\left(|A-\mu|^2\right)} \leq r, \tag{3.1}$$

then

$$(0 \leq) \sqrt{ber\left(|A|^2\right)} - ber\,(A) \leq \frac{1}{2}\frac{r}{|\mu|}. \tag{3.2}$$

Proof For any $\lambda \in \Omega$, we have from (3.1) that

$$\begin{aligned}\left\|A\widehat{k}_\lambda - \mu\widehat{k}_\lambda\right\| &= \left\|(A-\mu)\,\widehat{k}_\lambda\right\| = \left\langle (A-\mu)\,\widehat{k}_\lambda, (A-\mu)\,\widehat{k}_\lambda\right\rangle^{1/2} \\ &= \left\langle (A-\mu)^*\,(A-\mu)\,\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{1/2} \leq ber\left((A-\mu)^*\,(A-\mu)\right)^{1/2} \\ &= ber\left(|A-\mu|^2\right)^{1/2} \leq r,\end{aligned}$$

and hence

$$\left\|A\widehat{k}_\lambda - \mu\widehat{k}_\lambda\right\|^2 \leq r^2,$$

or equivalently,

$$\left\|A\widehat{k}_\lambda\right\|^2 + |\mu|^2 - 2\operatorname{Re}\left[\overline{\mu}\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right] \leq r^2.$$

Hence,

$$\left\|A\widehat{k}_\lambda\right\|^2 + |\mu|^2 \leq 2\,|\mu|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle + r^2. \tag{3.3}$$

Taking the supremum over $\lambda \in \Omega$ in (3.3), we have the following inequality:

$$ber\left(|A|^2\right) + |\mu|^2 \leq 2\,|\mu|\,ber\,(A) + r^2. \tag{3.4}$$

By arithmetic-geometric mean inequality,

$$ber\left(|A|^2\right) + |\mu|^2 \geq 2\,|\mu|\sqrt{ber\left(|A|^2\right)}, \tag{3.5}$$

and hence by (3.4) and (3.5) we deduce the desired inequality (3.2), because it is elementary to see that actually

$\sqrt{ber\left(|A|^2\right)} - ber\left(A\right) \geq 0$. Indeed,

$$|A(\lambda)| \leq \left\|A\widehat{k}_\lambda\right\| = \left\langle A^*A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{1/2} = \left(\widetilde{|A|^2}(\lambda)\right)^{\frac{1}{2}}$$
$$\leq ber\left(|A|^2\right)^{1/2}$$

for all $\lambda \in \Omega$, and hence $ber\left(A\right) \leq \sqrt{ber\left(|A|^2\right)}$. The theorem is proved. □

Corollary 1 *Let $A \in \mathcal{B}(\mathcal{H})$ be an operator and $\varphi, \psi \in \mathbb{C}$ with $\psi \neq -\varphi,\ \varphi$. If*

$$\operatorname{Re}\left((A-\varphi)^*(\psi - A)\right) \geq 0, \tag{3.6}$$

then

$$\sqrt{ber\left(|A|^2\right)} - ber\left(A\right) \leq \frac{1}{4}\frac{|\psi-\varphi|}{|\psi+\varphi|}. \tag{3.7}$$

Proof Utilizing the fact that in any Hilbert space the following two statements are equivalent,

(i) $\operatorname{Re}\langle y-x, x-z\rangle \geq 0,\ x, z, y \in H$;

(ii) $\left\|x - \frac{z+y}{2}\right\| \leq \frac{1}{2}\|y-z\|$,

we conclude that (3.6) is equivalent to

$$\left\|A\widehat{k}_\lambda - \frac{\psi+\varphi}{2}\widehat{k}_\lambda\right\| \leq \frac{1}{2}\|\psi - \varphi\|$$

for any $\lambda \in \Omega$, which in its turn is equivalent with the following inequality:

$$ber\left(\left|A - \frac{\psi+\varphi}{2}\right|^2\right)^{\frac{1}{2}} \leq \frac{1}{2}|\psi-\varphi|.$$

Now, applying Theorem 2 for $\mu = \dfrac{\psi+\varphi}{2}$ and $r = \dfrac{1}{2}|\psi-\varphi|$, we deduce the desired result (3.7). □

Corollary 2 *Assume that A, μ, r are as in Theorem 2. If, in addition, there exists $\rho \geq 0$ such that*

$$||\mu| - ber\left(A\right)| \geq \rho, \tag{3.8}$$

then

$$ber\left(|A|^2\right) - ber\left(A\right)^2 \leq r^2 - \rho^2. \tag{3.9}$$

Proof From (3.4) of Theorem 2, we have

$$ber\left(|A|^2\right) - ber\left(A\right)^2 \leq r^2 - ber\left(A\right)^2 + 2ber\left(A\right)|\mu| - |\mu|^2$$
$$= r^2 - \left(|\mu| - ber\left(A\right)\right)^2.$$

On utilizing (3.4) and (3.8), we deduce inequality (3.9), as desired. □

Remark 1 *In particular, if* $\sqrt{ber\left(|A-\mu|^2\right)} \leq r$ *and* $|\mu| = ber(A)$, $\mu \in \mathbb{C}$, *then* $ber\left(|A|^2\right) - ber(A)^2 \leq r^2$.

Theorem 3 *Let* $A \in \mathcal{B}(\mathcal{H})$ *be a nonzero operator and* $\mu \in \mathbb{C}\setminus\{0\}$, $r > 0$ *with* $|\mu| > r$. *If* $\sqrt{ber\left(|A-\mu|^2\right)} \leq r$, *then*

$$\sqrt{1 - \frac{r^2}{|\mu|^2}} \leq \frac{ber(A)}{\sqrt{ber\left(|A|^2\right)}}. \tag{3.10}$$

Proof From (3.4), we have

$$ber\left(|A|^2\right) + |\mu|^2 - r^2 \leq 2|\mu|\, ber(A),$$

which implies, on dividing with $\sqrt{|\mu|^2 - r^2} > 0$, that

$$\frac{ber\left(|A|^2\right)}{\sqrt{|\mu|^2 - r^2}} + \sqrt{|\mu|^2 - r^2} \leq \frac{2|\mu|\, ber(A)}{\sqrt{|\mu|^2 - r^2}}. \tag{3.11}$$

By the arithmetic-geometric mean inequality,

$$2\sqrt{ber\left(|A|^2\right)} \leq \frac{ber\left(|A|^2\right)}{\sqrt{|\mu|^2 - r^2}} + \sqrt{|\mu|^2 - r^2},$$

and by (3.11) we get

$$\sqrt{ber\left(|A|^2\right)} \leq \frac{ber(A)\,|\mu|}{\sqrt{|\mu|^2 - r^2}}.$$

This is equivalent to (3.10), which proves the theorem. □

Corollary 3 *Let* $\varphi, \psi \in \mathbb{C}$ *with* $\operatorname{Re}(\psi\overline{\varphi}) > 0$. *If* $A \in \mathcal{B}(\mathcal{H})$ *is an operator such that either* (3.6) *or*

$$(A^* - \overline{\varphi})(\psi - A) \geq 0 \tag{3.12}$$

holds true, then

$$\frac{2\sqrt{\operatorname{Re}(\psi\overline{\varphi})}}{|\psi + \varphi|} \leq \frac{ber(A)}{\sqrt{ber\left(|A|^2\right)}}$$

and

$$ber\left(|A|^2\right) - ber(A)^2 \leq \left|\frac{\psi - \varphi}{\psi + \varphi}\right| ber\left(|A|^2\right).$$

Proof If we put $\mu = \frac{\psi+\varphi}{2}$ and $r = \frac{1}{2}\left|\psi-\varphi\right|$, then $\left|\mu\right|^2 - r^2 = \left|\frac{\psi+\varphi}{2}\right|^2 - \left|\frac{\psi-\varphi}{2}\right|^2 = \operatorname{Re}\left(\psi\overline{\varphi}\right) > 0$. It is easy by applying Theorem 2 to see that under condition (3.12) we have

$$\sqrt{ber\left(\left|A\right|^2\right)} - ber\left(A\right) \leq \frac{1}{4}\frac{\left|\psi-\varphi\right|}{\left|\psi+\varphi\right|}.$$

By considering all these and applying Theorem 3, we obtain the desired results. □

The next result maybe of interest as well.

Theorem 4 *Let* $A : \mathcal{H}\left(\Omega\right) \rightarrow \mathcal{H}\left(\Omega\right)$ *be a nonzero bounded linear operator and* $\mu \in \mathbb{C}\backslash\left\{0\right\}$, $r > 0$ *with* $\left|\mu\right| > r$. *If*

$$\sqrt{ber\left(\left|A-\mu\right|^2\right)} \leq r,$$

then

$$ber\left(\left|A\right|^2\right) - ber\left(A\right)^2 \leq \frac{2r^2}{\left|\mu\right| + \sqrt{\left|\mu\right|^2 - r^2}} ber\left(A\right). \tag{3.13}$$

Proof From the proof of Theorem 2, we have

$$\left\|A\widehat{k}_\lambda\right\|^2 + \left|\mu\right|^2 \leq 2\operatorname{Re}\left[\overline{\mu}\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right] + r^2 \tag{3.14}$$

for all $\lambda \in \Omega$.

Now, after dividing (3.14) by $\left|\mu\right|\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|$ (which, by (3.14), is positive), we obtain

$$\frac{\left\|A\widehat{k}_\lambda\right\|^2}{\left|\mu\right|\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|} \leq \frac{2\operatorname{Re}\left[\overline{\mu}\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right]}{\left|\mu\right|\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|} + \frac{r^2}{\left|\mu\right|\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|} - \frac{\left|\mu\right|}{\left|\left\langle A\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|} \tag{3.15}$$

for all $\lambda \in \Omega$. Hence,

$$\frac{\widetilde{\left|A\right|^2}\left(\lambda\right)}{\left|\mu\right|\left|\widetilde{A}\left(\lambda\right)\right|} \leq \frac{2\operatorname{Re}\left[\overline{\mu}\widetilde{A}\left(\lambda\right)\right]}{\left|\mu\right|\left|\widetilde{A}\left(\lambda\right)\right|} + \frac{r^2}{\left|\mu\right|\left|\widetilde{A}\left(\lambda\right)\right|} - \frac{\left|\mu\right|}{\left|\widetilde{A}\left(\lambda\right)\right|} \tag{3.16}$$

for all $\lambda \in \Omega$. If we subtract in (3.16) the same quantity $\frac{\left|\widetilde{A}(\lambda)\right|}{|\mu|}$ from both sides, then we have

$$\frac{\widetilde{|A|^2}(\lambda)}{|\mu|\left|\widetilde{A}(\lambda)\right|} - \frac{\left|\widetilde{A}(\lambda)\right|}{|\mu|} \le \frac{2\operatorname{Re}\left[\overline{\mu}\widetilde{A}(\lambda)\right]}{|\mu|\left|\widetilde{A}(\lambda)\right|} + \frac{r^2}{|\mu|\left|\widetilde{A}(\lambda)\right|} - \frac{\left|\widetilde{A}(\lambda)\right|}{|\mu|} - \frac{|\mu|}{\left|\widetilde{A}(\lambda)\right|} \tag{3.17}$$

$$= \frac{2\operatorname{Re}\left[\overline{\mu}\widetilde{A}(\lambda)\right]}{|\mu|\left|\widetilde{A}(\lambda)\right|} - \frac{|\mu|^2 - r^2}{|\mu|\left|\widetilde{A}(\lambda)\right|} - \frac{\left|\widetilde{A}(\lambda)\right|}{|\mu|}$$

$$= \frac{2\operatorname{Re}\left[\overline{\mu}\widetilde{A}(\lambda)\right]}{|\mu|\left|\widetilde{A}(\lambda)\right|} - \left(\frac{\sqrt{|\mu|^2 - r^2}}{\sqrt{|\mu|\left|\widetilde{A}(\lambda)\right|}} - \frac{\sqrt{\left|\widetilde{A}(\lambda)\right|}}{\sqrt{|\mu|}}\right)^2 \frac{\left|\widetilde{A}(\lambda)\right|}{|\mu|} - 2\frac{\sqrt{|\mu|^2 - r^2}}{|\mu|}.$$

Since $\operatorname{Re}\left[\overline{\mu}\widetilde{A}(\lambda)\right] \le |\mu|\left|\widetilde{A}(\lambda)\right|$ and

$$\left(\frac{\sqrt{|\mu|^2 - r^2}}{\sqrt{|\mu|\left|\widetilde{A}(\lambda)\right|}} - \frac{\sqrt{\left|\widetilde{A}(\lambda)\right|}}{\sqrt{|\mu|}}\right)^2 \ge 0,$$

by (3.17) we obtain

$$\frac{\widetilde{|A|^2}(\lambda)}{|\mu|\left|\widetilde{A}(\lambda)\right|} - \frac{\left|\widetilde{A}(\lambda)\right|}{|\mu|} \le \frac{2\left(|\mu| - \sqrt{|\mu|^2 - r^2}\right)}{|\mu|},$$

which implies the inequality

$$\widetilde{|A|^2}(\lambda) \le \left|\widetilde{A}(\lambda)\right|^2 + 2\left|\widetilde{A}(\lambda)\right|\left(|\mu| - \sqrt{|\mu|^2 - r^2}\right)$$
$$\le ber(A)^2 + 2ber(A)\left(|\mu| - \sqrt{|\mu|^2 - r^2}\right)$$

for all $\lambda \in \Omega$, which implies that

$$ber\left(|A|^2\right) - ber(A)^2 \le 2ber(A)\frac{\left(|\mu| - \sqrt{|\mu|^2 - r^2}\right)\left(|\mu| + \sqrt{|\mu|^2 - r^2}\right)}{|\mu| + \sqrt{|\mu|^2 - r^2}}$$
$$= \frac{2r^2 ber(A)}{|\mu| + \sqrt{|\mu|^2 - r^2}},$$

as desired. The proof is complete. □

Corollary 4 *Let $\varphi, \psi \in \mathbb{C}$ with $\operatorname{Re}(\psi\overline{\varphi}) > 0$. If $A \in \mathcal{B}(\mathcal{H})$ is an operator such that either (3.6) or (3.12) holds true, then*

$$ber\left(|A|^2\right) - ber(A)^2 \leq \left[|\psi+\varphi| - 2\sqrt{\operatorname{Re}(\psi\overline{\varphi})}\right] ber(A).$$

Remark 2 *If $M \geq m > 0$ are such that either $\operatorname{Re}\left((A^* - m)(M - A)\right)^{\widetilde{}}(\lambda) \geq 0$ for all $\lambda \in \Omega$, or simply $(A^* - m)(M - A)$ is self-adjoint and $(A^* - m)(M - A) \geq 0$, then it follows from the first claim of Corollary 3 that*

$$\frac{\sqrt{ber\left(|A|^2\right)}}{ber(A)} \leq \frac{M+m}{2\sqrt{mM}},$$

which is equivalent to

$$\sqrt{ber\left(|A|^2\right)} - ber(A) \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} ber(A),$$

while we have from (3.13) that

$$ber\left(|A|^2\right) - ber(A)^2 \leq \left(\sqrt{M} - \sqrt{m}\right)^2 ber(A).$$

Also, (3.7) becomes

$$\sqrt{ber\left(|A|^2\right)} - ber(A) \leq \frac{1}{4}\frac{(M-m)^2}{M+m}.$$

Our next result is based on the following refinement of Schwartz's inequality obtained by Dragomir [11, Theorem 2] :

$$\|a\| \|b\| \geq |\langle a, b\rangle - \langle a, e\rangle \langle e, b\rangle| + |\langle a, e\rangle \langle e, b\rangle| \geq |\langle a, b\rangle|, \tag{3.18}$$

provided $a, b, e \in H$ and $\|e\| = 1$. Since

$$|\langle a, b\rangle - \langle a, e\rangle \langle e, b\rangle| \geq |\langle a, e\rangle \langle e, b\rangle| - |\langle a, b\rangle|,$$

by the first inequality in (3.18) we deduce that

$$\frac{1}{2}\left(\|a\| \|b\| + |\langle a, b\rangle|\right) \geq |\langle a, e\rangle \langle e, b\rangle|. \tag{3.19}$$

Inequality (3.19) was proved by a different method earlier by Buzano [7]. Now we are ready to state our result.

Theorem 5 *Let $A \in \mathcal{B}(\mathcal{H}(\Omega))$. Then*

$$ber(A)^2 \leq \frac{1}{2}\left[ber\left(A^2\right) + \sqrt{ber\left(|A|^2\right) ber\left(|A^*|^2\right)}\right].$$

Proof Let us choose in (3.19) $e=\widehat{k}_\lambda$, $a=A\widehat{k}_\lambda$, and $b=A^*\widehat{k}_\lambda$ to get

$$\frac{1}{2}\left(\left\|A\widehat{k}_\lambda\right\|\left\|A^*\widehat{k}_\lambda\right\|+\left|\left\langle A^2\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|\right)\geq\left|\left\langle A\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^2$$

for all $\lambda\in\Omega$. From this we have

$$\frac{1}{2}\left[\sqrt{ber\left(|A|^2\right)}\sqrt{ber\left(|A^*|^2\right)}+ber\left(A^2\right)\right]\geq\left|\widetilde{A}\left(\lambda\right)\right|^2$$

or all $\lambda\in\Omega$, which obviously implies the desired result. □

4. Other Berezin number inequalities for product of operators

The main goal of this section is to find upper bounds for the Berezin number of $A^\alpha XB^\alpha$ and $A^\alpha XB^{1-\alpha}$ for the case when $0\leq\alpha\leq 1$.

The following lemma is a consequence of the classical Jensen and Young inequalities [11]. By using this lemma, we prove the next results.

Lemma 3 *For $a,b\geq 0$, $0\leq\alpha\leq 1$, and $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have:*

(a) $a^\alpha b^{1-\alpha}\leq\alpha a+(1-\alpha)b\leq\left[\alpha a^r+(1-\alpha)b^r\right]^{\frac{1}{r}}$ *for* $r\geq 1$;

(b) $ab\leq\frac{a^p}{p}+\frac{b^q}{q}\leq\left(\frac{a^{pr}}{p}+\frac{b^{qr}}{q}\right)^{\frac{1}{r}}$ *for* $r\geq 1$.

Theorem 6 *Let $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $pr,qr\geq 2$. Let $A,B\in\mathcal{B}\left(\mathcal{H}\left(\Omega\right)\right)$ be positive operators. Then*

$$ber^r\left(A^\alpha XB^\alpha\right)\leq\|X\|^r\left(\frac{1}{p}ber\left(A^{pr}\right)+\frac{1}{q}ber\left(B^{qr}\right)\right)^\alpha$$

for all $0\leq\alpha\leq 1$.

Proof By using the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\left|\left\langle A^\alpha XB^\alpha\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle\right|^r&=\left|\left\langle XB^\alpha\widehat{k}_\lambda,A^\alpha\widehat{k}_\lambda\right\rangle\right|^r\\&\leq\left\|XB^\alpha\widehat{k}_\lambda\right\|^r\left\|A^\alpha\widehat{k}_\lambda\right\|^r\\&\leq\|X\|^r\left\langle A^{2\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^{r/2}\left\langle B^{2\alpha}\widehat{k}_\lambda,\widehat{k}_\lambda\right\rangle^{r/2},\end{aligned}$$

and so

$$\left|\widetilde{A^\alpha XB^\alpha}(\lambda)\right|^r\leq\|X\|^r\left[\widetilde{A^{2\alpha}}\left(\lambda\right)\right]^{r/2}\left[\widetilde{B^{2\alpha}}\left(\lambda\right)\right]^{r/2}$$

for all $\lambda \in \Omega$. From the McCarthy inequality and Lemma 3, we obtain

$$\begin{aligned} & \|X\|^r \left[\widetilde{A^{2\alpha}}(\lambda)\right]^{r/2} \left[\widetilde{B^{2\alpha}}(\lambda)\right]^{r/2} \\ & \leq \|X\|^r \left(\frac{1}{p}\left(\widetilde{A^{2\alpha}}(\lambda)\right)^{pr/2} + \frac{1}{q}\left[\widetilde{B^{2\alpha}}(\lambda)\right]^{qr/2}\right) \\ & \leq \|X\|^r \left(\frac{1}{p}\left(\widetilde{A^{pr}}(\lambda)\right)^{\alpha} + \frac{1}{q}\left(\widetilde{B^{qr}}(\lambda)\right)^{\alpha}\right) \end{aligned} \tag{4.1}$$

for all $\lambda \in \Omega$. From the concavity of t^α, we have

$$\|X\|^r \left(\frac{1}{p}\left(\widetilde{A^{pr}}(\lambda)\right)^{\alpha} + \frac{1}{q}\left(\widetilde{B^{qr}}(\lambda)\right)^{\alpha}\right) \leq \|X\|^r \left(\frac{1}{p}\widetilde{A^{pr}}(\lambda) + \frac{1}{q}\widetilde{B^{qr}}(\lambda)\right)^{\alpha} \tag{4.2}$$

for all $\lambda \in \Omega$.

Combining (4.1) and (4.2), we get

$$\left|\widetilde{A^\alpha X B^\alpha}(\lambda)\right|^r \leq \|X\|^r \left(\frac{1}{p}\widetilde{A^{pr}}(\lambda) + \frac{1}{q}\widetilde{B^{qr}}(\lambda)\right)^{\alpha}$$

for all $\lambda \in \Omega$. Taking the supremum in the last inequality, we get

$$ber^r\left(A^\alpha X B^\alpha\right) \leq \|X\|^r \left(\frac{1}{p} ber\left(A^{pr}\right) + \frac{1}{q} ber\left(B^{qr}\right)\right)^{\alpha}$$

for all positive operators $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$. This proves the theorem. □

Theorem 7 *Let $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$ be positive operators. Then*

$$ber^r\left(A^\alpha X B^{1-\alpha}\right) \leq \|X\|^r ber\left(\alpha A^r + (1-\alpha) B^r\right)$$

for all $r \geq 2$ and $0 \leq \alpha \leq 1$.

Proof By using the Cauchy–Schwarz inequality, as in the proof of Theorem 8, we have

$$\begin{aligned} \left|\left\langle A^\alpha X B^{1-\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle\right|^r &= \left|\left\langle X B^{1-\alpha}\widehat{k}_\lambda, A^\alpha \widehat{k}_\lambda\right\rangle\right|^r \\ &\leq \|X\|^r \left\|B^{1-\alpha}\widehat{k}_\lambda\right\|^r \left\|A^\alpha \widehat{k}_\lambda\right\|^r \\ &= \|X\|^r \left\langle B^{2(1-\alpha)}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{r/2} \left\langle A^{2\alpha}\widehat{k}_\lambda, \widehat{k}_\lambda\right\rangle^{r/2} \end{aligned}$$

and therefore

$$\left|\widetilde{A^\alpha X B^{1-\alpha}}(\lambda)\right|^r \leq \|X\|^r \left(\widetilde{B^{2(1-\alpha)}}(\lambda)\right)^{r/2} \left(\widetilde{A^{2\alpha}}(\lambda)\right)^{r/2}$$

for all $\lambda \in \Omega$. Then we get from the McCarthy inequality and Lemma 3 that

$$\begin{aligned} \left|\widetilde{A^\alpha X B^{1-\alpha}}(\lambda)\right|^r &\leq \|X\|^r \left(\widetilde{A^r}(\lambda)\right)^{\alpha} \left(\widetilde{B^r}(\lambda)\right)^{1-\alpha} \\ &\leq \|X\|^r \left(\alpha A^r + \widetilde{(1-\alpha)} B^r(\lambda)\right) \end{aligned}$$

for all $\lambda \in \Omega$. Taking the supremum in the last inequality, we obtain

$$ber^r\left(A^\alpha X B^{1-\alpha}\right) \leq \|X\|^r\, ber\left(\alpha A^r + (1-\alpha) B^r\right)$$

for all positive operators $A, B \in \mathcal{B}(\mathcal{H}(\Omega))$. This proves the theorem. □

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