Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 1953 – 1966
© TÜBİTAK
doi:10.3906/mat-1812-16

Weak-stability and saddle point theorems for a multiobjective optimization problem with an infinite number of constraints

Ta Quang SON[1], **Ching-Feng WEN**[2,3,*]
[1]Faculty of Mathematics and Applications, Saigon University, Ho Chi Minh City, Vietnam
[2]Center for Fundamental Science and Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Kaohsiung, Taiwan
[3]Department of Medical Research, Kaohsiung Medical University Hospital, Kaohsiung, Taiwan

Received: 13.12.2018 • **Accepted/Published Online:** 30.05.2019 • **Final Version:** 31.07.2019

Abstract: In this paper, we focus on weak-stability and saddle point theorems of multiobjective optimization problems that have an infinite number of constraints. The obtained results are based on the notion of weak-subdifferentials for vector functions. Some properties of weak stability for the problems are introduced. Relationships between strong duality and saddle points of the augmented Lagrange vector functions associated to the problems are investigated. Connections between weak-stability and saddle point theorems of the problems are established. An example is given.

Key words: Weak-subdifferential, Weak-stability, Strong duality, Saddle point theorem

1. Introduction

It is well known that the notion of subdifferentials for convex functions is important in convex analysis, optimization theory, and variational analysis [13]. Optimality conditions for convex optimization problems are usually represented in terms of subdifferentials of convex functions. Note that this notion may not be suitable for the cases of nonconvex optimization problems. Hence, it is necessary to generalize this concept for nonconvex functions. Such generalized subdifferentials for nonconvex functions were proposed by several mathematicians, such as Clarke [3], Rockafellar [12, 14], Mordukhovich [11], and Azimov and Gazimov [2]. One of the generalized subdifferentials mentioned above, which has attracted several researchers recently, is the notion of weak-subdifferentials introduced by Azimov and Gazimov. In [2], based on the notion of supporting cones, the authors proposed the notion of weak-subdifferentials for scalar nonconvex functions. Moreover, with a type of augmented Lagrange function, which was constructed by using supporting cones to the epigraph of a perturbation function, the relationships between stability and duality of nonconvex problems via the augmented Lagrangians are investigated. Several papers related to this topic were published, where the obtained results were based on the use of weak-subdifferentials [4–10, 15]. Recently, this notion was developed for vector functions defined on a normed space [9]. Inspired by this work, very recently, it was extended to vector functions defined on a linear space and was applied to a multiobjective semiinfinite optimization problem [16]. Then dual problems via augmented Lagrange vector functions were studied. In that paper, a concept of weak-stability for vector functions was proposed and the relationships between weak-stability and strong duality for the problems were investigated.

*Correspondence: cfwen@kmu.edu.tw

In optimization theory, there exists a close connection between the notion of strong duality and saddle point theorems (see [17] for more details). Hence, it is worth studying such relationships for the case of multiobjective semiinfinite optimization. Note that the relationships between weak-stability, strong duality, and saddle point theorems for a scalar nonconvex optimization problem based on weak-subdifferentials for scalar functions were introduced in [2]. Furthermore, the results were extended for scalar semiinfinite optimization problems in [15]. Very recently, the results were developed for multiobjective problems. In [16], the relationships between weak-stability and strong duality for semiinfinite optimization problems are studied. Motivated by the observations above, in this research, based on the notion of weak-subdifferentials for vector functions, we study the relationships between weak-stability and saddle point theorems for multiobjective optimization problems that have an infinite number of constraints. In our research, besides studying the relationships between weak-stability and saddle point theorems, connections between strong duality and the existence of saddle points of the problems are also investigated. Hence, the relations among weak stability, strong duality, and saddle points theorems for the problems are studied.

The results in this paper are divided into two parts. The first part is devoted to investigating some properties of weak-stability for the multiobjective optimization problem. The second part concerns saddle point theorems for augmented Lagrange vector functions. We also show that there exists a close connection between the pairs of efficient solutions of a multiobjective optimization problem and its dual problem with saddle points of the corresponding vector Lagrange function. Then saddle point theorems for the multiobjective optimization problems are given.

The paper is organized as follows. In the second section, we recall the results on weak-subgradients for vector functions and the theorem that connects the property of weak-stability with the property of L-lower Lipschitz. The main results are given in Sections 3 and 4, where results on weak-stability for a multiobjective optimization problem are given and saddle point theorems are introduced. An example is given in the last part.

2. Preliminaries

Let T be an arbitrary index set. We use the following notations: $\mathbb{R}^T := \{u = (u_t)_t \mid u_t \in \mathbb{R}, t \in T\}, \mathbb{R}^T_+ := \{u \in \mathbb{R}^T \mid u_t \geq 0, t \in T\}$ and $\mathbb{R}^T_- := \{u \in \mathbb{R}^T \mid u_t \leq 0, t \in T\}$. Let $\lambda : T \to \mathbb{R}$ be a real function such that $\lambda(t) = \lambda_t \in \mathbb{R}$ for each $t \in T$ but only finitely many λ_t differ from zero. We need the following linear space:

$$\mathbb{R}^{(T)} := \{\lambda = (\lambda_t)_{t\in T} \mid \lambda_t = 0 \text{ for all } t \in T \text{ but only finitely many } \lambda_t \neq 0\}.$$

For $\lambda \in \mathbb{R}^{(T)}$, its supporting set, denoted by $T(\lambda)$, is defined by

$$T(\lambda) := \{t \in T \mid \lambda_t \neq 0\}.$$

It is obvious that, for a given $\lambda \in \mathbb{R}^{(T)}$, the set $T(\lambda)$ is a finite set. The subset of $\mathbb{R}^{(T)}$ where its elements satisfy $\lambda_t \geq 0$ for all $t \in T$ is denoted by $\mathbb{R}^{(T)}_+$. It is clear that $\mathbb{R}^{(T)}_+$ is a convex cone of $\mathbb{R}^{(T)}$. For $\lambda, \bar{\lambda} \in \mathbb{R}^{(T)}$, $u \in \mathbb{R}^T$, and $\alpha \in \mathbb{R}$, we define

$$\begin{aligned} \alpha\lambda &:= (\alpha\lambda_t)_{t\in T}, \\ \lambda \pm \bar{\lambda} &:= (\lambda_t \pm \bar{\lambda}_t)_{t\in T}, \end{aligned}$$

$$\langle \lambda, u\rangle := \sum_{t\in T} \lambda_t u_t = \begin{cases} \sum_{t\in T(\lambda)} \lambda_t u_t & \text{if} \quad T(\lambda) \neq \emptyset, \\ 0 & \text{if} \quad T(\lambda) = \emptyset. \end{cases}$$

Let us define the function $\rho : \mathbb{R}^T \times \mathbb{R}^{(T)} \to \mathbb{R}$ as follows:

$$\rho(u,\lambda) := \begin{cases} \sum_{t \in T(\lambda)} |u_t|, & T(\lambda) \neq \emptyset, \\ 0, & T(\lambda) = \emptyset. \end{cases}$$

The notation $\|u\|_\lambda$ is used instead of the value of the function ρ at (u,λ).

Let us denote $D := \mathbb{R}^m_+$. The notations D°, D^*, and 0_D stand for the interior of D, $D \setminus \{0_D\}$, and $(0,0,\dots,0)$, respectively. Then, for $y,z \in \mathbb{R}^m$, we understand that

$$\begin{aligned} &y = z \Leftrightarrow y - z = 0_D, \\ &y \leq_{D^\circ} z \Leftrightarrow z - y \in D^\circ, \\ &y \leq_{D^*} z \Leftrightarrow z - y \in D^*, \\ &y \leq_D z \Leftrightarrow z - y \in D \end{aligned}$$

and

$$\begin{aligned} &y \nleq_{D^\circ} z \Leftrightarrow z - y \notin D^\circ, \\ &y \nleq_{D^*} z \Leftrightarrow z - y \notin D^*, \\ &y \nleq_D z \Leftrightarrow z - y \notin D. \end{aligned}$$

We need some more notations below:

$$\begin{aligned} &\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}, \\ &\overline{\mathbb{R}}^m := \overline{\mathbb{R}} \times \overline{\mathbb{R}} \times \ldots \times \overline{\mathbb{R}}, (m \text{ component}), \\ &\mathcal{L} := \{(\lambda^1,\dots,\lambda^m), \lambda^i \in \mathbb{R}^{(T)}, i \in M\}, M := \{1,2,\dots,m\}, \\ &\mathcal{L}_+ := \{(\lambda^1,\dots,\lambda^m), \lambda^i \in \mathbb{R}^{(T)}_+, i \in M\}, \\ &\mathcal{L}_- := \{(\lambda^1,\dots,\lambda^m), \lambda^i \in \mathbb{R}^{(T)}_-, i \in M\}. \end{aligned}$$

For $u \in \mathbb{R}^T$, $\Lambda = (\lambda^1,\dots,\lambda^m) \in \mathcal{L}$, we define

$$\langle \Lambda, u \rangle := \big(\langle \lambda^1, u\rangle, \dots, \langle \lambda^m, u \rangle\big)$$

and

$$\|u\|_\Lambda := \|u\|_{\bar\lambda} = \begin{cases} \sum_{t \in T(\bar\lambda)} |u_t|, & T(\bar\lambda) \neq \emptyset, \\ 0, & T(\bar\lambda) = \emptyset, \end{cases}$$

where $\bar\lambda := \sum_{i=1}^m \lambda^i$.

For convenience, we recall some basic concepts and results.

Definition 2.1 [16, Definition 3.2] *Given $\Lambda \in \mathcal{L}$ and $u_0 \in \mathbb{R}^T$, the Λ-neighborhood of $u_0 \in \mathbb{R}^T$ with the radius $\epsilon > 0$ is defined by*

$$\mathcal{U}^\Lambda_\epsilon(u_0) := \{u \in \mathbb{R}^T : \|u - u_0\|_\Lambda < \epsilon\}.$$

We note that the two following definitions are developed from the original ones for real valued functions, which were introduced in [2].

Definition 2.2 [16, Definition 3.3] *Let $h : \mathbb{R}^T \to \overline{\mathbb{R}}^m$ be a vector function such that* $\text{dom}h \neq \emptyset$, *i.e.* $\text{dom}h_i \neq \emptyset$ *for all $i \in M$, and let $u_0 \in \text{dom}h$. The weak-subdifferential of the function h at u_0 is denoted and defined by the set*

$$\partial^w h(u_0) := \{(\Lambda,k) \in \mathcal{L} \times \mathbb{R}^m_+ \mid h(u) - h(u_0) \geq_D \langle \Lambda, u - u_0 \rangle - k\|u - u_0\|_\Lambda, \forall u \in \mathbb{R}^T\}.$$

Each pair $(\Lambda, k) \in \partial^w h(u_0)$ *is said to be a weak-subgradient of* h *at* u_0*.*
If $\partial^w h(u_0) \neq \emptyset$*, then we say that* h *is weakly subdifferentiable at* u_0*.*
If h *is defined on the set* $S \subset \mathbb{R}^T$*, we use the notation* $\partial^w_S h(u_0)$ *instead of* $\partial^w h(u_0)$*.*

Definition 2.3 [16, Definition 3.4] *Given* $\Lambda \in \mathcal{L}$*, a vector function* $h : \mathbb{R}^T \to \overline{\mathbb{R}}^m, \mathrm{dom}h \neq \emptyset,$ *is called* Λ*-lower locally Lipschitz at* $u_0 \in \mathrm{dom}h$ *if there exist a vector* $L \in \mathbb{R}^m_+$ *and a* Λ*-neighborhood* $\mathcal{U}^\Lambda_\epsilon(u_0)$ *such that*

$$h(u) - h(u_0) \geq_D -L\|u - u_0\|_\Lambda, \forall u \in \mathcal{U}^\Lambda_\epsilon(u_0).$$

The vector function h *is called* Λ*-lower Lipschitz if the inequality above holds for all* $u \in \mathrm{dom}h$*.*

The following proposition and corollary are developed with inspiration from Theorem 1 and Corollary 2, respectively, in [2] (see also [1]).

Proposition 2.4 [16, Theorem 3.2] *Let* $h : \mathbb{R}^T \to \overline{\mathbb{R}}^m$ *be a vector function such that* $\mathrm{dom}h \neq \emptyset$ *and* $u_0 \in \mathrm{dom}h$*. The following statements are equivalent:*

a) h *is weakly subdifferentiable at* u_0*,*

b) h *is* Λ*-lower Lipschitz at* u_0*,*

c) h *is* Λ*-lower locally Lipschitz at* u_0 *and there exist numbers* $p \in \mathbb{R}^m_+$ *and* $q \in \mathbb{R}^m$ *such that*

$$h(u) \geq_D -p\|u\|_\Lambda + q, \forall u \in \mathbb{R}^T.$$

We understand that a vector function $h : \mathbb{R}^T \to \overline{\mathbb{R}}^m$ is said to be bounded below if each component function of h is bounded below.

Corollary 2.5 *Let* $h : \mathbb{R}^T \to \overline{\mathbb{R}}^m$ *be a* Λ*-lower Lipschitz function at* $u_0 \in \mathrm{dom}h$*. If the function* h *is bounded below, then it is weakly subdifferentiable at* u_0*.*

Proof If the function h is bounded below then there exists a vector $m \in \mathbb{R}^m$ such that

$$h(u) \geq m, \forall u \in \mathrm{dom}h.$$

Hence,

$$h(u) \geq -0\|u\|_\Lambda + m, \forall u \in \mathrm{dom}h.$$

Combining this and the assumption that h is a Λ-lower Lipschitz function at $u_0 \in \mathrm{dom}h$, we get the desired conclusion by applying Proposition 2.4. □

Let us consider the following problem:

$$\begin{array}{lll} \text{(MP)} & \text{Minimize} & f(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \\ & \text{subject to} & g_t(x) \leq 0,\ t \in T, \\ & & x \in S, \end{array}$$

where f_i with $i \in M$ and g_t with $t \in T$ are real valued functions defined on a linear space X, T is as above, and S is a nonempty subset of X. Denote by A the feasible set of (MP). Set

$$\mathbb{R}^{(T)}_\lambda := \{e \in \mathbb{R}^{(T)} : \ |e_t| \leq 1, t \in T(\lambda)\}$$

and

$$F := \{(\lambda, k) \in \mathbb{R}^{(T)} \times \mathbb{R}_+ \mid \exists e \in \mathbb{R}^{(T)}_{\lambda}, ke - \lambda \in \mathbb{R}^{(T)}_+\}.$$

We need the following function (see [15]):

$$\omega(u, \lambda, k) : \mathbb{R}^T \times \mathbb{R}^{(T)} \times \mathbb{R}_+ \to \mathbb{R},$$

where

$$\omega(u, \lambda, k) := \begin{cases} \sup_{e \in \mathbb{R}^{(T)}_{\lambda}} \{\langle ke, u\rangle \mid ke - \lambda \in \mathbb{R}^{(T)}_+\}, & T(\lambda) \neq \emptyset, \\ 0, & T(\lambda) = \emptyset. \end{cases} \tag{2.1}$$

The augmented Lagrange vector function associated to (MP) is defined by

$$L(x, \Lambda, k) := \big(L_1(x, \lambda^1, k^1), \ldots, L_m(x, \lambda^m, k^m)\big), \tag{2.2}$$

where

$$L_i(x, \lambda^i, k^i) := f_i(x) - \langle \lambda^i, (g_t(x))_t\rangle + \omega((g_t(x)), \lambda^i, k^i), i \in M. \tag{2.3}$$

The perturbation function $\phi_i : X \times \mathbb{R}^T \to \overline{\mathbb{R}}$ associated to f_i, $i \in M$, is defined by

$$\phi_i(x, u) := \begin{cases} f_i(x) \text{ if } x \in S \text{ and } g_t(x) \leq u_t, t \in T, \\ +\infty, \text{ otherwise.} \end{cases} \tag{2.4}$$

Hence, the perturbation vector function $\phi : X \times \mathbb{R}^T \to \overline{\mathbb{R}}^m$ is

$$\phi(x, u) := \big(\phi_1(x, u), \ldots, \phi_m(x, u)\big). \tag{2.5}$$

We need the following lemmas. They are quoted from the ones in [16].

Lemma 2.6 *For each $i \in M$, let L_i be the augmented Lagrange function defined by (2.3) and let ϕ_i be the perturbation defined by (2.4). One has*

$$\inf_{u \in \mathbb{R}^T} \{\phi_i(x, u) - \langle \lambda, u\rangle + k\|u\|_{\lambda}\} = \begin{cases} L_i(x, \lambda, k) & \text{if} \quad x \in S \text{ and } (\lambda, k) \in F, \\ -\infty & \text{if} \quad x \in S \text{ but } (\lambda, k) \notin F, \\ +\infty & \text{if} \quad x \notin S. \end{cases}$$

Lemma 2.7 *For each $i \in M$ and for every $x \in S$, it holds that*

$$\sup_{(\lambda, k) \in F} L_i(x, \lambda, k) = \begin{cases} f_i(x) \text{ if } g_t(x) \leq 0, \forall t \in T, \\ +\infty \quad \text{otherwise.} \end{cases}$$

Definition 2.8 *For the problem* (MP)*, a point $z \in A$ is said to be:*

i) *an efficient solution of* (MP) *if there is no $x \in A$ such that $f(x) \leq_{D^*} f(z)$.*

ii) *a weakly-efficient solution of* (MP) *if there is no $x \in A$ such that $f(x) \leq_{D^\circ} f(z)$.*

The efficient solution set of (MP) is denoted by E(MP). Assume that $\mathrm{E(MP)} \neq \emptyset$.

We note that the following duality scheme is also a generalization of the duality scheme given in [2]. The dual problem for (MP) is formulated as follows (see [16]):

$$\begin{array}{lll}\text{(MD)} & \text{Maximize} & H(\Lambda,k)\\ & \text{subject to} & (\Lambda,k)\in F^m,\end{array}$$

where $H : F^m \to \overline{\mathbb{R}}^m$ is defined by

$$H(\Lambda,k) := \big(H_1(\Lambda,k),\dots,H_m(\Lambda,k)\big),$$

and

$$H_i(\Lambda,k) := \inf_S L_i(x,\lambda^i,k^i), (\lambda^i,k^i)\in F.$$

Definition 2.9 *For the problem* (MD)*, a pair* $(\bar\Lambda,\bar k)\in F^m$ *is said to be:*

i) *an efficient solution of* (MD) *if there is no* $(\Lambda,k)\in F^m$ *such that*

$$H(\bar\Lambda,\bar k) \leq_{D^*} H(\Lambda,k),$$

ii) *a weakly efficient solution of* (MD) *if there is no* $(\Lambda,k)\in F^m$ *such that*

$$H(\bar\Lambda,\bar k) \leq_{D^\circ} H(\Lambda,k).$$

The set of efficient solutions of (MD) is called the efficient solution set of (MD) and is denoted by $\mathrm{E(MD)}$.

Definition 2.10 *The strong duality between* (MP) *and* (MD) *holds if there exist* $\bar x\in \mathrm{E(MP)}$ *and* $(\bar\Lambda,\bar k)\in \mathrm{E(MD)}$ *such that* $f(\bar x) = H(\bar\Lambda,\bar k)$.

In the next section, we establish a connection between weak-stability and the saddle point theorem for (MP).

3. Weak-stability of (MP) and related properties

For the function ϕ defined by (2.5), let us consider the following vector function G defined as follows:

$$G(u) := \big(\inf_S \phi_1(x,u), \inf_S \phi_2(x,u),\dots,\inf_S \phi_m(x,u)\big). \tag{3.1}$$

Suppose that $\mathrm{dom} G\neq\emptyset$. It is obvious that $\inf_S \phi_i(x,0) = \inf_A f_i(x)$ for each $i\in M$. We recall the notation of weak-stable introduced in [16].

Definition 3.1 *The problem* (MP) *is said to be weak-stable if* $\partial^w G(0)\neq\emptyset$ *and* $G_i(0)$ *is finite for any* $i\in M$.

Theorem 3.2 *For the problem* (MP)*:*

i) *If there exist* $(\Lambda,k)\in F^m$ *and* $\bar x\in A$ *such that* $L(x,\Lambda,k)\geq\phi(\bar x,0)$ *for all* $x\in S$ *and* $\inf_A f_i(x)$ *is finite for all* $i\in M$*, then* (MP) *is weak-stable.*

ii) *If* (MP) *is weak-stable and* $(\inf_A f_1(x), \ldots, \inf_A f_m(x)) \in f(A)$*, then there exist* $(\Lambda, k) \in \mathcal{L} \times \overline{\mathbb{R}}_+^m$ *and* $\bar{x} \in A$ *such that*

$$L(x, \Lambda, k) \geq \phi(\bar{x}, 0), \forall x \in S.$$

Proof i) Suppose that there exist $(\Lambda, k) \in \mathcal{L}_+ \times \overline{\mathbb{R}}_+^m$ and $\bar{x} \in A$ such that

$$L(x, \Lambda, k) \geq \phi(\bar{x}, 0), \forall x \in S.$$

Then, for any $i \in M$, we get

$$L_i(x, \lambda^i, k^i) \geq \phi_i(\bar{x}, 0), \forall x \in S.$$

By Lemma 2.6, we get

$$\inf_{u \in \mathbb{R}^T} \{\phi_i(x, u) - \langle \lambda^i, u \rangle + k^i \|u\|_{\lambda^i}\} \geq \phi_i(\bar{x}, 0), \forall x \in S.$$

Hence,

$$\inf_{x \in S} \{\phi_i(x, u) - \langle \lambda^i, u \rangle + k^i \|u\|_{\lambda^i}\} \geq \phi_i(\bar{x}, 0), \forall u \in \mathbb{R}^T.$$

Thus,

$$\inf_S \phi_i(x, u) - \langle \lambda^i, u \rangle + k^i \|u\|_{\lambda^i} \geq \phi_i(\bar{x}, 0) \geq \inf_S \phi_i(x, 0), \forall u \in \mathbb{R}^T.$$

We get

$$G_i(u) - G_i(0) \geq \langle \lambda^i, u \rangle - k^i \|u\|_{\lambda^i}, i \in M.$$

Note that $G_i(0) = \inf_A f_i(x)$ is finite. Hence, for each $i \in M$, G_i is weak-subdifferentiable at $u = 0$. Thus, the vector function G is weak-subdifferentiable at $u = 0$. The problem (MP) is weak-stable.

ii) Suppose that (MP) is weak-stable and $(\inf_A f_1(x), \ldots, \inf_A f_m(x)) \in f(A)$. By applying Theorem 4.3 in [16], the strong duality between (MP) and (MD) holds. Using an argument as in the proof of Theorem 4.3 in [16], there exists $(\Lambda, k) \in F^m$ and $\bar{x} \in A$ such that

$$H(\Lambda, k) = f(\bar{x}).$$

Since $f(\bar{x}) = \phi(\bar{x}, 0)$ and $H_i(\Lambda, k) = \inf_S L_i(x, \lambda^i, k^i)$, it is easy to deduce that

$$L(x, \Lambda, k) \geq \phi(\bar{x}, 0), \forall x \in S.$$

□

The following theorem is developed from Theorem 6 in [2].

Theorem 3.3 *For the problem* (MP)*, suppose that* $(\inf_A f_1(x), \ldots, \inf_A f_m(x)) \in f(A)$*. Then* (MP) *is weak-stable if and only if both the following conditions hold:*

i) *There exist* $\Lambda \in \mathcal{L}$*,* $L \in \mathbb{R}_+^m$*,* $\mathcal{U}^\Lambda(0)$*, and* $x^* \in S$ *such that*

$$\phi(x, u) - \phi(x^*, 0) \geq_D -L\|u\|_\Lambda, \forall x \in S, \forall u \in \mathcal{U}^\Lambda(0). \tag{3.2}$$

ii) *There exists vectors* $p \in \mathbb{R}_+^m$ *and* $q \in \mathbb{R}^m$ *such that*

$$\phi(x, u) \geq_D -p\|u\|_\Lambda + q, \forall x \in S, \forall u \in \mathbb{R}^T. \tag{3.3}$$

Proof If (MP) is weak-stable, then G is weak-subdifferentiable at $u=0$ and $G_i(0)$ is finite for each $i \in M$. By Proposition 2.4, there exists $\Lambda \in \mathcal{L}$ such that the function G is Λ-locally lower Lipschitz at $u=0$ and there exist vectors $p \in \mathbb{R}^m_+$ and $q \in \mathbb{R}^m$ such that

$$G(u) := (\inf_S \phi_1(x,u), \ldots, \inf_S \phi_m(x,u)) \geq_D -p\|u\|_\Lambda + q, \forall u \in \mathbb{R}^T.$$

Hence,

$$\phi(x,u) \geq_D -p\|u\|_\Lambda + q, \forall u \in \mathbb{R}^T, \forall x \in S. \tag{3.4}$$

On the other hand, since G is Λ-lower locally Lipschitz at $u=0$, there exist a Λ-neighborhood $\mathcal{U}^\Lambda(0)$ and a vector $L \in \mathbb{R}^m_+$ such that

$$G(u) - G(0) \geq_D -L\|u\|_\Lambda, \forall u \in \mathcal{U}^\Lambda(0).$$

Note that

$$G(0) = (\inf_S \phi_1(x,0), \ldots, \inf_S \phi_1(x,0)) = (\inf_A f_1(x), \ldots, \inf_A f_m(x)) \in f(A).$$

Then there exists $x^* \in A$ such that $(\inf_A f_1(x), \ldots, \inf_A f_m(x)) = f(x^*)$. This implies that

$$\phi(x,u) - \phi(x^*,0) \geq_D -L\|u\|_\Lambda, \forall x \in S, \forall u \in \mathcal{U}^\Lambda(0).$$

Conversely, suppose that conditions i) and ii) hold. Note that

$$G(0) = (\inf_S \phi_1(x,0), \ldots, \inf_S \phi_m(x,0)) \leq_D (\phi_1(x,0), \ldots, \phi_m(x,0)) = \phi(x,0), \forall x \in S$$

and $G_i(u) = \inf_S \phi_i(x,u)$ for all $i \in M$. Hence, from (3.2), we get

$$G(u) - G(0) \geq_D -L\|u\|_\Lambda, \forall u \in \mathcal{U}^\Lambda(0).$$

Thus, $G(u)$ is Λ-locally lower Lipschitz at $u=0$. On the other hand, since (3.3) holds for all $x \in S$, by taking the infimum over S, we get

$$G(u) \geq_D -p\|u\|_\Lambda + q, \forall u \in \mathbb{R}^T.$$

Thus, by applying Proposition 2.4, G is weak-subdifferentiable at $u=0$, and (MP) is weak-stable. □

Theorem 3.4 *For the problem* (MP)*, suppose that* $(\inf_A f_1(x), \ldots, \inf_A f_m(x)) \in f(A)$. *Then* (MP) *is weak-stable if and only if there exist* $\Lambda \in \mathcal{L}$, $L \in \mathbb{R}^m_+$ *and* $x^* \in S$ *such that*

$$\phi(x,u) - \phi(x^*,0) \geq_D -L\|u\|_\Lambda, \forall x \in S, \forall u \in \mathbb{R}^T. \tag{3.5}$$

Proof Suppose that there exist $\Lambda \in \mathcal{L}$, $L \in \mathbb{R}^m_+$, and $x^* \in S$ such that (3.5) holds. By (3.1), we obtain

$$G(u) - \phi(x^*,0) \geq_D -L\|u\|_\Lambda, \forall x \in S, \forall u \in \mathbb{R}^T.$$

Moreover, since $\phi(x^*,0) \geq G(0)$, the inequality above implies that

$$G(u) - G(0) \geq_D -L\|u\|_\Lambda, \forall u \in \mathbb{R}^T.$$

The function G is Λ-lower Lipschitz at $u=0$. From this it can be deduced that the problem (MP) is weak-stable by Proposition 2.4.

Conversely, suppose that (MP) is weak-stable. Hence, the function G is Λ-lower Lipschitz at $u=0$, i.e. there exists $\Lambda \in \mathbb{R}^m_+$ such that

$$G(u) - G(0) \geq_D -L\|u\|_\Lambda, \forall u \in \mathbb{R}^T.$$

Hence,

$$\phi(x,u) - G(0) \geq_D -L\|u\|_\Lambda, \forall x \in S, \forall u \in \mathbb{R}^T.$$

Since $G(0) = (\inf_A f_1(x), \ldots, \inf_A f_m(x)) \in f(A)$, there exist $x^* \in A$ such that $G(0) = f(x^*) = \phi(x^*, 0)$. The desired conclusion is derived □

4. Saddle point theorems

Let us consider the vector Lagrange function L given by (2.2), where $\Lambda \in \mathcal{L}$ and $k \in \mathbb{R}^m_+$.

Definition 4.1 *A pair* $(\bar{x}, \bar{\Lambda}) \in S \times \mathcal{L}_-$ *is called a saddle point of the vector function* L *if the following inequalities are satisfied for some* $k \in \mathbb{R}^m_+$*:*

$$L(\bar{x}, \Lambda, k) \not\geq_D L(\bar{x}, \bar{\Lambda}, k), \forall (\Lambda, k) \in F^m, \tag{4.1a}$$

$$L(\bar{x}, \bar{\Lambda}, k) \not\geq_D L(x, \bar{\Lambda}, k), \forall x \in S. \tag{4.1b}$$

That is to say that there exist no $\lambda^i \in \mathbb{R}^{(T)}_-$ *and* $x \in S$*, such that:*

i) $-\langle \lambda^i, (g_t(\bar{x}))_t \rangle + \omega((g_t(\bar{x})), \lambda^i, k^i) \geq -\langle \bar{\lambda}^i, (g_t(\bar{x}))_t \rangle + \omega((g_t(\bar{x})), \bar{\lambda}^i, k^i), i \in M,$ *with at least one strict inequality;*

ii) $f_i(\bar{x}) - \langle \bar{\lambda}^i, (g_t(\bar{x}))_t \rangle + \omega((g_t(\bar{x})), \bar{\lambda}^i, k^i) \geq f_i(x) - \langle \bar{\lambda}^i, (g_t(x))_t \rangle + \omega((g_t(x)), \bar{\lambda}^i, k^i), i \in M,$ *with at least one strict inequality.*

Theorem 4.2 *If* $(\bar{x}, \bar{\Lambda}) \in S \times \mathcal{L}_-$ *is a saddle point of the vector function* L *and* $g_t(x) \leq g_t(\bar{x})$ *for all* $x \in S$ *and for all* $t \in T$*, then* $\bar{x}$ *is an efficient solution of* (MP) *and* $(\bar{\Lambda}, k)$ *is an efficient solution of* (MD)*.*

Proof Suppose that $(\bar{x}, \bar{\Lambda}) \in S \times \mathcal{L}_-$ is a saddle point of the vector function L for some $k \in \mathbb{R}^m_+$. From Definition 4.1, there exists no $\lambda^i \in \mathbb{R}^{(T)}_-$ such that

$$\langle \bar{\lambda}^i - \lambda^i, g_t(\bar{x}) \rangle + \omega((g_t(\bar{x})), \lambda^i, k^i) - \omega((g_t(\bar{x})), \bar{\lambda}^i, k^i) \geq 0, i \in M, \tag{4.2}$$

with at least one strict inequality, and there exists no $x \in S$ such that

$$f_i(\bar{x}) - f_i(x) - \langle \bar{\lambda}^i, (g_t(\bar{x}) - g_t(x)) \rangle + \omega((g_t(\bar{x})), \bar{\lambda}^i, k^i) - \omega((g_t(x)), \bar{\lambda}^i, k^i) \geq 0, i \in M, \tag{4.3}$$

with at least one strict inequality.

First, we claim that $\bar{x} \in A$. Assume to the contrary that $\bar{x} \notin A$. Then there exists $t_0 \in T$ such that $g_{t_0}(\bar{x}) > 0$. For $i \in M$, choose λ^{i*} such that $\lambda^{i*}_{t_0} < 0$ and $\lambda^*_t = \bar{\lambda}^i_t$ for all $t \in T \setminus \{t_0\}$. Since $\lambda^{i*}_{t_0} < 0$ is chosen arbitrarily, by letting $\lambda^{i*}_{t_0} \to -\infty$, we get

$$\langle \bar{\lambda}^i - \lambda^i, g_t(\bar{x}) \rangle = (\bar{\lambda}^i_{t_0} - \lambda^{i*}_{t_0}) g_{t_0}(\bar{x}) \to +\infty,$$

i.e. there exists $\lambda^i \in \mathbb{R}^{(T)}_-$ such that the left-hand side of (4.2) tends to $+\infty$, a contradiction.

Next, we prove that $\bar{x} \in \mathrm{E(MP)}$. It needs to be proved that there is no $x \in A$ such that $f(x) \leq_{D^*} f(\bar{x})$. Suppose to the contrary that there exists $x \in A$ such that $f(x) \leq_{D^*} f(\bar{x})$. Then

$$f_i(x) \leq f_i(\bar{x}), \forall i \in M, \tag{4.4}$$

with at least one strict inequality. Since $g_t(x) \leq g_t(\bar{x})$ for all $t \in T$,

$$-\langle \bar{\lambda}^i, (g_t(x)) \rangle \leq -\langle \bar{\lambda}^i, (g_t(\bar{x})) \rangle. \tag{4.5}$$

On the other hand, if $g_t(x) \leq g_t(\bar{x})$ for all $t \in T$ then

$$\sup_{e \in \mathbb{R}^{(T)}_{\bar{\lambda}^i}} \{\langle ke, g_t(x) \rangle \mid ke - \bar{\lambda}^i \in \mathbb{R}^{(T)}_+\} \leq \sup_{e \in \mathbb{R}^{(T)}_{\bar{\lambda}^i}} \{\langle ke, g_t(\bar{x}) \rangle \mid ke - \bar{\lambda}^i \in \mathbb{R}^{(T)}_+\}.$$

This is equivalent to

$$\omega((g_t(x)), \bar{\lambda}^i, k^i) \leq \omega((g_t(\bar{x})), \bar{\lambda}^i, k^i). \tag{4.6}$$

Adding side by side (4.4), (4.5), and (4.6), we can see that there exists $x \in A$ such that (4.3) holds, a contradiction.

To complete the proof, we prove that $(\bar{\Lambda}, k)$ is an efficient solution of (MD). It needs to be proved that there exists no (Λ^*, k) such that

$$H(\bar{\Lambda}, k) \leq_{D^*} H(\Lambda^*, k). \tag{4.7}$$

Suppose to the contrary that there exists (Λ^*, k) such that (4.7) holds. There exists $i \in M$ such that

$$H_i(\bar{\Lambda}, k) < H_i(\Lambda^*, k),$$

or in other words,

$$\inf_S L_i(x, \bar{\lambda}^i, k^i) < \inf_S L_i(x, \lambda^{*i}, k^i).$$

Thus, there exists $x \in S$ such that

$$L_i(x, \bar{\lambda}^i, k^i) < L_i(x, \lambda^{*i}, k^i).$$

Thus,

$$-\langle \bar{\lambda}^i, g_t(x) \rangle + \omega((g_t(x)), \bar{\lambda}^i, k^i) < -\langle \lambda^{*i}, g_t(x) \rangle + \omega((g_t(x)), \lambda^{*i}, k^i),$$

or in other words,

$$0 < \langle \bar{\lambda}^i - \lambda^{*i}, g_t(x) \rangle + \omega((g_t(x)), \lambda^{*i}, k^i) - \omega((g_t(x)), \bar{\lambda}^i, k^i),$$

a contradiction to (4.2). □

Theorem 4.3 *If the strong duality between* (MP) *and* (MD) *holds then the function* L *given by* (2.2) *has a saddle point.*

Proof Let $\bar{x}$ be an efficient solution of (MP) and $(\bar{\Lambda}, k)$ be an efficient solution of (MD) with some $k \in \mathbb{R}^m_+$. Since the strong duality between (MP) and (MD) holds,

$$f(\bar{x}) = H(\bar{\Lambda}, k).$$

Hence, for every $i \in M$, we get

$$f_i(\bar{x}) = \inf_S L_i(x, \bar{\lambda}^i, k^i).$$

Thus,

$$f_i(\bar{x}) \leq L_i(x, \bar{\lambda}^i, k^i), \forall x \in S, i \in M. \tag{4.8}$$

By Lemma 2.7, from the above inequality, we get

$$f_i(\bar{x}) \leq L_i(\bar{x}, \bar{\lambda}^i, k^i) \leq \sup_{(\lambda^i, k^i) \in F} L_i(\bar{x}, \lambda^i, k^i) = f_i(\bar{x}), i \in M. \tag{4.9}$$

Thus, $f_i(\bar{x}) = L_i(\bar{x}, \bar{\lambda}^i, k^i)$. We claim that (4.1b) holds. Suppose to the contrary that there exists $x' \in S$ such that

$$L(x', \bar{\Lambda}, k) < L(\bar{x}, \bar{\Lambda}, k).$$

Combining this with (4.9) and (4.8), we deduce that

$$L_i(x', \bar{\lambda}^i, k^i) < L_i(\bar{x}, \bar{\lambda}^i, k^i) = f_i(\bar{x}) \leq L_i(x, \bar{\lambda}^i, k^i), \forall x \in S,$$

a contradiction. To complete the proof, we need to show that (4.1a) holds. Suppose there exists $(\Lambda', k') \in F^m$ such that

$$L(\bar{x}, \Lambda', k') > L(\bar{x}, \bar{\Lambda}, k). \tag{4.10}$$

By (4.9), we get

$$L_i(\bar{x}, \lambda^i, k^i) \leq f_i(\bar{x}), i \in M, \forall (\lambda^i, k^i) \in F.$$

This implies that

$$L_i(\bar{x}, \lambda^i, k^i) \leq f_i(\bar{x}) = L_i(\bar{x}, \bar{\lambda}^i, k^i), \forall (\lambda^i, k^i) \in F, i \in M. \tag{4.11}$$

Since inequality (4.11) holds for all $(\lambda^i, k^i) \in F, i \in M$, we get a contradiction to (4.10). □

Corollary 4.4 *For* (MP)*, suppose that* $(\inf_A f_1(x), \ldots, \inf_A f_m(x) \in f(A)$*. If the problem* (MP) *is weak-stable then the function* L *given by* (2.2) *has a saddle point.*

Proof The desired conclusion can be obtained by combining Theorem 4.3 in [16] and Theorem 4.3. □

Corollary 4.5 *For the problem* (MP)*, and the function* L *given by* (2.2)*, if there exist* $(\Lambda, k) \in F^m$ *and* $\bar{x} \in A$ *such that* $L(x, \Lambda, k) \geq \phi(\bar{x}, 0)$ *for all* $x \in S$ *then the function* L *given by* (2.2) *has a saddle point.*

Proof The desired conclusion can be obtained by combining Theorem 3.2 and Theorem 4.3. □

Corollary 4.6 *For the problem* (MP) *and the function* L *given by* (2.2)*, suppose that both of the following conditions hold:*

i) *There exist* $\Lambda \in \mathcal{L}$, $L \in \mathbb{R}^m_+$, $\mathcal{U}^\Lambda(0)$*, and* $x^* \in S$ *such that*

$$\phi(x, u) - \phi(x^*, 0) \geq_D -L\|u\|_\Lambda, \forall x \in S, \forall u \in \mathcal{U}^\Lambda(0). \tag{4.12}$$

ii) *There exists vectors* $p \in \mathbb{R}^m_+$ *and* $q \in \mathbb{R}^m$ *such that*

$$\phi(x,u) \geq_D -p\|u\|_\Lambda + q, \forall x \in S, \forall u \in \mathbb{R}^T. \tag{4.13}$$

Then the function L *has a saddle point.*

Proof The desired result follows by combining Theorem 3.3 and Theorem 4.3. □

Corollary 4.7 *For the problem* (MP)*, if there exists* $\Lambda \in \mathcal{L}$ *such that function* G *is* Λ*-lower Lipschitz at* $u=0$ *and* $(\inf_A f_1(x), \ldots, \inf_A f_m(x)) \in f(A)$*, then the function* L *has a saddle point.*

Proof Suppose that there exists $\Lambda \in \mathcal{L}$ such that function G is Λ-lower Lipschitz at $u=0$. By Corollary 2.5, the function G is weakly subdifferentiable at $u=0$. Since $(\inf_A f_1(x), \ldots, \inf_A f_m(x) \in f(A)$, $G_i(0)$ is finite for all $i \in M$. Hence, the problem (MP) is weak-stable by Definition 3.1. The conclusion follows by Corollary 4.4. □

The following example illustrates Theorem 4.3.

Example

$$\begin{array}{lll}(\mathrm{MP}_1) & \text{Minimize} & (-x, x^2) \\ & & tx - 1 \leq 0, t \in T = [0,1], \\ & & x \in S := [0, \infty).\end{array}$$

It is easy to check that the feasible set of (MP_1) is $A=[0,1]$ and $\bar{x}=0$ is an efficient solution of (MP_1). The dual problem of (MP_1) is formulated as follows:

$$\begin{array}{lll}(\mathrm{MD}_1) & \text{Maximize} & H(\Lambda, k) \\ & & (\Lambda, k) \in F^2,\end{array}$$

where $H(\Lambda,k) = \big(H_1(\lambda^1,k^1), H_2(\lambda^2,k^2)\big)$ and $H_i(\lambda^i,k^i) = \inf_{x\in S} L_i(x,\lambda^i,k^i)$ with i=1,2. We have

$$L_1(x,\lambda^2,k^2) = -x - \langle \lambda^1, (tx-1)_t \rangle + \omega((tx-1)_t, \lambda^1, k^1), t \in T,$$

$$L_2(x,\lambda^1,k^1) = x^2 - \langle \lambda^2, (tx-1)_t \rangle + \omega((tx-1)_t, \lambda^2, k^2), t \in T.$$

Note that $\mathbb{R}^{(T)}_\lambda$ is compact. By (2.1), there exists $\bar{e}^1, \bar{e}^2 \in \mathbb{R}^{(T)}_\lambda$ such that

$$\begin{aligned} L_1(x,\lambda^1,k^1) &= -x + \langle k^1\bar{e}^1 - \lambda^1, (tx-1)_t \rangle, t \in T, k^1\bar{e}^1 - \lambda^1 \geq 0 \\ &= -x + [\textstyle\sum_{t\in T} t(k^1\bar{e}^1_t - \lambda^1_t)]x - \sum_{t\in T}(k^1\bar{e}^1_t - \lambda^1_t) \\ &= [\textstyle\sum_{t\in T} t(k^1\bar{e}^1_t - \lambda^1_t) - 1]x - \sum_{t\in T}(k^1\bar{e}^1_t - \lambda^1_t) \end{aligned}$$

and

$$\begin{aligned} L_2(x,\lambda^2,k^2) &= x^2 + \langle k^2\bar{e}^2 - \lambda^2, (tx-1)_t \rangle, t \in T, k^2\bar{e}^2 - \lambda^2 \geq 0 \\ &= x^2 + [\textstyle\sum_{t\in T} t(k^2\bar{e}^2_t - \lambda^2_t)]x - \sum_{t\in T}(k^2\bar{e}^2_t - \lambda^2_t). \end{aligned}$$

It is easy to check that

$$\inf_S L_1(x,\lambda^1,k^1) = \begin{cases} -\sum_{\in T}(k^1\bar{e}^1_t - \lambda^1_t) & \text{if} \quad \sum_{t\in T}(k^1\bar{e}^1_t - \lambda^1_t)t \geq 1 \\ -\infty & \text{if} \quad \sum_{t\in T}(k^1\bar{e}^1_t - \lambda^1_t)t < 1, \end{cases}$$

and

$$\inf_S L_2(x, \lambda^2, k^2) = -\sum_{t\in T}(k^2\bar{e}_t^2 - \lambda_t^2).$$

It is obvious that $(\bar{\Lambda}, \bar{k}) = \big((\bar{\lambda}^1, \bar{k}^1), (\bar{\lambda}^2, \bar{k}^2)\big) = ((0_t, 0), (0_t, 0)) \in \mathrm{E(MD)}$ and $H(\bar{\Lambda}, \bar{k}) = (0, 0) = f(\bar{x})$, where $\bar{x} = 0 \in \mathrm{E(MP)}$. The strong duality between (MP$_1$) and (MD$_1$) holds.

We claim that $(\bar{x}, \bar{\Lambda})$ is a saddle point of the function $L(x, \Lambda, k) = \big(L(x, \lambda^1, k^1), L(x, \lambda^2, k^2)\big)$, i.e. there exist no $\lambda^i \in \mathbb{R}_-^{(T)}$ and $x \in S$, such that:

i) $-\langle \lambda^i, (g_t(\bar{x}))_t\rangle + \omega((g_t(\bar{x})), \lambda^i, k^i) \geq -\langle \bar{\lambda}^i, (g_t(\bar{x}))_t\rangle + \omega((g_t(\bar{x})), \bar{\lambda}^i, k^i), i \in M$, with at least one strict inequality, and

ii) $f_i(\bar{x}) - \langle \bar{\lambda}^i, (g_t(\bar{x}))_t\rangle + \omega((g_t(\bar{x})), \bar{\lambda}^i, k^i) \geq f_i(x) - \langle \bar{\lambda}^i, (g_t(x))_t\rangle + \omega((g_t(x)), \bar{\lambda}^i, k^i), i \in M$, with at least one strict inequality.

Suppose to the contrary that there exists $\lambda^i \in \mathbb{R}_-^{(T)}$ such that i) is satisfied with a strict inequality. Then

$$-\langle \lambda^i, (-1)_t\rangle + \omega((-1)_t, \lambda^i, k^i) > \omega((-1)_t, 0, k^i) \tag{4.14}$$

with some $i \in \{1, 2\}$. Note that $\lambda^i \in \mathbb{R}_-^{(T)}$. It is easy to check that

$$\omega((-1)_t, \lambda^i, k^i) \leq \omega((-1)_t, 0, k^i).$$

This and (4.14) allow us to deduce that $-\langle \lambda^i, (-1)_t\rangle > 0$, a contradiction

Next, assume to the contrary that there exists $x \in S$ such that ii) is satisfied with a strict inequality. Then we get

$$f_i(0) + \omega((-1)_t, 0, k^i) > f_i(x) + \omega(tx - 1)_t, 0, k^i),$$

with at least an index $i \in \{1, 2\}$. Note that $\bar{x} = 0$ is an efficient solution of (MP$_1$). Since $\bar{\lambda}^i = 0$, by (2.1), we get $T(\lambda^i) = \emptyset$. Hence, from the inequality above, it is deduced that

$$f_i(0) > f_i(x).$$

If $x \in S \setminus A$ then $f_i(x) > 0 = f_i(0)$, a contradiction. If $x \in A$, it contradicts the fact that $\bar{x} = 0$ is an efficient solution of (MP$_1$). The proof is completed.

Acknowledgments

The first author was partially supported by the Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.01-2017.08. The second author was partially supported by the grant MOST 106-2115-M-037001, and a grant from the Research Center for Nonlinear Analysis and Optimization, Kaohsiung Medical University, Taiwan.

References

[1] Azimov AY, Gasimov RN. On weak conjugacy, weak subdifferentials and duality with zero gap in nonconvex optimization. International Journal of Applied Mathematics 1999; 4: 171-192.

[2] Azimov AY, Gasimov RN. Stability and duality of nonconvex problems via augmented Lagrangian. Cybernetics and Systems Analysis 2002; 38: 412-421.

[3] Clarke FH. Generalized gradient and applications. Transactions of the American Mathematical Society 1975; 205: 247-262.

[4] Gasimov RN. Augmented Lagrangian duality and nondifferentiable optimization methods in nonconvex programming. Journal of Global Optimization 2002; 24: 187-203.

[5] Gasimov RN, Rubinov AM. On augmented Lagrangians for optimization problems with a single constraint. Journal of Global Optimization 2004; 28: 153-173.

[6] Kasımbeyli R, İnceoğlu G. The properties of the weak subdifferentials. Gazi University Journal of Science 2010; 23: 49-52.

[7] Kasimbeyli R, Mamadov M. On weak subdifferentials, directional derivatives and radial epiderivatives for nonconvex functions. SIAM Journal on Optimization 2009; 20: 841-855.

[8] Kasimbeyli R, Mamadov M. Optimality conditions in nonconvex optimization via weak subdifferentials. Nonlinear Analysis 2011; 74: 2534-2547.

[9] Küçük Y, Ataserver I, Küçük M. Generalized weak subdifferentials. Optimization 2011; 60: 537-552.

[10] Küçük Y, Ataserver I, Küçük M. Weak Fenchel and weak Fenchel-Lagrange conjugate duality for nonconvex scalar optimization problems. Journal of Global Optimization 2012; 54: 813-830.

[11] Mordukhovich BS, Shao Y. On nonconvex subdifferentials calculus in Banach spaces. Journal of Convex Analysis 1995; 2: 211-227.

[12] Rockafellar RT. Proximal subgradients, marginal values, and augmented Lagrangians in nonconvex optimization. Mathematics of Operations Research 1981; 6: 424-436.

[13] Rockafellar RT. The Theory of Subgradients and Its Applications to Problems of Optimization: Convex and Nonconvex Functions. Berlin, Germany: Helderman, 1981.

[14] Rockafellar RT. Extensions of subgradient calculus with applications in optimization. Nonlinear Analysis 1985; 9: 665-698.

[15] Son TQ, Kim DS, Tam NN. Weak stability and strong duality of a class of nonconvex infinite programs via augmented Lagrangian. Journal of Global Optimization 2012; 53: 163-184.

[16] Son TQ, Wen CF. Weak-subdifferentials for vector functions and applications to multiobjective semi-infinite optimization problems. Applicable Analysis (in press). doi: 10.1080/00036811.2018.1514018

[17] Walk M. Theory of Duality in Mathematical Programming. Berlin, Germany: Springer-Verlag, 1989.