

An integral-boundary value problem for a partial differential equation of second order

Anar T. ASSANOVA* 

Department of Differential Equations, Institute of Mathematics and Mathematical Modeling, Almaty, Kazakhstan

Received: 30.03.2019

Accepted/Published Online: 30.05.2019

Final Version: 31.07.2019

Abstract: An integral-boundary value problem for a hyperbolic partial differential equation in two independent variables is considered. By introducing additional functional parameters, we investigate the solvability of the problem and develop an algorithm for finding its approximate solutions. The problem is reduced to an equivalent one, consisting of the Goursat problem for a hyperbolic equation with parameters and boundary value problems with an integral condition for ODEs with respect to the parameters entered. We propose an algorithm to find an approximate solution to the original problem, which is based on the algorithm for finding a solution to the equivalent problem. The convergence of the algorithms is proved. A coefficient criterion for the unique solvability of the integral-boundary value problem is established.

Key words: Hyperbolic equation of second order, integral-boundary value problem, parameter, algorithm, approximate solution

1. Introduction

On the domain $\Omega = [0, T] \times [0, \omega]$, we consider the integral-boundary value problem for the hyperbolic equation of second order

$$\frac{\partial^2 u}{\partial t \partial x} = A(t, x) \frac{\partial u}{\partial x} + B(t, x) \frac{\partial u}{\partial t} + C(t, x)u + f(t, x), \quad (1)$$

$$P(t)u(t, 0) + \int_0^a K(t, \xi)u(t, \xi)d\xi = \psi(t), \quad t \in [0, T], \quad (2)$$

$$S(x)u(0, x) + \int_0^b M(\tau, x)u(\tau, x)d\tau = \varphi(x), \quad x \in [0, \omega], \quad (3)$$

where $u(t, x)$ is an unknown function; the functions $A(t, x)$, $B(t, x)$, $C(t, x)$, and $f(t, x)$ are continuous on Ω ; the functions $K(t, x)$ and $P(t)$, $\psi(t)$, are continuously differentiable with respect to t on Ω and $[0, T]$, respectively; the functions $M(t, x)$ and $S(x)$, $\varphi(x)$, are continuously differentiable with respect to x on Ω and $[0, \omega]$, respectively; and $0 < a \leq \omega$, $0 < b \leq T$. The compatibility condition is given below.

Let $C(\Omega, R)$ be a space of continuous functions $u : \Omega \rightarrow R$ with norm $\|u\|_0 = \max_{(t,x) \in \Omega} |u(t, x)|$.

*Correspondence: assanova@math.kz

2010 AMS Mathematics Subject Classification: 35L51, 35L53, 35R30, 34B10

A function $u(t, x) \in C(\Omega, R)$, whose partial derivatives $\frac{\partial u(t, x)}{\partial x}$, $\frac{\partial u(t, x)}{\partial t}$, and $\frac{\partial^2 u(t, x)}{\partial t \partial x}$ belong to $C(\Omega, R)$, is called a classical solution to problem (1)–(3), if it satisfies equation (1) and integral conditions (2) and (3).

Integral-boundary value problems for hyperbolic equations often arise in the mathematical modeling of processes of heat distribution, plasma physics, moisture transfer in capillary-porous media, clean technology of silicon ores, etc. [8–15, 17–26, 28–31]. Some types of integral-boundary value problems for hyperbolic equations were studied in [8–31]. Solvability conditions for these problems are established in different terms. Problem (1)–(3) for $P(t) = 0$, $S(x) = 0$, $K(t, x) = M(t, x)$, and $K(t, x) = M(t, x) = 1$ was investigated in [10, 13, 22–26]. Under the assumption of continuous differentiability of the equation coefficients, the conditions for the unique solvability of that problem have been obtained. In [31], the contractive mapping principle is used to study problem (1)–(3) for $P(t) = 0$, $S(x) = 0$ and $K(t, x) = K(x)$, $M(t, x) = M(t)$. In [7], the unique solvability conditions for the problem, where $P(t) = 0$, $S(x) = 0$, and $K(t, x) = M(t, x)$, were established in terms of initial data.

A boundary value problem for hyperbolic equations subject to general integral conditions is one of the rarely studied problems of mathematical physics. This formulation of the problem is considered for the first time.

The aim of this work is to develop an algorithm for finding a solution to problem (1)–(3) and establish conditions for the existence and uniqueness of its classical solution.

In Section 2, a scheme of the method from [4–6] is presented, which will be used to investigate the problem. We introduce new unknown functions as linear combinations of the solutions' values on the characteristics. The problem (1)–(3) is reduced to an equivalent one consisting of the Goursat problem for hyperbolic equations with functional parameters and boundary value problems with integral conditions for ODEs with respect to the parameters entered. An algorithm for finding an approximate solution to the problem under investigation is proposed. The algorithm consists of two parts. First, we solve two boundary value problems with integral conditions for ODEs. Such problems have been intensively studied in recent years [1–3] and they occur in numerous areas of applied mathematics. In the second part of the algorithm, we solve the Goursat problem for a hyperbolic equation with parameters. In Section 3, the conditions for the existence of a unique solution to the boundary value problem with integral condition for ODEs are provided. In Section 3, the convergence of the algorithm is proved, and the conditions for the unique solvability of problem (1)–(3) are given in terms of initial data.

2. The description of the method and the algorithm

We introduce the following notations: $\mu(t) = u(t, 0) - \frac{1}{2}u(0, 0)$, $\lambda(x) = u(0, x) - \frac{1}{2}u(0, 0)$, $\tilde{u}(t, x)$, where the latter is a new unknown function.

We make the following replacement of desired function $u(t, x)$ in problem (1)–(3):

$u(t, x) = \tilde{u}(t, x) + \mu(t) + \lambda(x)$, and we move to the problem:

$$\frac{\partial^2 \tilde{u}}{\partial t \partial x} = A(t, x) \frac{\partial \tilde{u}}{\partial x} + B(t, x) \frac{\partial \tilde{u}}{\partial t} + C(t, x) \tilde{u} + A(t, x) \dot{\lambda}(x) + B(t, x) \dot{\mu}(t) + C(t, x) \lambda(x) + C(t, x) \mu(t) + f(t, x), \quad (4)$$

$$\tilde{u}(t, 0) = 0, \quad t \in [0, T], \quad (5)$$

$$\tilde{u}(0, x) = 0, \quad x \in [0, \omega], \quad (6)$$

$$\left[P(t) + \int_0^a K(t, \xi) d\xi \right] \mu(t) + P(t)\lambda(0) + \int_0^a K(t, \xi)\lambda(\xi) d\xi + \int_0^a K(t, \xi)\tilde{u}(t, \xi) d\xi = \psi(t), \quad t \in [0, T], \quad (7)$$

$$\left[S(x) + \int_0^b M(\tau, x) d\tau \right] \lambda(x) + S(x)\mu(0) + \int_0^b M(\tau, x)\mu(\tau) d\tau + \int_0^b M(\tau, x)\tilde{u}(\tau, x) d\tau = \varphi(x), \quad x \in [0, \omega]. \quad (8)$$

A triple of functions $(\tilde{u}(t, x), \mu(t), \lambda(x))$ satisfying the hyperbolic equation (4), the conditions on characteristics (5) and (6), and the functional relations (7) and (8) for $\mu(0) = \lambda(0)$ is called a solution to problem (4)–(8) if the function $\tilde{u}(t, x) \in C(\Omega, R)$ has the partial derivatives $\frac{\partial \tilde{u}(t, x)}{\partial x}$, $\frac{\partial \tilde{u}(t, x)}{\partial t}$, $\frac{\partial^2 \tilde{u}(t, x)}{\partial t \partial x}$ in $C(\Omega, R)$, and the functions $\mu(t)$ and $\lambda(x)$ are continuously differentiable on $[0, T]$ and $[0, \omega]$, respectively.

The relation $\mu(0) = \lambda(0)$ is a compatibility condition of data.

Problem (4)–(8) is equivalent to problem (1)–(3). If the function $u^*(t, x)$ is a solution to problem (1)–(3), then the triple of functions $(\tilde{u}^*(t, x), \mu^*(t), \lambda^*(x))$, where $\tilde{u}^*(t, x) = u^*(t, x) - \mu^*(t) - \lambda^*(x)$, $\mu^*(t) = u^*(t, 0) - \frac{1}{2}u^*(0, 0)$, $\lambda^*(x) = u^*(0, x) - \frac{1}{2}u^*(0, 0)$, is a solution to problem (4)–(8). The converse is also true. If the triple of functions $(\tilde{u}^{**}(t, x), \mu^{**}(t), \lambda^{**}(x))$ is a solution to problem (4)–(8), then the function $u^{**}(t, x)$ defined by the equality

$$u^{**}(t, x) = \tilde{u}^{**}(t, x) + \mu^{**}(t) + \lambda^{**}(x),$$

where $u^{**}(t, 0) - \frac{1}{2}u^{**}(0, 0) = \mu^{**}(t)$, $u^{**}(0, x) - \frac{1}{2}u^{**}(0, 0) = \lambda^{**}(x)$, is a solution to problem (1)–(3).

For fixed $\mu(t)$, $\lambda(x)$, problem (4)–(6) is the Goursat problem with respect to the function $\tilde{u}(t, x)$ on the domain Ω . Relations (7) and (8) allow us to determine the unknown parameters $\mu(t)$, $\lambda(x)$ satisfying the condition $\mu(0) = \lambda(0)$.

By virtue of conditions (5) and (6), relations (7) at $t = 0$ and (8) at $x = 0$ yield

$$\left[P(0) + \int_0^a K(0, \xi) d\xi \right] \mu(0) + P(0)\lambda(0) + \int_0^a K(0, \xi)\lambda(\xi) d\xi = \psi(0), \quad (9)$$

$$\left[S(0) + \int_0^b M(\tau, 0) d\tau \right] \lambda(0) + S(0)\mu(0) + \int_0^b M(\tau, 0)\mu(\tau) d\tau = \varphi(0). \quad (10)$$

Taking into account $\lambda(0) = \mu(0)$, we get

$$\left[2P(0) + \int_0^a K(0, \xi) d\xi \right] \lambda(0) + \int_0^a K(0, \xi)\lambda(\xi) d\xi = \psi(0), \quad (11)$$

$$\left[2S(0) + \int_0^b M(\tau, 0) d\tau \right] \mu(0) + \int_0^b M(\tau, 0)\mu(\tau) d\tau = \varphi(0). \quad (12)$$

Assumptions on the data of problem (1)–(3) allow us to differentiate (7) and (8) with respect to t and x , respectively. We then obtain

$$\left[P(t) + \int_0^a K(t, \xi) d\xi \right] \dot{\mu}(t) = - \left[\dot{P}(t) + \int_0^a \frac{\partial K(t, \xi)}{\partial t} d\xi \right] \mu(t) - \int_0^a \frac{\partial K(t, \xi)}{\partial t} \tilde{u}(t, \xi) d\xi -$$

$$-\int_0^a K(t, \xi) \frac{\partial \tilde{u}(t, \xi)}{\partial t} d\xi - \dot{P}(t)\lambda(0) - \int_0^a \frac{\partial K(t, \xi)}{\partial t} \lambda(\xi) d\xi + \dot{\psi}(t), \quad t \in [0, T], \quad (13)$$

$$\begin{aligned} \left[S(x) + \int_0^b M(\tau, x) d\tau \right] \dot{\lambda}(x) = & - \left[\dot{S}(x) + \int_0^b \frac{\partial M(\tau, x)}{\partial x} d\tau \right] \lambda(x) - \int_0^b \frac{\partial M(\tau, x)}{\partial x} \tilde{u}(\tau, x) d\tau - \\ & - \int_0^b M(\tau, x) \frac{\partial \tilde{u}(\tau, x)}{\partial x} d\tau - \dot{S}(x)\mu(0) - \int_0^b \frac{\partial M(\tau, x)}{\partial x} \mu(\tau) d\tau + \dot{\varphi}(x), \quad x \in [0, \omega]. \end{aligned} \quad (14)$$

We introduce new unknown functions $\tilde{v}(t, x) = \frac{\partial \tilde{u}(t, x)}{\partial x}$, $\tilde{w}(t, x) = \frac{\partial \tilde{u}(t, x)}{\partial t}$ and the following notations:

$$B_1(t) = P(t) + \int_0^a K(t, \xi) d\xi, \quad B_2(x) = S(x) + \int_0^b M(\tau, x) d\tau, \quad C_1(t) = \dot{P}(t) + \int_0^a \frac{\partial K(t, \xi)}{\partial t} d\xi,$$

$$C_2(x) = \dot{S}(x) + \int_0^b \frac{\partial M(\tau, x)}{\partial x} d\tau, \quad G_1(t, \tilde{u}, \tilde{w}) = \int_0^a \frac{\partial K(t, \xi)}{\partial t} \tilde{u}(t, \xi) d\xi + \int_0^a K(t, \xi) \tilde{w}(t, \xi) d\xi,$$

$$G_2(x, \tilde{u}, \tilde{v}) = \int_0^b \frac{\partial M(\tau, x)}{\partial x} \tilde{u}(\tau, x) d\tau + \int_0^b M(\tau, x) \tilde{v}(\tau, x) d\tau,$$

$$L_1(t, \lambda) = \dot{P}(t)\lambda(0) + \int_0^a \frac{\partial K(t, \xi)}{\partial t} \lambda(\xi) d\xi, \quad L_2(x, \mu) = \dot{S}(x)\mu(0) + \int_0^b \frac{\partial M(\tau, x)}{\partial x} \mu(\tau) d\tau.$$

Then equations (13) and (14) can be written in the following forms:

$$B_1(t)\dot{\mu}(t) = -C_1(t)\mu(t) - G_1(t, \tilde{u}, \tilde{w}) - L_1(t, \lambda) + \dot{\psi}(t), \quad t \in [0, T], \quad (15)$$

$$B_2(x)\dot{\lambda}(x) = -C_2(x)\lambda(x) - G_2(x, \tilde{u}, \tilde{v}) - L_2(x, \mu) + \dot{\varphi}(x), \quad x \in [0, \omega]. \quad (16)$$

Thus, we have a closed system of equations (4)–(6), (15), (12), (16), and (11) for determining unknown functions $\tilde{v}(t, x)$, $\tilde{w}(t, x)$, $\tilde{u}(t, x)$, $\dot{\lambda}(x)$, $\lambda(x)$, $\dot{\mu}(t)$, and $\mu(t)$.

Relation (15) in conjunction with (12) presents a boundary value problem with integral condition for a differential equation with respect to $\mu(t)$, and the relation (16) in conjunction with (11) presents a boundary value problem with integral condition for a differential equation with respect to $\lambda(x)$.

The boundary value problem with integral condition (15) and (12) is equivalent to relation (7), and the boundary value problem with integral condition (16) and (11) is equivalent to relation (8) for $\mu(0) = \lambda(0)$.

If $\dot{\mu}(t)$, $\dot{\lambda}(x)$, $\mu(t)$, $\lambda(x)$ are known, we can find the functions $\tilde{v}(t, x)$, $\tilde{w}(t, x)$, $\tilde{u}(t, x)$ from (4)–(6). Conversely, if we know the functions $\tilde{v}(t, x)$, $\tilde{w}(t, x)$, $\tilde{u}(t, x)$, then we can find $\dot{\mu}(t)$, $\mu(t)$, $\dot{\lambda}(x)$, $\lambda(x)$ from boundary value problems (15), (12) and (16), (11). The unknowns are both $\tilde{v}(t, x)$, $\tilde{w}(t, x)$, $\tilde{u}(t, x)$, and $\dot{\mu}(t)$, $\mu(t)$, $\dot{\lambda}(x)$, $\lambda(x)$. Therefore, to find a solution of problem (4)–(8), we use an iterative method: determine the triple $(\tilde{u}^*(t, x), \mu^*(t), \lambda^*(x))$ as the limit of the sequence $(\tilde{u}^{(m)}(t, x), \mu^{(m)}(t), \lambda^{(m)}(x))$, $m = 0, 1, 2, \dots$, according to the following algorithm:

Step 0. 1) Assuming $\tilde{u}(t, x) = 0$, $\tilde{w}(t, x) = 0$, $\lambda(x) = 0$, on the right-hand side of equation (15), we find initial approximations $\dot{\mu}^{(0)}(t)$, $\mu^{(0)}(t)$, $t \in [0, T]$, from the boundary value problem with integral condition (15) and (12). Assuming $\tilde{u}(t, x) = 0$, $\tilde{v}(t, x) = 0$, $\mu(t) = 0$ on the right-hand side of equation (16), we find initial

approximations $\dot{\lambda}^{(0)}(x)$, $\lambda^{(0)}(x)$, $x \in [0, \omega]$, from the boundary value problem with integral condition (16) and (11).

2) Find $\tilde{v}^{(0)}(t, x)$, $\tilde{w}^{(0)}(t, x)$, $\tilde{u}^{(0)}(t, x)$, $(t, x) \in \Omega$ from the Goursat problem (4)–(6) for $\dot{\lambda}(x) = \dot{\lambda}^{(0)}(x)$, $\dot{\mu}(t) = \dot{\mu}^{(0)}(t)$, $\lambda(x) = \lambda^{(0)}(x)$, $\mu(t) = \mu^{(0)}(t)$.

Step 1. 1) Assuming $\tilde{u}(t, x) = \tilde{u}^{(0)}(t, x)$, $\tilde{w}(t, x) = \tilde{w}^{(0)}(t, x)$, $\lambda(x) = \lambda^{(0)}(x)$ on the right-hand side of equation (15), we find $\dot{\mu}^{(1)}(t)$, $\mu^{(1)}(t)$, $t \in [0, T]$ from the boundary value problem with integral condition (15) and (12). Assuming $\tilde{u}(t, x) = \tilde{u}^{(0)}(t, x)$, $\tilde{v}(t, x) = \tilde{v}^{(0)}(t, x)$, $\mu(t) = \mu^{(0)}(t)$ on the right-hand side of equation (16), we find $\dot{\lambda}^{(1)}(x)$, $\lambda^{(1)}(x)$, $x \in [0, \omega]$, from the boundary value problem with integral condition (16) and (11).

2) Find $\tilde{v}^{(1)}(t, x)$, $\tilde{w}^{(1)}(t, x)$, $\tilde{u}^{(1)}(t, x)$, $(t, x) \in \Omega$ from the Goursat problem (4)–(6) for $\dot{\lambda}(x) = \dot{\lambda}^{(1)}(x)$, $\dot{\mu}(t) = \dot{\mu}^{(1)}(t)$, $\lambda(x) = \lambda^{(1)}(x)$, $\mu(t) = \mu^{(1)}(t)$, and so on.

Step m. 1) Assuming $\tilde{u}(t, x) = \tilde{u}^{(m-1)}(t, x)$, $\tilde{w}(t, x) = \tilde{w}^{(m-1)}(t, x)$, $\lambda(x) = \lambda^{(m-1)}(x)$ on the right-hand side of equation (15), we find $\dot{\mu}^{(m)}(t)$, $\mu^{(m)}(t)$, $t \in [0, T]$, from the boundary value problem with integral condition (15) and (12). Assuming $\tilde{u}(t, x) = \tilde{u}^{(m-1)}(t, x)$, $\tilde{v}(t, x) = \tilde{v}^{(m-1)}(t, x)$, $\mu(t) = \mu^{(m-1)}(t)$ on the right-hand side of equation (16), we find $\dot{\lambda}^{(m)}(x)$, $\lambda^{(m)}(x)$, $x \in [0, \omega]$, from the boundary value problem with integral condition (16) and (11).

2) Find $\tilde{v}^{(m)}(t, x)$, $\tilde{w}^{(m)}(t, x)$, $\tilde{u}^{(m)}(t, x)$, $(t, x) \in \Omega$, from the Goursat problem (4)–(6) for $\dot{\lambda}(x) = \dot{\lambda}^{(m)}(x)$, $\dot{\mu}(t) = \dot{\mu}^{(m)}(t)$, $\lambda(x) = \lambda^{(m)}(x)$, $\mu(t) = \mu^{(m)}(t)$, $m = 1, 2, \dots$

The constructed algorithm consists of two parts: we solve the boundary value problems with integral condition for the ordinary differential equations (15), (12) and (16), (11) in the first part, and we solve the Goursat problem for hyperbolic equations with functional parameters in the second part.

3. Boundary value problems with integral conditions for the ordinary differential equations

Consider the boundary value problem with integral condition for the ordinary differential equations

$$\dot{\mu}(t) = A_1(t)\mu(t) + g_1(t), \quad t \in [0, T], \tag{17}$$

$$\left[2S(0) + \int_0^b M(\tau, 0)d\tau \right] \mu(0) + \int_0^b M(\tau, 0)\mu(\tau)d\tau = \varphi(0), \tag{18}$$

where the functions $A_1(t)$ and $g_1(t)$ are continuous on $[0, T]$, the function $M(t, x)$ is continuous on Ω , and $S(0)$ and $\varphi(0)$ are some constants, $0 < b \leq T$.

The function $\mu(t) \in C([0, T], R)$ having the derivative $\dot{\mu}(t) \in C([0, T], R)$ is called a solution to problem (17)–(18), if it satisfies ordinary differential equation (17) and boundary condition (18).

We also consider the boundary value problem with integral condition for the ordinary differential equation of the following type:

$$\dot{\lambda}(x) = A_2(x)\lambda(x) + g_2(x), \quad x \in [0, \omega], \tag{19}$$

$$\left[2P(0) + \int_0^a K(0, \xi)d\xi \right] \lambda(0) + \int_0^a K(0, \xi)\lambda(\xi)d\xi = \psi(0), \tag{20}$$

where the functions $A_2(x)$ and $g_2(x)$ are continuous on $[0, \omega]$, the function $K(t, x)$ is continuous on Ω , and $P(0)$ and $\psi(0)$ are some constants, $0 < a \leq \omega$.

The function $\lambda(x) \in C([0, \omega], R)$ having the derivative $\dot{\lambda}(x) \in C([0, \omega], R)$ is called a solution to problem (19)–(20), if it satisfies ordinary differential equation (19) and boundary condition (20).

Below we give conditions for the unique solvability of boundary value problems with integral condition (17), (18) and (19), (20).

Suppose

$$a_1(t) = \int_0^t A_1(\tau) d\tau, \quad a_2(x) = \int_0^x A_2(\xi) d\xi, \quad \alpha_1 = \max_{t \in [0, T]} |A_1(t)|, \quad \alpha_2 = \max_{x \in [0, \omega]} |A_2(x)|,$$

$$\tilde{a}_1 = 2S(0) + \int_0^b M(\tau, 0) [1 + e^{a_1(\tau)}] d\tau, \quad \tilde{a}_2 = 2P(0) + \int_0^a K(0, \xi) [1 + e^{a_2(\xi)}] d\xi.$$

Assume that $|\tilde{a}_1| \geq \delta_1 > 0$, $|\tilde{a}_2| \geq \delta_2 > 0$.

Then the solutions to problems (17), (18) and (19), (20) can be written in the following forms:

$$\mu(t) = \frac{e^{a_1(t)}}{\tilde{a}_1} \cdot \varphi(0) - \frac{e^{a_1(t)}}{\tilde{a}_1} \cdot \int_0^b M(\tau, 0) e^{a_1(\tau)} \int_0^\tau e^{-a_1(\tau_1)} g_1(\tau_1) d\tau_1 d\tau + e^{a_1(t)} \int_0^t e^{-a_1(\tau)} g_1(\tau) d\tau, \quad t \in [0, T], \quad (21)$$

and

$$\lambda(x) = \frac{e^{a_2(x)}}{\tilde{a}_2} \cdot \psi(0) - \frac{e^{a_2(x)}}{\tilde{a}_2} \cdot \int_0^a K(0, \xi) e^{a_2(\xi)} \int_0^\xi e^{-a_2(\xi_1)} g_2(\xi_1) d\xi_1 d\xi + e^{a_2(x)} \int_0^x e^{-a_2(\xi)} g_2(\xi) d\xi, \quad x \in [0, \omega], \quad (22)$$

respectively. Relations (21) and (22) follow from the representation of the solution to the Cauchy problem for the ordinary differential equations according to the qualitative theory of differential equations.

The following statements are true.

Theorem 1 Suppose that $|\tilde{a}_1| = \left| 2S(0) + \int_0^b M(\tau, 0) [1 + e^{a_1(\tau)}] d\tau \right| \geq \delta_1 > 0$, where $a_1(t) = \int_0^t A_1(\tau) d\tau$.

Then problem (17)–(18) has a unique solution $\mu^*(t) \in C([0, T], R)$ representable in the form of (21), and the estimate

$$\max_{t \in [0, T]} |\mu^*(t)| \leq \mathcal{K}_1 \max \left(\max_{t \in [0, T]} |g_1(t)|, |\varphi(0)| \right) \quad (23)$$

holds, where $\mathcal{K}_1 = \frac{1}{\delta_1} e^{\alpha_1 T} \left[1 + \max_{t \in [0, b]} |M(t, 0)| \cdot \frac{b^2}{2} e^{\alpha_1 b} \right] + T e^{\alpha_1 T}$.

Theorem 2 Suppose that $|\tilde{a}_2| = \left| 2P(0) + \int_0^a K(0, \xi) [1 + e^{a_2(\xi)}] d\xi \right| \geq \delta_2 > 0$, where $a_2(x) = \int_0^x A_2(\xi) d\xi$.

Then problem (19)–(20) has a unique solution $\lambda^*(x) \in C([0, \omega], R)$ representable in the form of (22), and the estimate

$$\max_{x \in [0, \omega]} |\lambda^*(x)| \leq \mathcal{K}_2 \max \left(\max_{x \in [0, \omega]} |g_2(x)|, |\psi(0)| \right) \quad (24)$$

holds, where $\mathcal{K}_2 = \frac{1}{\delta_2} e^{\alpha_2 \omega} \left[1 + \max_{x \in [0, a]} |K(0, x)| \cdot \frac{a^2}{2} e^{\alpha_2 a} \right] + \omega e^{\alpha_2 \omega}$.

4. Conditions for convergence of the algorithm and the main result

In Section 2, an algorithm for finding a solution to problem (4)–(8), which is equivalent to problem (1)–(3), is constructed. To formulate the main result, assume that

$$B_1(t) = P(t) + \int_0^a K(t, \xi)d\xi \neq 0 \text{ for all } t \in [0, T] \text{ and } B_2(x) = S(x) + \int_0^b M(\tau, x)d\tau \neq 0 \text{ for all } x \in [0, \omega].$$

We introduce the following notations:

$$\begin{aligned} \alpha &= \max_{(t,x) \in \Omega} |A(t, x)|, \quad \beta = \max_{(t,x) \in \Omega} |B(t, x)|, \quad \gamma = \max_{(t,x) \in \Omega} |C(t, x)|, \quad H = \alpha + \beta + \gamma, \quad \beta_1 = \max_{t \in [0, T]} |[B_1(t)]^{-1}|, \\ \beta_2 &= \max_{x \in [0, \omega]} |[B_2(x)]^{-1}|, \quad A_1(t) = -[B_1(t)]^{-1}C_1(t), \quad A_2(x) = -[B_2(x)]^{-1}C_2(x), \quad \kappa_1 = \max_{(t,x) \in \Omega} |K(t, x)|, \\ \kappa_2 &= \max_{(t,x) \in \Omega} \left| \frac{\partial K(t, x)}{\partial t} \right|, \quad \sigma_1 = \max_{(t,x) \in \Omega} |M(t, x)|, \quad \sigma_2 = \max_{(t,x) \in \Omega} \left| \frac{\partial M(t, x)}{\partial x} \right|, \quad \chi_1 = \max_{t \in [0, T]} |\dot{P}(t)|, \quad \chi_2 = \max_{x \in [0, \omega]} |\dot{S}(x)|, \\ l_1(a) &= \beta_1 \left\{ a(\kappa_1 + \kappa_2) \max(T, \omega)e^{H(T+\omega)} + a\kappa_2 + \chi_1 \right\}, \quad l_2(b) = \beta_2 \left\{ b(\sigma_1 + \sigma_2) \max(T, \omega)e^{H(T+\omega)} + b\sigma_2 + \chi_2 \right\}. \end{aligned}$$

In Section 3, the conditions for unique solvability of boundary value problems with integral condition (17), (18) and (19), (20) are established. For fixed $\tilde{v}(t, x)$, $\tilde{w}(t, x)$, $\tilde{u}(t, x)$ in each step of the algorithm we solve the boundary value problems with integral condition (15), (12) and (16), (11). For fixed $\dot{\lambda}(x)$, $\dot{\mu}(t)$, $\lambda(x)$, $\mu(t)$, we solve the Goursat problem (4), (6).

The following statement provides the conditions for the convergence of the proposed algorithm and the existence of a unique solution to problem (4)–(8).

Theorem 3 *Let:*

- (1) functions $A(t, x)$, $B(t, x)$, $C(t, x)$, and $f(t, x)$ be continuous on Ω ;
- (2) functions $K(t, x)$ and $P(t)$, $\psi(t)$ be continuously differentiable with respect to t on Ω and $[0, T]$, respectively, and functions $M(t, x)$ and $S(x)$, $\varphi(x)$ be continuously differentiable with respect to x on Ω and $[0, \omega]$, respectively;

- (3) $B_1(t) = P(t) + \int_0^a K(t, \xi)d\xi \neq 0$ for all $t \in [0, T]$, and $B_2(x) = S(x) + \int_0^b M(\tau, x)d\tau \neq 0$ for all $x \in [0, \omega]$;

(4) the following conditions be fulfilled:

$$|\tilde{a}_1| = \left| 2S(0) + \int_0^b M(\tau, 0)[1 + e^{a_1(\tau)}]d\tau \right| \geq \delta_1 > 0, \quad |\tilde{a}_2| = \left| 2P(0) + \int_0^a K(0, \xi)[1 + e^{a_2(\xi)}]d\xi \right| \geq \delta_2 > 0,$$

where $a_1(t) = -\int_0^t [B_1(\tau)]^{-1} \left[\dot{P}(\tau) + \int_0^a \frac{\partial K(\tau, \xi)}{\partial \tau} d\xi \right] d\tau$, $a_2(x) = -\int_0^x [B_2(\xi)]^{-1} \left[\dot{S}(\xi) + \int_0^b \frac{\partial M(\tau, \xi)}{\partial \xi} d\tau \right] d\xi$;

- (5) the inequality $q = \max(\mathcal{K}_1 l_1(a) + \mathcal{K}_2 l_2(b), (\alpha_1 \mathcal{K}_1 + 1)l_1(a), (\alpha_2 \mathcal{K}_2 + 1)l_2(b)) < 1$ hold.

Then problem (4)–(8) has a unique solution.

Proof Let conditions (1)–(3) of Theorem 3 be fulfilled. Use the 0th step of the algorithm and consider the following boundary value problem with integral condition

$$\dot{\mu}(t) = A_1(t)\mu(t) - [B_1(t)]^{-1}\dot{\psi}(t), \quad t \in [0, T], \tag{25}$$

$$\left[2S(0) + \int_0^b M(\tau, 0) d\tau \right] \mu(0) + \int_0^b M(\tau, 0) \mu(\tau) d\tau = \varphi(0). \tag{26}$$

$$\dot{\lambda}(x) = A_2(x)\lambda(x) - [B_2(x)]^{-1}\dot{\varphi}(x), \quad x \in [0, \omega], \tag{27}$$

$$\left[2P(0) + \int_0^a K(0, \xi) d\xi \right] \lambda(0) + \int_0^a K(0, \xi) \lambda(\xi) d\xi = \psi(0). \tag{28}$$

Condition (4) including the conditions of Theorems 1 and 2 yields the unique solvability of problems (17), (18) and (19), (20). Find initial approximations $\mu^{(0)}(t)$ and $\lambda^{(0)}(x)$ from the boundary value problems (17), (18) and (19), (20). Then, similarly to the estimates (23) and (24), for the functions $\mu^{(0)}(t)$, $\lambda^{(0)}(x)$ and their derivatives $\dot{\mu}^{(0)}(t)$, $\dot{\lambda}^{(0)}(x)$ the following estimates hold:

$$\max_{t \in [0, T]} |\mu^{(0)}(t)| \leq \mathcal{K}_1 \max \left(\max_{t \in [0, T]} |[B_1(t)]^{-1}\dot{\psi}(t)|, |\varphi(0)| \right), \tag{29}$$

$$\max_{t \in [0, T]} |\dot{\mu}^{(0)}(t)| \leq [\alpha_1 \mathcal{K}_1 + 1] \max \left(\max_{t \in [0, T]} |[B_1(t)]^{-1}\dot{\psi}(t)|, |\varphi(0)| \right). \tag{30}$$

$$\max_{x \in [0, \omega]} |\lambda^{(0)}(x)| \leq \mathcal{K}_2 \max \left(\max_{x \in [0, \omega]} |[B_2(x)]^{-1}F_1(x)|, |\psi(0)| \right), \tag{31}$$

$$\max_{x \in [0, \omega]} |\dot{\lambda}^{(0)}(x)| \leq (\alpha_2 \mathcal{K}_2 + 1) \max \left(\max_{x \in [0, \omega]} |[B_2(x)]^{-1}\dot{\varphi}(x)|, |\psi(0)| \right). \tag{32}$$

Solving the Goursat problem (4)–(6) for the found values of parameters, we find $\tilde{v}^{(0)}(t, x)$, $\tilde{w}^{(0)}(t, x)$, $\tilde{u}^{(0)}(t, x)$ for all $(t, x) \in \Omega$.

The following inequalities are valid:

$$|\tilde{v}^{(0)}(t, x)| \leq \max(T, \omega) e^{H(T+\omega)} \max_{(t,x) \in \Omega} |\tilde{f}(t, x)|,$$

$$|\tilde{w}^{(0)}(t, x)| \leq \max(T, \omega) e^{H(T+\omega)} \max_{(t,x) \in \Omega} |\tilde{f}(t, x)|,$$

$$|\tilde{u}^{(0)}(t, x)| \leq \max(T, \omega) e^{H(T+\omega)} \max_{(t,x) \in \Omega} |\tilde{f}(t, x)|,$$

where $\tilde{f}(t, x) = A(t, x)\dot{\lambda}^{(0)}(x) + B(t, x)\dot{\mu}^{(0)}(t) + C(t, x) [\lambda^{(0)}(x) + \mu^{(0)}(t)] + f(t, x)$.

Successively, we determine the functions $\mu^{(m)}(t)$, $\lambda^{(m)}(x)$, $\dot{\mu}^{(m)}(t)$, $\dot{\lambda}^{(m)}(x)$, $\tilde{v}^{(m)}(t, x)$, $\tilde{w}^{(m)}(t, x)$, $\tilde{u}^{(m)}(t, x)$ on the m th step of the algorithm and obtain $\mu^{(m+1)}(t)$, $\lambda^{(m+1)}(x)$, $\dot{\mu}^{(m+1)}(t)$, $\dot{\lambda}^{(m+1)}(x)$, $\tilde{v}^{(m+1)}(t, x)$, $\tilde{w}^{(m+1)}(t, x)$, $\tilde{u}^{(m+1)}(t, x)$ on the $(m + 1)$ th step, $m = 1, 2, \dots$

Evaluating the corresponding differences of successive approximations, we obtain

$$\max_{t \in [0, T]} |\mu^{(m+1)}(t) - \mu^{(m)}(t)| \leq$$

$$\leq \mathcal{K}_1 \max_{t \in [0, T]} |[B_1(t)]^{-1}| \left[|L_1(t, \lambda^{(m)} - \lambda^{(m-1)})| + |G_1(t, \tilde{u}^{(m)} - \tilde{u}^{(m-1)}, \tilde{w}^{(m)} - \tilde{w}^{(m-1)})| \right], \quad (33)$$

$$\begin{aligned} & \max_{x \in [0, \omega]} |\lambda^{(m+1)}(x) - \lambda^{(m)}(x)| \leq \\ & \leq \mathcal{K}_2 \max_{x \in [0, \omega]} |[B_2(x)]^{-1}| \left[|L_2(x, \mu^{(m)} - \mu^{(m-1)})| + |G_2(x, \tilde{u}^{(m)} - \tilde{u}^{(m-1)}, \tilde{v}^{(m)} - \tilde{v}^{(m-1)})| \right], \end{aligned} \quad (34)$$

$$\begin{aligned} \max_{t \in [0, T]} |\dot{\mu}^{(m+1)}(t) - \dot{\mu}^{(m)}(t)| & \leq [\alpha_1 \mathcal{K}_1 + 1] \max_{t \in [0, T]} |[B_1(t)]^{-1}| \left[|L_1(t, \lambda^{(m)} - \lambda^{(m-1)})| + \right. \\ & \left. + |G_1(t, \tilde{u}^{(m)} - \tilde{u}^{(m-1)}, \tilde{w}^{(m)} - \tilde{w}^{(m-1)})| \right], \end{aligned} \quad (35)$$

$$\begin{aligned} \max_{x \in [0, \omega]} |\dot{\lambda}^{(m+1)}(x) - \dot{\lambda}^{(m)}(x)| & \leq [\alpha_2 \mathcal{K}_2 + 1] \max_{x \in [0, \omega]} |[B_2(x)]^{-1}| \left[|L_2(x, \mu^{(m)} - \mu^{(m-1)})| + \right. \\ & \left. + |G_2(x, \tilde{u}^{(m)} - \tilde{u}^{(m-1)}, \tilde{v}^{(m)} - \tilde{v}^{(m-1)})| \right], \end{aligned} \quad (36)$$

$$\begin{aligned} & |\tilde{v}^{(m+1)}(t, x) - \tilde{v}^{(m)}(t, x)| \leq \\ & \leq \max(T, \omega) e^{H(T+\omega)} \left\{ \alpha \max_{x \in [0, \omega]} |\dot{\lambda}^{(m+1)}(x) - \dot{\lambda}^{(m)}(x)| + \beta \max_{t \in [0, T]} |\dot{\mu}^{(m+1)}(t) - \dot{\mu}^{(m)}(t)| + \right. \\ & \left. + \gamma \left[\max_{x \in [0, \omega]} |\lambda^{(m+1)}(x) - \lambda^{(m)}(x)| + \max_{t \in [0, T]} |\mu^{(m+1)}(t) - \mu^{(m)}(t)| \right] \right\}, \end{aligned} \quad (37)$$

$$\begin{aligned} & |\tilde{w}^{(m+1)}(t, x) - \tilde{w}^{(m)}(t, x)| \leq \\ & \leq \max(T, \omega) e^{H(T+\omega)} \left\{ \alpha \max_{x \in [0, \omega]} |\dot{\lambda}^{(m+1)}(x) - \dot{\lambda}^{(m)}(x)| + \beta \max_{t \in [0, T]} |\dot{\mu}^{(m+1)}(t) - \dot{\mu}^{(m)}(t)| + \right. \\ & \left. + \gamma \left[\max_{x \in [0, \omega]} |\lambda^{(m+1)}(x) - \lambda^{(m)}(x)| + \max_{t \in [0, T]} |\mu^{(m+1)}(t) - \mu^{(m)}(t)| \right] \right\}, \end{aligned} \quad (38)$$

$$\begin{aligned} & |\tilde{u}^{(m+1)}(t, x) - \tilde{u}^{(m)}(t, x)| \leq \\ & \leq \max(T, \omega) e^{H(T+\omega)} \left\{ \alpha \max_{x \in [0, \omega]} |\dot{\lambda}^{(m+1)}(x) - \dot{\lambda}^{(m)}(x)| + \beta \max_{t \in [0, T]} |\dot{\mu}^{(m+1)}(t) - \dot{\mu}^{(m)}(t)| + \right. \\ & \left. + \gamma \left[\max_{x \in [0, \omega]} |\lambda^{(m+1)}(x) - \lambda^{(m)}(x)| + \max_{t \in [0, T]} |\mu^{(m+1)}(t) - \mu^{(m)}(t)| \right] \right\}. \end{aligned} \quad (39)$$

Suppose

$$\begin{aligned} \Delta_{m+1} & = \max \left(\max_{x \in [0, \omega]} |\lambda^{(m+1)}(x) - \lambda^{(m)}(x)| + \max_{t \in [0, T]} |\mu^{(m+1)}(t) - \mu^{(m)}(t)|, \right. \\ & \left. \max_{x \in [0, \omega]} |\dot{\lambda}^{(m+1)}(x) - \dot{\lambda}^{(m)}(x)|, \max_{t \in [0, T]} |\dot{\mu}^{(m+1)}(t) - \dot{\mu}^{(m)}(t)| \right). \end{aligned}$$

Then, from relations (33)–(36), taking into account estimations (37)–(39) and using the notation introduced, we obtain the main inequality:

$$\Delta_{m+1} \leq q\Delta_m. \tag{40}$$

Condition (5) of Theorem 3 leads to the convergence of the sequence Δ_m to Δ_* as $m \rightarrow \infty$. This gives the uniform convergence of the sequences $\lambda^{(m)}(x)$, $\dot{\lambda}^{(m)}(x)$, $\mu^{(m)}(t)$, $\dot{\mu}^{(m)}(t)$ to $\lambda^*(x)$, $\dot{\lambda}^*(x)$, $\mu^*(t)$, $\dot{\mu}^*(t)$, respectively, as $m \rightarrow \infty$. The functions $\lambda^*(x)$ and $\mu^*(t)$ are continuous and continuously differentiable on $[0, \omega]$ and $[0, T]$, respectively. Based on estimates (37)–(39), we establish the uniform convergence of the sequences $\tilde{v}^{(m)}(t, x)$, $\tilde{w}^{(m)}(t, x)$, $\tilde{u}^{(m)}(t, x)$ to the functions $\tilde{v}^*(t, x)$, $\tilde{w}^*(t, x)$, $\tilde{u}^*(t, x)$, respectively, with respect to $(t, x) \in \Omega$. Obviously, the functions $\tilde{u}^*(t, x)$, $\tilde{v}^*(t, x)$, and $\tilde{w}^*(t, x)$ are continuous on Ω . Solving the problems on the $(m + 1)$ th step of the algorithm and passing to the limit as $m \rightarrow \infty$, we obtain that the functions $\tilde{u}^*(t, x)$, $\lambda^*(x)$, $\mu^*(t)$ together with their derivatives satisfy the Goursat problem (4)–(6) and boundary value problems with integral condition (15), (12) and (16), (11).

Carry out the inverse transition from problem (15), (12) to relation (7), and pass from problem (16), (11) to relation (8). Then the triple of functions $(\tilde{u}^*(t, x), \lambda^*(x), \mu^*(t))$ is a solution to problem (4)–(8).

Prove the uniqueness of a solution to problem (4)–(8). Let the function triples $(\tilde{u}^*(t, x), \lambda^*(x), \mu^*(t))$ and $(\tilde{u}^{**}(t, x), \lambda^{**}(x), \mu^{**}(t))$ be two solutions to the problem. We introduce the following notation:

$$\begin{aligned} \tilde{\Delta} = \max & \left(\max_{x \in [0, \omega]} |\lambda^*(x) - \lambda^{**}(x)| + \max_{t \in [0, T]} |\mu^*(t) - \mu^{**}(t)|, \right. \\ & \left. \max_{x \in [0, \omega]} |\dot{\lambda}^*(x) - \dot{\lambda}^{**}(x)|, \max_{t \in [0, T]} |\dot{\mu}^*(t) - \dot{\mu}^{**}(t)| \right). \end{aligned}$$

After calculation, analogically with (34)–(39), we get

$$\tilde{\Delta} \leq q\tilde{\Delta}. \tag{41}$$

By condition (5) of Theorem 3, we have $q < 1$. Then inequality (41) takes place only for $\tilde{\Delta} \equiv 0$, which implies $\lambda^*(x) = \lambda^{**}(x)$, $\mu^*(t) = \mu^{**}(t)$, and $\tilde{u}^*(t, x) = \tilde{u}^{**}(t, x)$. Therefore, the solution to problem (4)–(8) is unique. Theorem 3 is proved. □

From equivalence of problem (1)–(3) to problem (4)–(8), the next assertion follows.

Theorem 4 *Let conditions (1)–(5) of Theorem 3 be fulfilled.*

Then problem (1)–(3) has a unique classical solution.

Proof Conditions (1)–(5) of Theorem 3 imply the existence of a unique solution $(\tilde{u}^*(t, x), \lambda^*(x), \mu^*(t))$ to problem (4)–(8). According to the algorithm offered above, for each $m = 0, 1, 2, \dots$, this triple is determined as the limit of the triples sequence $(\tilde{u}^{(m)}(t, x), \mu^{(m)}(t), \lambda^{(m)}(x))$ as $m \rightarrow \infty$.

Then the solution to problem (1)–(3), the function $u^*(t, x)$, exists and is determined by the equality

$$u^*(t, x) = \tilde{u}^*(t, x) + \lambda^*(x) + \mu^*(t).$$

Theorem 4 is proved. □

Acknowledgments

The author thanks the referees for careful reading of the manuscript and for giving constructive comments that substantially helped improve the quality of the paper.

The results of the present article were partially supported by a grant from the Ministry of Education and Science of the Republic of Kazakhstan, No. AP05131220.

References

- [1] Abramov AA, Yukhno LF. Nonlinear eigenvalue problem for a system of ordinary differential equations subject to a nonlocal condition. *Computational Mathematics and Mathematical Physics* 2012; 52 (2): 213-218. doi: 10.1134/S0965542512020029
- [2] Abramov AA, Yukhno LF. Solving a system of linear ordinary differential equations with redundant conditions. *Computational Mathematics and Mathematical Physics* 2014; 54 (4): 598-603. doi: 10.1134/S0965542514040022
- [3] Abramov AA, Yukhno LF. A solution method for a nonlocal problem for a system of linear differential equations. *Computational Mathematics and Mathematical Physics* 2014; 54 (11): 1686-1689. doi: 10.1134/S0965542514110025
- [4] Asanova AT, Dzhumabaev DS. Unique solvability of the boundary value problem for systems of hyperbolic equations with data on the characteristics. *Computational Mathematics and Mathematical Physics* 2002; 42 (11): 1609-1621.
- [5] Asanova AT, Dzhumabaev DS. Unique solvability of nonlocal boundary value problems for systems of hyperbolic equations. *Differential Equations* 2003; 39 (10): 1414-1427. doi: 10.1023/B:DIEQ.0000017915.18858.d4
- [6] Asanova AT, Dzhumabaev DS. Well-posedness of nonlocal boundary value problems with integral condition for the system of hyperbolic equations. *Journal of Mathematical Analysis and Applications* 2013; 402 (1): 167-178. doi: 10.1016/j.jmaa.2013.01.012
- [7] Assanova AT. Nonlocal problem with integral conditions for the system of hyperbolic equations in the characteristic rectangle. *Russian Mathematics (Iz VUZ)* 2017; 61(5): 7-20. doi: 10.3103/S1066369X17050024
- [8] Bouziani A. Solution forte d'un probleme mixte avec conditions non locales pour une classe d'equations hyperboliques. *Bulletins de l'Academie royale des sciences, des lettres et des beaux-arts de Belgique* 1997; 8: 53-70 (in French).
- [9] Byszewski L. Existence and uniqueness of solutions of nonlocal problems for hyperbolic equation $u_{xt} = F(x, t, u, u_x)$. *Journal of Applied Mathematics and Stochastic Analysis* 1990; 3 (3): 163-168.
- [10] Golubeva ND, Pul'kina LS. A nonlocal problem with integral conditions. *Mathematical Notes* 1996; 59 (3): 326-328. doi: 10.1007/BF02308548
- [11] Il'kiv VS, Nytrebych ZM, Pukach PY. Boundary-value problems with integral conditions for a system of Lamé equations in the space of almost periodic functions. *Electronic Journal of Differential Equations* 2016; 2016 (304): 1-12.
- [12] Il'kiv VS, Ptashnyk BI. Problems for partial differential equations with nonlocal conditions. Metric approach to the problem of small denominators. *Ukrainian Mathematical Journal* 2006; 58 (12): 1847-1875. doi: 10.1007/s11253-006-0172-8
- [13] Kechina OM. Nonlocal problem for hyperbolic equation with conditions given into characteristic rectangle. *Vestnik SamGU Estestvonauchn Ser* 2009; 72 (6): 50-56 (in Russian).
- [14] Kiguradze T. Some boundary value problems for systems of linear partial differential equations of hyperbolic type. *Memoirs on Differential Equations and Mathematical Physics* 1994. 1: 1-144.
- [15] Kozhanov AI, Pul'kina LS. On the solvability of boundary value problems with a nonlocal boundary condition of integral form for multidimensional hyperbolic equations. *Differential Equations* 2006; 42 (9): 1233-1246. doi: 10.1134/S0012266106090023

- [16] Moiseev EI, Korzyuk VI, Kozlovskaya IS. Classical solution of a problem with an integral condition for the one-dimensional wave equation. *Differential Equations* 2014; 50 (10): 1364-1377. doi: 10.1134/S0012266114100103
- [17] Nakhushev AM. Approximate method of solving boundary-value problems for differential equations and its application to the dynamics of soil moisture and groundwater. *Differential Equations* 1982; 18 (1): 60-67.
- [18] Nakhushev AM. *Problems with Shift for Partial Differential Equations*. Moscow, Russia: Nauka, 2006 (in Russian).
- [19] Nakhusheva ZA. On a nonlocal problem for partial differential equations. *Differential Equations* 1986; 22 (1): 171-174 (in Russian).
- [20] Ptashnyk BI. *Ill-posed boundary value problems for partial differential equations*. Kyiv, Ukraine: Naukova Dumka, 1984 (in Russian).
- [21] Ptashnyk BYo, Il'kiv VS, Kmit' IYa, Polishchuk VM. *Nonlocal Boundary Value Problems for Partial Differential Equations*. Kyiv, Ukraine: Naukova Dumka, 2002 (in Ukrainian).
- [22] Pulkina LS. A nonlocal problem with integral conditions for the quasilinear hyperbolic equation. *Electronic Journal of Differential Equations* 1999; 1999 (45): 1-6.
- [23] Pulkina LS. The L_2 solvability of a nonlocal problem with integral conditions for a hyperbolic equation. *Differential Equations* 2000; 36 (2): 316-318.
- [24] Pulkina LS. A nonlocal problem with integral conditions for hyperbolic equations. *Mathematical Notes* 2001; 70 (1): 79-85.
- [25] Pul'kina LS. A nonlocal problem for a hyperbolic equation with integral conditions of the 1st kind with time-dependent kernels. *Russian Mathematics (Iz VUZ)* 2012; 58 (10): 26-37.
- [26] Pul'kina LS, Kechina OM. A nonlocal problem with integral conditions for hyperbolic equations in characteristic rectangle. *Vestnik SamGU Estestvonauchn Ser* 2009; 68 (2): 80-88 (in Russian).
- [27] Sabitov KB. Boundary value problem for a parabolic-hyperbolic equation with a nonlocal integral condition. *Differential Equations* 2010; 46 (10): 1472-1481. doi: 10.1134/S0012266110100113
- [28] Sabitova YuK. Nonlocal initial-boundary-value problem for a degenerate hyperbolic equation. *Russian Mathematics (Iz VUZ)* 2009; 53 (12): 49-58.
- [29] Sabitova YuK. Boundary-value problem with nonlocal integral condition for mixed type equations with degeneracy on the transition line. *Mathematical Notes* 2015; 98 (3): 454-465. doi: 10.1134/S0001434615090114
- [30] Tkach BP, Urmancheva LB. Numerical-analytical method for finding solutions of systems with distributed parameters and integral condition. *Nonlinear Oscillations* 2009; 12 (1): 110-119.
- [31] Zhestkov SV. The Goursat problem with integral boundary conditions. *Ukrainian Mathematical Journal* 1990; 42 (1): 119-122.