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Some applications of differential subordination for certain starlike functions

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Abstract: Let $\mathcal{S}^*(q_c)$ denote the class of functions f analytic in the open unit disc Δ, normalized by the condition $f(0) = 0 = f'(0) - 1$ and satisfying the following inequality

$$\left| \left(\frac{zf'(z)}{f(z)} \right)^2 - 1 \right| < c \quad (z \in \Delta, 0 < c \leq 1).$$

By use of the subordination principle for the univalent functions we have

$$f \in \mathcal{S}^*(q_c) \Leftrightarrow \frac{zf'(z)}{f(z)} \prec \sqrt{1 + cz} \quad (z \in \Delta, 0 < c \leq 1).$$

In the present paper, for an analytic function p in Δ with $p(0) = 1$ we give some conditions which imply $p(z) \prec \sqrt{1 + cz}$. These conditions are then used to obtain some corollaries for certain subclasses of analytic functions.

Key words: Analytic, univalent, subordination, Janowski starlike functions, Bernoulli lemniscate

1. Introduction

Let Δ be the open unit disc in the complex plane $\mathbb{C}$, i.e. $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{H}(\Delta)$ be the class of functions that are analytic in Δ. Also, let $\mathcal{A} \subset \mathcal{H}(\Delta)$ be the class of functions that have the following Taylor–Maclaurin series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \Delta).$$

Thus, if $f \in \mathcal{A}$, then it satisfies the following normalization condition

$$f(0) = 0 = f'(0) - 1.$$

The set of all univalent (one–to–one) functions f in Δ is denoted by $\mathcal{U}$. Let f and g belong to class $\mathcal{H}(\Delta)$. Then we say that a function f is subordinate to g, written by

$$f(z) \prec g(z) \quad \text{or} \quad f \prec g,$$

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if there exists a Schwarz function w with the following properties

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \Delta),$$

such that $f(z) = g(w(z))$ for all $z \in \Delta$. In particular, if $g \in \mathcal{U}$, then we have

$$f(z) \prec g(z) \Leftrightarrow (f(0) = g(0) \quad \text{and} \quad f(\Delta) \subset g(\Delta)).$$

Furthermore, we say that the function $f \in \mathcal{U}$ is starlike if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \Delta).$$

The familiar class of starlike functions in Δ is denoted by $\mathcal{S}^*$. Also the function $f \in \mathcal{U}$ is called convex if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \Delta).$$

We denote by $\mathcal{K}$ the class of convex functions in Δ. A function $f \in \mathcal{A}$ is said to be close-to-convex, if there exists a function $g \in \mathcal{K}$ such that

$$\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\} > 0 \quad (z \in \Delta).$$

The class of close-to-convex functions is denoted by $\mathcal{C}$. Note that $\mathcal{C} \subset \mathcal{U}$.

Let $c \in (0, 1]$. We say that the function $f \in \mathcal{A}$ belongs to the class $\mathcal{S}^*(q_c)$, if it satisfies the following condition

$$\left|\left(\frac{zf'(z)}{f(z)}\right)^2 - 1\right| < c \quad (z \in \Delta).$$

The class $\mathcal{S}^*(q_c)$ was introduced by Sokół, see [19]. Also, the class $\mathcal{S}^*(q_1) \equiv \mathcal{SL}^*$ was considered in [20]. In Geometric Function Theory there are many interesting subclasses of starlike functions which have been defined by subordination, see for example [3–8, 14, 16–18]. In the sequel we give a necessary and sufficient condition for the class $\mathcal{S}^*(q_c)$ by using the subordination.

Define

$$q_c(z) := \sqrt{1 + cz} \quad (z \in \Delta, c \in (0, 1]) \tag{1.1}$$

and Ω_c by

$$\Omega_c := \{\zeta \in \mathbb{C} : \operatorname{Re}\{\zeta\} > 0, |\zeta^2 - 1| < c\}.$$

Then we have $q_c(\Delta) = \Omega_c$, see [19]. Indeed, the function $q_c(z)$ maps Δ onto a set bounded by Bernoulli lemniscate. It is easy to see that $f \in \mathcal{S}^*(q_c)$ if and only if it satisfies the following differential subordination

$$\frac{zf'(z)}{f(z)} \prec q_c(z) \quad (z \in \Delta, c \in (0, 1]),$$

where q_c is defined by (1.1) and the branch of the square root is chosen to be $q_c(0) = 1$. Noting to the above we have $\mathcal{S}^*(q_c) \subset \mathcal{S}^*$. Another class that we are interested to study is the class $\mathcal{U}(c)$ which is defined as follows:

$$\mathcal{U}(c) := \left\{f \in \mathcal{A} : \left|\left(\frac{z}{f(z)}\right)^2 f'(z) - 1\right| \leq c, \ 0 < c \leq 1, z \in \Delta\right\}.$$

For each $c \in (0,1]$ we have $\mathcal{U}(c) \subset \mathcal{U}$, see [12]. Let A and B be two fixed constants such that $-1 \leq B < A \leq 1$. We denote by $\mathcal{S}^*[A,B]$ the class of Janowski starlike functions $f \in \mathcal{A}$ and satisfying the condition

$$\frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \Delta).$$

This class was introduced by Janowski [2]. We remark that $\mathcal{S}^*[1,-1]$ becomes the class of starlike functions.

Next, we recall a lemma, called Jack's lemma.

Lemma 1.1 (*see [1], see also [15, Lemma 1.3, p. 28]*) *Let w be a nonconstant function meromorphic in Δ with $w(0)=0$. If*

$$|w(z_0)| = \max\{|w(z)| : |z| \leq |z_0|\} \quad (z \in \Delta),$$

then there exists a real number k $(k \geq 1)$ such that $z_0 w'(z_0) = kw(z_0)$.

In this paper, for an analytic function $p(z)$ in the unit disk Δ we find some conditions that imply $p(z) \prec \sqrt{1+cz}$. Also, some interesting corollaries are obtained.

2. Main Results

We start with the following.

Theorem 2.1 *Let p be an analytic function in Δ with $p(0)=1$, $|A| \leq 1$, $|B| < 1$, $0 < c \leq 1$. Also let γ satisfy the following inequality*

$$\gamma \geq \frac{2(|A|+|B|)}{c(1-|B|)}(1+c). \tag{2.1}$$

Then the following subordination

$$1 + \gamma \frac{zp'(z)}{p(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \Delta) \tag{2.2}$$

implies that

$$p(z) \prec \sqrt{1+cz} \quad (z \in \Delta).$$

Proof Let γ satisfy the condition (2.1) and consider

$$F(z) := 1 + \gamma \frac{zp'(z)}{p(z)} \tag{2.3}$$

for all $z \in \Delta$. Define the function w by the relation

$$p(z) = \sqrt{1+cw(z)} = 1 + p_1 z + p_2 z^2 + \cdots, \tag{2.4}$$

or $w(z) = (p^2(z)-1)/c = w_1 z + \cdots$. By the hypothesis, since p is analytic and $p(0)=1$, thus w is meromorphic in Δ and $w(0)=0$. We shall show that $|w(z)| < 1$ in Δ. With a simple calculation (2.4) gives

$$\gamma \frac{zp'(z)}{p(z)} = \frac{c\gamma zw'(z)}{2(1+cw(z))}.$$

Using the last equality in (2.3) we get

$$F(z)=1+\frac{c\gamma zw'(z)}{2(1+cw(z))}$$

and thus by computation we obtain

$$\frac{F(z)-1}{A-BF(z)}=\frac{c\gamma zw'(z)}{2A(1+cw(z))-B[2(1+cw(z))+c\gamma zw'(z)]}.$$

Now assume that there exists a point $z_0\in\Delta$ such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=1.$$

Therefore, by Lemma 1.1, there exists a number $k\ge 1$ such that $z_0w'(z_0)=kw(z_0)$. Without loss of generality we may assume that $w(z_0)=e^{i\delta}$ where $\delta\in[-\pi,\pi]$. For this z_0, we have

$$\begin{aligned}\left|\frac{F(z_0)-1}{A-BF(z_0)}\right| &= \left|\frac{ck\gamma e^{i\delta}}{2A(1+ce^{i\delta})-B[2(1+ce^{i\delta})+c\gamma ke^{i\delta}]}\right|\\ &\ge \frac{ck\gamma}{2|A||1+ce^{i\delta}|+|B||2+(2c+c\gamma k)e^{i\delta}|}\\ &= \frac{ck\gamma}{2|A|p_1(\delta)+|B|p_2(\delta)}\\ &=: H(\cos\delta)\end{aligned}$$

where the expressions $p_1(\delta)$ and $p_2(\delta)$ have a form

$$p_1(\delta)=\sqrt{1+2c\cos\delta+c^2},$$

$$p_2(\delta)=\sqrt{4+c(2+\gamma k)[4\cos\delta+c(2+\gamma k)]}$$

and

$$H(t)=\frac{ck\gamma}{2|A|\sqrt{1+2ct+c^2}+|B|\sqrt{4+4(2c+c\gamma k)t+(2c+c\gamma k)^2}}.$$

By a simple computation it can be easily seen that $H'(t)<0$. Thus, H is a decreasing function when $-1\le t=\cos\delta\le 1$ and consequently

$$H(t)\ge H(1)=\frac{ck\gamma}{2|A|(1+c)+|B|(2+2c+c\gamma k)}. \tag{2.5}$$

Now consider the function

$$L(k)=\frac{ck\gamma}{2|A|(1+c)+|B|(2+2c+c\gamma k)}\quad (k\ge 1). \tag{2.6}$$

It is easy to see that $L'(k)>0$. In conclusion,

$$L(k)\ge L(1)=\frac{c\gamma}{2|A|(1+c)+|B|(2+2c+c\gamma)}. \tag{2.7}$$

Finally from the definition of H and from (2.5)–(2.7), it follows that

$$\left|\frac{F(z_0)-1}{A-BF(z_0)}\right| \geq \frac{c\gamma}{2|A|(1+c)+|B|(2(1+c)+c\gamma)} =: T(A,B,c,\gamma).$$

The inequality (2.1) implies that $T(A,B,c,\gamma)>1$. However, this is a contradiction with the assumption (2.2). This is the end of the proof. □

If we put $p(z)=zf'(z)/f(z)$ in Theorem 2.1, then we obtain the following result:

Corollary 2.1 *Let* $|A|\leq 1$, $|B|<1$, $0<c\leq 1$ *and let*

$$\gamma \geq \frac{2(|A|+|B|)}{c(1-|B|)}(1+c).$$

If f *satisfies the subordination*

$$1+\gamma\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz} \quad (z\in\Delta),$$

then $f\in\mathcal{S}^*(q_c)$.

If we let $c=1$ in Corollary 2.1, then we have:

Corollary 2.2 *Let* $|A|\leq 1$, $B|<1$ *and let*

$$\gamma \geq \frac{4(|A|+|B|)}{1-|B|}.$$

If f *satisfies the following subordination*

$$1+\gamma\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right) \prec \frac{1+Az}{1+Bz} \quad (z\in\Delta),$$

then $f\in\mathcal{SL}^*$.

Taking $A=1$ and $B=0$ in Corollary 2.2, we obtain:

Corollary 2.3 *Let* $\gamma\geq 4$. *If* f *satisfies the following inequality*

$$\mathrm{Re}\left\{1+\gamma\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right\}>0$$

for all $z\in\Delta$, *then* $f\in\mathcal{SL}^*$.

If we put $p(z)=z\sqrt{f'(z)}/f(z)$ in Theorem 2.1, then we have:

Corollary 2.4 *Let* $|A|\leq 1$, $|B|<1$, $0<c\leq 1$ *and let*

$$\gamma \geq \frac{2(|A|+|B|)}{c(1-|B|)}(1+c).$$

If the function f satisfies the following condition

$$1+\gamma\left(1+\frac{1}{2}\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\prec\frac{1+Az}{1+Bz}\quad(z\in\Delta),$$

then

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|\leq c\quad(z\in\Delta).$$

This means that $f\in\mathcal{U}(c)$, hence it is univalent in Δ.

If we put $p(z)=\sqrt{f'(z)}$ and $c=1$ in Theorem 2.1, then we have the following result:

Corollary 2.5 *Assume that $|A|\leq 1$, $|B|<1$ and that*

$$\gamma\geq\frac{4(|A|+|B|)}{1-|B|}.$$

If

$$1+\gamma\left(\frac{1}{2}\frac{zf''(z)}{f'(z)}\right)\prec\frac{1+Az}{1+Bz}\quad(z\in\Delta),$$

then f is univalent in Δ by [13].

If we put $p(z)=f(z)/z$ in Theorem 2.1, then we obtain the following result.

Corollary 2.6 *Let $|A|\leq 1$, $|B|<1$, $0<c\leq 1$ and let*

$$\gamma\geq\frac{2(|A|+|B|)}{c(1-|B|)}(1+c).$$

If the function f satisfies the following condition

$$1+\gamma\left(\frac{zf'(z)}{f(z)}-1\right)\prec\frac{1+Az}{1+Bz}\quad(z\in\Delta),$$

then

$$\left|\left(\frac{f(z)}{z}\right)^2-1\right|<c$$

for all $z\in\Delta$.

Taking $A=1$ and $B=0$ in Corollary 2.6, we obtain:

Corollary 2.7 *Let $\gamma\geq 2(1+1/c)$ with $c\in(0,1]$. If the following inequality holds*

$$\mathrm{Re}\left\{1+\gamma\left(\frac{zf'(z)}{f(z)}-1\right)\right\}>0\quad(z\in\Delta),$$

then

$$\left|\left(\frac{f(z)}{z}\right)^2-1\right|<c\quad(z\in\Delta).$$

For the proofs of next theorems we need a couple of lemmas.

Lemma 2.1 *([11]) Let q be univalent in the unit disk Δ and θ and ϕ be analytic in a domain $\mathbb{U}$ containing $q(\Delta)$ with $\phi(w) \neq 0$ when $w \in q(\Delta)$. Set $Q(z) = zq'(z)\phi(q(z))$, $h(z) = \theta(q(z)) + Q(z)$. Suppose that Q is starlike (univalent) in Δ, and*

$$\mathrm{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0 \quad (z \in \Delta).$$

If p is analytic in Δ, with $p(0) = q(0)$, $p(\Delta) \subset \mathbb{U}$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \tag{2.8}$$

then $p(z) \prec q(z)$, and q is the best dominant of (2.8).

Lemma 2.2 *(see [9], see also [10, p. 24]) Assume that $\mathcal{Q}$ is the set of analytic functions that are injective on $\overline{\Delta}\backslash E(f)$, where $E(f) : \{\omega : \omega \in \partial\Delta$ and $\lim_{z\to\omega} f(z) = \infty\}$, and are such that $f'(\omega) \neq 0$ for $(\omega \in \partial\Delta\backslash E(f)$. Let $\psi \in \mathcal{Q}$ with $\psi(0) = a$ and let $\varphi(z) = a + a_m z^m + \cdots$ be analytic in Δ with $\varphi(z) \not\equiv a$ and $m \in \mathbb{N}$. If $\varphi \not\prec \psi$ in Δ, then there exist points $z_0 = r_0 e^{i\theta} \in \Delta$ and $\omega_0 \in \partial\Delta\backslash E(\psi)$, for which $\varphi(|z| < r_0) \subset \psi(\Delta)$, $\varphi(z_0) = \psi(\omega_0)$ and $z_0\varphi'(z_0) = k\omega_0\psi'(\omega_0)$, for some $k \geq m$.*

Next we prove the following.

Theorem 2.2 *Let $p \in \mathcal{H}(\Delta)$ with $p(0) = 1$ and $c \in (0, 1]$. If the function p satisfies the subordination*

$$\frac{1}{3}p^3(z) + zp'(z) \prec \frac{1}{3}\left(\sqrt{1+cz}\right)^3 + \frac{cz}{2\sqrt{1+cz}} \quad (z \in \Delta), \tag{2.9}$$

then

$$p(z) \prec \sqrt{1+cz} \quad (z \in \Delta),$$

and the function $\sqrt{1+cz}$ is the best dominant of (2.9).

Proof Consider

$$q_c(z) = \sqrt{1+cz}, \quad \theta(\omega) = \frac{1}{3}\omega^3, \quad \phi(\omega) = 1.$$

We know that q_c is analytic and univalent in Δ. Also $q_c(0) = p(0) = 1$. Moreover, both functions $\theta(\omega)$ and $\phi(\omega)$ are analytic in the ω–plane with $\phi(\omega) \neq 0$. The function

$$Q(z) = zq_c'(z)\phi(q(z)) = \frac{cz}{2\sqrt{1+cz}} = zq_c'(z),$$

is a starlike function, because q_c is convex. If we put

$$h(z) = \theta(q_c(z)) + Q(z) = \frac{1}{3}q_c^3(z) + zq_c'(z), \tag{2.10}$$

then we have

$$\operatorname{Re}\left\{\frac{zh'(z)}{Q(z)}\right\} = \operatorname{Re}\left\{1+cz+\left(1+\frac{zq_c''(z)}{q_c'(z)}\right)\right\} > 1-c \geq 0$$

for all $z\in\Delta$. Therefore, the function h given by (2.10) is close-to-convex and univalent in Δ. Thus, by the Lemma 2.1 and (2.9), we find that $p(z)\prec q_c(z)$ and $q_c(z)$ is the best dominant of (2.9) so the desired conclusion follows. □

If we put $p(z)=zf'(z)/f(z)$, then we have the following result:

Corollary 2.8 *Let $c\in(0,1]$. If a function f satisfies the subordination*

$$\frac{1}{3}\left(\frac{zf'(z)}{f(z)}\right)^3+\left(1+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\left(\frac{zf'(z)}{f(z)}\right)\prec\frac{1}{3}\left(\sqrt{1+cz}\right)^3+\frac{cz}{2\sqrt{1+cz}},$$

then $f\in\mathcal{S}^*(q_c)$ where $z\in\Delta$.

Finally we prove the following:

Theorem 2.3 *Let $k\geq 1$ and $0<c\leq 1$. If $p\in\mathcal{H}(\Delta)$ with $p(0)=1$ and it satisfies the condition*

$$\operatorname{Re}\{p(z)(p(z)+zp'(z))\} > 1+c(1+k/2) \quad (z\in\Delta), \tag{2.11}$$

then

$$p(z)\prec\sqrt{1+cz} \quad (z\in\Delta).$$

Proof Suppose that $p(z)\not\prec q_c(z)=\sqrt{1+cz}$. Then there exist points z_0, $|z_0|<1$ and ω_0, $|\omega_0|=1$, $\omega_0\neq 1$ satisfying the following conditions

$$p(z_0)=q_c(\omega_0), \quad p(|z|<|z_0|)\subset q_c(\Delta) \quad \text{and} \quad |\omega_0|=1.$$

From Lemma 2.2, we find that there exists a number $k\geq 1$ such that

$$\{p(z_0)(p(z_0)+zp'(z_0))\} = \{q_c(\omega_0)(q_c(\omega_0)+k\omega_0 q_c'(\omega_0))\} = 1+c(1+k/2)\omega_0. \tag{2.12}$$

By setting $\omega_0=e^{i\delta}$, $\delta\in[-\pi,\pi]$ in (2.12), it can be easily seen that

$$\operatorname{Re}\{1+c(1+k/2)\omega_0\} = 1+c(1+k/2)\cos\delta \leq 1+c(1+k/2).$$

However, it contradicts our assumption (2.11) and consequently $p(z)\prec q_c(z)$ in Δ. □

If we let $p(z)=zf'(z)/f(z)$, then we have the following result:

Corollary 2.9 *Let $0<c\leq 1$ and let $k\geq 1$. If f satisfies the following inequality*

$$\operatorname{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)^2\left(2+\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right)\right\} > 1+c(1+k/2) \quad (z\in\Delta)$$

then $f\in\mathcal{S}^*(q_c)$.

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