Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 2000 – 2009
© TÜBİTAK
doi:10.3906/mat-1803-54

Multiplication modules with prime spectrum

Ortaç ÖNEŞ*, **Mustafa ALKAN**
Department of Mathematics, Faculty of Science, Akdeniz University, Antalya, Turkey

Received: 09.03.2018 • **Accepted/Published Online:** 19.06.2019 • **Final Version:** 31.07.2019

Abstract: The subject of this paper is the Zariski topology on a multiplication module M over a commutative ring R. We find a characterization for the radical submodule $rad_M(0)$ and also show that there are proper ideals $I_1, ..., I_n$ of R such that $rad_M(0) = rad_M((I_1...I_n)M)$. Finally, we prove that the spectrum $Spec(M)$ is irreducible if and only if M is the finite sum of its submodules, whose $\mathcal{T}$-radicals are prime in M.

Key words: Multiplication module, prime submodule, spectrum of module

1. Introduction

Throughout this study, R and M denote a commutative ring with identity and a unitary R-module, respectively. We also use $Spec(M)$ for the spectrum of prime submodules. In [4], the author investigated some properties of Zariski topology of multiplication modules. Motivated by this study, we generalize some important results in [4] and also give a characterization for the intersection of all prime submodules of M. Then M is said to be a multiplication R-module if for each submodule N of M, there exists an ideal I of R such that $N = IM$. For example, invertible ideals and projective ideals of R are multiplication R-modules. Since every cyclic module is a multiplication module and every finitely generated Artinian multiplication module is cyclic, there is a close relationship between multiplication modules and cyclic modules and so there are many studies related to these important concepts in module theory ([1, 4, 6, 8]).

A proper submodule P of an R-module M is said to be prime if for $a \in R$ and $m \in M$, $am \in P$ implies that $m \in P$ or $aM \subseteq P$. The radical of a submodule N in M denoted by $rad_M(N)$ is defined as the intersection of all prime submodules of M containing N.

In [11], $V(N)$ was defined as the set $\{P \in Spec(M) : N \subseteq P\}$ for any submodule N of an R-module M. Note that $V(M) = \emptyset$, $V(0) = Spec(M)$ and $\bigcap_{i\in\Lambda} V(N_i)$ is equivalent to $V\left(\sum_{i\in\Lambda} N_i\right)$ for any family of submodules N_i of M.

Let $\Gamma(M) = \{V(N) : N \text{ is a submodule of } M\}$. If $\Gamma(M)$ is closed under finite union, $\Gamma(M)$ satifies the axioms of closed subsets of a topological space. Then it is said that M is a module with a Zariski topology.

A topological space X is said to be Noetherian if the closed subsets of X satisfy the descending chain condition. X is said to be irreducible if $X \neq \emptyset$ and for every decomposition $X = X_1 \cup X_2$ with closed subsets X_1, $X_2 \subseteq X$, we have $X = X_1$ or $X = X_2$. $D \subseteq X$ is said to be dense in X if for every nonempty open

*Correspondence: ortacns@gmail.com
2010 *AMS Mathematics Subject Classification:* 13C13, 16D99

set $U \subseteq X$, $U \cap D \neq \emptyset$ holds. X is said to be quasi-compact if every open cover of X has a finite subcover ([7, 10]).

The aim of this paper is to study Zariski topology of multiplication modules over commutative ring with identity.

Section 2 is devoted to the study of a subspace associated with a submodule. We begin by giving a base for complement Zariski topology of a submodule N in a module M. We show that $rad_M(N) = rad_M(Rm_1 + ... + Rm_n)$, where $m_i \in M$ if $\mathcal{X}_N$ is quasicompact (Theorem 2.4). We also prove that $\mathcal{N}_N(0)$ is a prime submodule of M if and only if $\mathcal{X}_N$ is irreducible (Theorem 2.8). Moreover, we give equivalent conditions for $Spec(M)$ (Theorem 2.10).

In Section 3, we are interested in the relationships between the complement Zariski topologies and submodules of a module M to find some algebraic and topological tools for submodules and find some characterizations for the modules. We show that there are proper ideals $I_1, ..., I_n$ of R such that $rad_M(0) = rad_M((I_1...I_n)M)$, where M is a finitely generated multiplication R-module satisfying the $\mathcal{T}$-condition for every submodule (Theorem 3.9). Consequently, we prove that $\mathcal{X} = \bigcup_{i=1}^{n} \mathcal{X}_{I_iM}$, where $\mathcal{X}_{I_iM}$ is irreducible if and only if $M = \left(\sum_{i=1}^{n} I_i\right) M$ and $\mathcal{N}_{I_iM}(0)$ is a prime submodule of M (Theorem 3.10).

2. The subspace associated with a submodule

Let M be a multiplication R-module and let $N = IM$ be a submodule of M, where I is an ideal of R. Let $\mathcal{X}_N = Spec(M)\backslash V(IM)$ and $\tilde{V}(JM) = V(JM)\backslash V(IM)$, where J is an ideal of R. Then

$$\Gamma_N = \left\{\tilde{V}(JM) : J \text{ is an ideal of } R\right\}$$

satisfies the axioms for closed sets of a topological space on $\mathcal{X}_N$. We name this topology as the complement Zariski topology of N in M.

Example 2.1 *Let $R = \mathbb{Z}$, $M = 6\mathbb{Z}$ and $N = 30\mathbb{Z}$. Then M is a multiplication $\mathbb{Z}$-module. It is clear that $Spec(M) = \{6a\mathbb{Z} : a \in \mathbb{P}\}$, $V(30\mathbb{Z}) = \{30\mathbb{Z}\}$, $V(36\mathbb{Z}) = \{12\mathbb{Z}, 18\mathbb{Z}\}$ and $V(90\mathbb{Z}) = \{18\mathbb{Z}, 30\mathbb{Z}\}$, where $\mathbb{P}$ is the set of prime numbers. Thus we have*

$$\mathcal{X}_N = Spec(M)\backslash V(30\mathbb{Z}) = \{6a\mathbb{Z} : a \in \mathbb{P}\backslash\{5\}\},$$

$$\tilde{V}(36\mathbb{Z}) = V(36\mathbb{Z})\backslash V(30\mathbb{Z}) = \{12\mathbb{Z}, 18\mathbb{Z}\}\backslash\{30\mathbb{Z}\} = \{12\mathbb{Z}, 18\mathbb{Z}\},$$

$$\tilde{V}(90\mathbb{Z}) = V(90\mathbb{Z})\backslash V(30\mathbb{Z}) = \{18\mathbb{Z}, 30\mathbb{Z}\}\backslash\{30\mathbb{Z}\} = \{18\mathbb{Z}\}.$$

We fix the submodule N as $N = IM$, where I is an ideal of R, and the module M as a multiplication module in this section.

Lemma 2.2 *Let $N = IM$ be a submodule of a multiplication R-module M, where I is an ideal of R. For any ideal J of R, the set $(\mathcal{X}_N)^{JM} = \mathcal{X}_N\backslash\tilde{V}(JM)$ forms a base for the complement Zariski topology of N in M on $\mathcal{X}_N$.*

Proof If $\mathcal{X}_{IM}$ is empty, then $(\mathcal{X}_{IM})^{JM} = \emptyset$, which is the trivial case. Therefore, we can assume that $\mathcal{X}_{IM} \neq \emptyset$.

Let $U \subset \mathcal{X}_N$ be an open set. Let $\sum_{i\in\Lambda} Rm_i = N$, where $m_i \in M$. Then $U = \mathcal{X}_N \backslash \tilde{V}(JM)$, where JM is a submodule of M. Thus,

$$\begin{aligned} U &= \mathcal{X}_{IM} \backslash \tilde{V}\left(\sum_{i\in\Lambda} Rm_i\right) = \mathcal{X}_{IM} \backslash \bigcap_{i\in\Lambda} \tilde{V}(Rm_i) \\ &= \bigcup_{i\in\Lambda} (\mathcal{X}_{IM} \backslash \tilde{V}(Rm_i)) = \bigcup_{i\in\Lambda} (\mathcal{X}_{IM})^{Rm_i}. \end{aligned}$$

Thus, it is proved that $(\mathcal{X}_{IM})^{JM}$ is a base for the complement Zariski topology of N in M. □

We are interested in the properties of the complement Zariski topology of a submodule N of a module M. The following proposition reveals some connections between $\mathcal{X}_{IM}$ and a submodule IM. One can easily prove Proposition 2.3.

Proposition 2.3 *Let $N = IM$ be a submodule of a multiplication R-module M, where I is an ideal of R. The following statements hold:*

i) $(\mathcal{X}_N)^{JM} = \mathcal{X}_N \backslash \tilde{V}(JM) = Spec(M) \backslash V(IJM)$ for J is an ideal of R.

ii) $(\mathcal{X}_N)^{J_1M} \cap (\mathcal{X}_N)^{J_2M} = (\mathcal{X}_N)^{(J_1J_2)M}$ for every ideal J_1, J_2 of R.

iii) $(\mathcal{X}_N)^{JM} = \emptyset$ if and only if $rad_M(IJM) \subseteq rad_M(0)$ for every ideal J of R.

iv) $(\mathcal{X}_N)^{J_1M} = (\mathcal{X}_N)^{J_2M}$ if and only if $rad_M(IJ_1M) = rad_M(IJ_2M)$ for every ideal J_1, J_2 of R.

v) If $(\mathcal{X}_N)^{JM} = \mathcal{X}_N$, then we have $rad_M(IJM) = rad_M(IM)$ for every ideal J of R.

Let M be an R-module and let N be a proper submodule of M. Then we will say that N satisfies the condition $(*)$ if there is a finite subset Δ of Λ such that $rad_M(\langle\{m_i \in M : i \in \Lambda\}\rangle) = rad_M(\langle\{m_j : j \in \Delta\}\rangle)$, whenever $rad_M(N) \subseteq rad_M(\langle\{m_i \in M : i \in \Lambda\}\rangle)$. It is clear that if $M/rad_M(N)$ is a Noetherian module, N satisfies the condition $(*)$.

In the following theorem, we give an algebraic property belonging to a submodule N and a topological property belonging to $\mathcal{X}_N$.

Theorem 2.4 *Let $N = IM$ be a proper submodule of a multiplication R-module M, where I is an ideal of R. Let $(\mathcal{X}_N)^{JM} = \mathcal{X}_N \backslash \tilde{V}(JM) = Spec(M) \backslash V(IJM)$ for any ideal J of R. Then the following statements are true.*

i) $(\mathcal{X}_N)^{JM}$ is quasicompact for every ideal J of R.

ii) If $\mathcal{X}_N$ is quasicompact, then $rad_M(N) = rad_M(Rm_1 + ... + Rm_n)$, where $m_i \in M$.

iii) If N satisfies the condition $(*)$, then $\mathcal{X}_N$ is quasicompact.

Proof *i)* Obvious.

ii) Let $\mathcal{X}_N$ be quasicompact.

Let $N = \langle m_i : i \in \Lambda \rangle$. Then $V(\langle \{m_i : i \in \Lambda\} \rangle) = V(N)$ and so $\tilde{V}\left(\sum_{i \in \Lambda} Rm_i\right) = \emptyset$. Thus,

$$\begin{aligned} \mathcal{X}_N &= \mathcal{X}_N \backslash \emptyset = \mathcal{X}_N \backslash \tilde{V}\left\{\sum_{i \in \Lambda} Rm_i\right\} = \mathcal{X}_N \backslash \left(\bigcap_{i \in \Lambda} \tilde{V}(Rm_i)\right) \\ &= \bigcup_{i \in \Lambda} \left(\mathcal{X}_N \backslash \tilde{V}(Rm_i)\right) = \bigcup_{i \in \Lambda} (\mathcal{X}_N)^{Rm_i}. \end{aligned}$$

Since $\mathcal{X}_N$ is quasicompact, there is a finite set $\Delta = \{1, 2, ...n\} \subseteq \Lambda$ such that $\mathcal{X}_N = \bigcup_{i \in \Delta} (\mathcal{X}_N)^{Rm_i} = \mathcal{X}_N \backslash \tilde{V}(\langle m_1, m_2, ..., m_n \rangle)$. Then $V(\langle m_1, m_2, ..., m_n \rangle) \subseteq V(N)$ and so $rad_M(N) \subseteq rad_M(\langle m_1, m_2, ..., m_n \rangle)$. On the other hand, we have $rad_M(\langle m_1, m_2, ..., m_n \rangle) \subseteq rad_M(N)$, which means $rad_M(N) = rad_M(Rm_1 + ... + Rm_n)$.

iii) Let N satisfy the condition $(*)$.

Let $\{A_i : i \in \Lambda\}$ be an open cover of $\mathcal{X}_N$. Since A_i can be expressed as a union of the sets of $(\mathcal{X}_N)^{Rm_i}$, we may assume that $A_i = (\mathcal{X}_N)^{Rm_i}$ for every $i \in \Lambda$. Then

$$\begin{aligned} \mathcal{X}_N &= \bigcup_{i \in \Lambda} (\mathcal{X}_N)^{Rm_i} = \bigcup_{i \in \Lambda} \left(\mathcal{X}_N \backslash \tilde{V}(Rm_i)\right) \\ &= \mathcal{X}_N \backslash \bigcap_{i \in \Lambda} \tilde{V}(Rm_i) \\ &= \mathcal{X}_N \backslash \tilde{V}\left(\sum_{i \in \Lambda} Rm_i\right). \end{aligned}$$

Thus, $\tilde{V}\left(\sum_{i \in \Lambda} Rm_i\right) = \emptyset$ and so $\tilde{V}\left(\sum_{i \in \Lambda} Rm_i\right) \subseteq V(N)$.

In this case, $rad_M(N) \subseteq rad_M\left(\sum_{i \in \Lambda} Rm_i\right)$. By the condition $(*)$, there is a finite subset $\Delta \subseteq \Lambda$ such that $rad_M\left(\sum_{i \in \Lambda} Rm_i\right) = rad_M\left(\sum_{i \in \Delta} Rm_i\right)$. Then $V\left(\sum_{i \in \Delta} Rm_i\right) \subseteq V(N)$ and so $\tilde{V}\left(\sum_{i \in \Delta} Rm_i\right) = \emptyset$. Then

$$\begin{aligned} \mathcal{X}_N &= \mathcal{X}_N \backslash \tilde{V}\left(\sum_{i \in \Delta} Rm_i\right) = \mathcal{X}_N \backslash \bigcap_{i \in \Delta} \tilde{V}(Rm_i) \\ &= \bigcup_{i \in \Delta} \left(\mathcal{X}_N \backslash \tilde{V}(Rm_i)\right) = \bigcup_{i \in \Delta} (\mathcal{X}_N)^{Rm_i}. \end{aligned}$$

Since $\mathcal{X}_N$ is covered by a finite number $(\mathcal{X}_N)^{Rm_i}$, $\mathcal{X}_N$ is quasicompact. □

Theorem 2.4 also generalizes Theorem 3.7 in [4].

We now introduce the new submodule class which is a generalization of radical submodule of a module.

Definition 2.5 *Let N be a submodule of an R-module M. The set $\mathcal{N}_N(T)$ is defined as the intersection of all prime submodules containing submodule T which does not contain N.*

It is clear that $\mathcal{N}_N(T)$ is equivalent to the radical of a submodule T when $M = N$. Then $\mathcal{N}_N(T)$ is a generalization of radical submodule.

Example 2.6 *Let $M = \mathbb{Z}$ be an $\mathbb{Z}$-module. Let $N = 12\mathbb{Z}$ and $T = 20\mathbb{Z}$ be submodules of M. Then $\mathcal{N}_{12\mathbb{Z}}(20\mathbb{Z}) = 5\mathbb{Z}$ but $rad_{\mathbb{Z}}(20\mathbb{Z}) = 10\mathbb{Z}$. Thus $\mathcal{N}_{12\mathbb{Z}}(20\mathbb{Z})$ is different from $rad_{\mathbb{Z}}(20\mathbb{Z})$.*

The following lemma deals with algebraic properties of submodule $\mathcal{N}_N(T)$.

Lemma 2.7 *Let $N = IM$ be a proper submodule of a multiplication R-module M, where I is an ideal of R. The following statements are true:*

i) $\mathcal{N}_N(T)$ is a submodule of M.

ii) $\mathcal{N}_{N/K}(T/K) = \mathcal{N}_N(T)/K$, where $K \subseteq T$ is a submodule of M.

iii) $\mathcal{N}_N(0) = \mathcal{N}_{rad_M(N)}(0)$.

Proof The proof is straightforward. □

The following theorem gives a connection between topological property of the complement Zariski topology $\mathcal{X}_N$ and algebraic property of submodule $\mathcal{N}_N(0)$.

Theorem 2.8 *Let $N = IM$ be a proper submodule of a multiplication R-module M and $rad_M(IM) \neq rad_M(0)$. Then $\mathcal{N}_N(0)$ is a prime submodule of M if and only if $\mathcal{X}_N$ is irreducible.*

Proof Let $\mathcal{N}_N(0)$ be a prime submodule of M and K be a nonempty open subset of $\mathcal{X}_{IM}$. Then $K = \mathcal{X}_N \backslash \tilde{V}(JM) = Spec(M) \backslash (V(IM) \cup V(JM))$, where JM is a submodule of M. Take $P \in K$. Then we have $P \notin V(IM) \cup V(JM)$, which means that $IM \nsubseteq P$ and $JM \nsubseteq P$. Thus, $\mathcal{N}_N(0) \subseteq P$, so $JM \nsubseteq \mathcal{N}_N(0) \subseteq P$. This implies that $\mathcal{N}_N(0) \notin V(JM)$ and by the definition of $\mathcal{N}_N(0)$, we get $\mathcal{N}_N(0) \notin V(IM)$. Thus, $\mathcal{N}_N(0) \in K$. Therefore, any nonempty open subset of $\mathcal{X}_N$ contains $\mathcal{N}_N(0)$. This means that $\mathcal{X}_N$ is irreducible.

Let $\mathcal{X}_N$ be irreducible. Suppose that $\mathcal{N}_N(0)$ is not a prime submodule of M. Then there exists elements $a \in R$ and $m \in M$ such that $am \in \mathcal{N}_N(0)$, $m \notin \mathcal{N}_N(0)$ and $aM \subseteq \mathcal{N}_N(0)$.

Since $rad_M(N) = rad_M(IM) \neq rad_M(0)$ and $m \in M \backslash \mathcal{N}_N(0)$, it follows that $\tilde{V}(Rm) \neq \emptyset$ and $\tilde{V}(Rm) \neq \mathcal{X}_N$, which implies $(\mathcal{X}_N)^{Rm} \neq \emptyset$. This can also be used to prove that $(\mathcal{X}_N)^{aM}$ is a nonempty open subset. Therefore, we get

$$\begin{aligned} (\mathcal{X}_N)^{aM} \cap (\mathcal{X}_N)^{Rm} &= (\mathcal{X}_N)^{Rm \cap aM} \subseteq \mathcal{X}_N \backslash \tilde{V}(am) \\ &\subseteq \mathcal{X}_N \backslash \tilde{V}(\mathcal{N}_N(0)) \\ &= Spec(M) \backslash (V(\mathcal{N}_N(0)) \cup V(N)) = \emptyset. \end{aligned}$$

This contradicts the hypothesis. Thus, $\mathcal{N}_N(0)$ is a prime submodule of M. □

We now need a condition on the submodules $\mathcal{N}_N$, which helps us out with going further in finding more connections between topological space and module.

A module M is said to satisfy $\mathcal{T}$-condition for a submodule N, if for any chain $\mathcal{N}_N(U_1M) \subseteq \mathcal{N}_N(U_2M) \subseteq \mathcal{N}_N(U_3M) \subseteq \ldots$, where U_i is an ideal of R, there is an integer m such that $\mathcal{N}_N(U_mM) = \mathcal{N}_N(U_{m+i}M)$ for all positive integers i.

Theorem 2.9 *Let $N = IM$ be a proper submodule of a multiplication R-module M, where I is an ideal of R. Then the following statements are equivalent:*

i) M satisfies the $\mathcal{T}$-condition.

ii) $\mathcal{X}_N$ is a Noetherian topological space.

Proof $(i) \Rightarrow (ii)$ Assume that M satisfies the $\mathcal{T}$-condition. Take the sequence $\tilde{V}(U_1M) \supseteq \tilde{V}(U_2M) \supseteq \tilde{V}(U_3M) \supseteq ...$, where U_iM is a submodule of M. Then we have the sequence $\mathcal{N}_N(U_1M) \subseteq \mathcal{N}_N(U_2M) \subseteq \mathcal{N}_N(U_3M) \subseteq ...$ and there exists an integer m such that $\mathcal{N}_N(U_mM) = \mathcal{N}_N(U_{m+i}M)$ for all positive integers i since M satisfies the $\mathcal{T}$-condition. Therefore, we have $\tilde{V}(U_mM) = \tilde{V}(U_{m+i}M)$ for all positive integers i. Thus, $\mathcal{X}_N$ is Noetherian.

$(ii) \Rightarrow (i)$ Let $\mathcal{X}_N$ be a Noetherian topological space. Take the sequence $\mathcal{N}_N(U_1M) \subseteq \mathcal{N}_N(U_2M) \subseteq \mathcal{N}_N(U_3M)...$, where U_iM is a submodule of M. Then this yields the sequence $\tilde{V}(U_1M) \supseteq \tilde{V}(U_2M) \supseteq \tilde{V}(U_3M) \supseteq ....$ Since $\mathcal{X}_N$ is Noetherian, there exists an integer m such that $\tilde{V}(U_mM) = \tilde{V}(U_{m+i}M)$ for all positive integers i. This implies $\mathcal{N}_N(U_mM) = \mathcal{N}_N(U_{m+i}M)$ for all positive integers i. Therefore M satisfies the $\mathcal{T}$-condition. □

We close this section with the following theorem, which reveals the connections between algebraic and topological properties.

Theorem 2.10 *Let $N = IM$ be a submodule of a multiplication R-module M, where I is an ideal of R. Then the following are equivalent:*

i) $\mathcal{X}$ is a Noetherian topological space.

ii) $\mathcal{X}_N$ is a Noetherian topological space for every submodule N of M.

iii) M satisfies the $\mathcal{T}$-condition.

iv) M satisfies ascending chain condition on the radical submodules of M.

Proof $(i) \Rightarrow (ii)$, $(ii) \Leftrightarrow (iii)$ and $(iv) \Leftrightarrow$ (i) are clear.

$(ii) \Rightarrow (i)$ Take the sequence $V(U_1M) \supseteq V(U_2M) \supseteq V(U_3M) \supseteq ...$, where U_i is an ideal of R. Let $I = \cap U_i$ be an ideal of R. Consider the complement Zariski topology $\mathcal{X}_{IM}$. Then we have the sequence $\tilde{V}(U_1M) \supseteq \tilde{V}(U_2M) \supseteq \tilde{V}(U_3M) \supseteq ....$ Since $\mathcal{X}_N$ is Noetherian, there exists an integer m such that $\tilde{V}(U_mM) = \tilde{V}(U_{m+i}M)$ for all positive integers i. Thus, we have $V(U_mM) = V(U_{m+i}M)$ for all positive integers i. Thus, $\mathcal{X}$ is Noetherian. □

3. The connections between subspaces and submodules

This section deals with the relationships between the complement Zariski topologies and submodules of a module to find some algebraic and topological tools for submodules and find some characterizations for modules.

Theorem 3.1 *Let M be a multiplication R-module and let I, J, and K be proper ideals of R. Then we have the following.*

i) Any open set of $\mathcal{X}$ is of the form $\mathcal{X}_{IM}$.

ii) $\mathcal{X}_{IM} = \mathcal{X}_{JM}$ if and only if $rad_M(IM) = rad_M(JM)$.

iii) $\mathcal{X}_{IM} \cap \mathcal{X}_{JM} = \mathcal{X}_{KM}$ if and only if $rad_M(IJM) = rad_M(KM)$.

iv) $\mathcal{X}_{IM} \subseteq \mathcal{X}_{JM}$ *if and only if* $rad_M(IM) \subseteq rad_M(JM)$.

Proof It is straightforward. □

Corollary 3.2 *Let* M *be a multiplication* R*-module and let* I, J *be proper ideals of* R. *Then* $\mathcal{X}_{IM} \cap \mathcal{X}_{JM} = \emptyset$ *if and only if* $rad_M(IJM) = rad_M(0)$.

Theorem 3.3 *Let* M *be a multiplication* R*-module and let* I *be a proper ideal of* R. *Then* $\mathcal{X}_{IM}$ *is dense in* $\mathcal{X}$ *if and only if* $rad_M(IJM) \neq rad_M(0)$ *for every proper ideal* J *such that* JM *is not contained in* $rad_M(0)$.

Proof Let $\mathcal{X}_{IM}$ be dense in $\mathcal{X}$ and let J be any proper ideal of R, where JM is not in $rad_M(0)$. Then $\mathcal{X}_{JM} = Spec(M)\backslash V(JM)$ is a nonempty open set in the Zariski topology and by the hypothesis, the intersection of $\mathcal{X}_{IM}$ and $\mathcal{X}_{JM}$ is nonempty. Thus, $rad_M(IJM) \neq rad_M(0)$ by Corollary 3.2.

Let $rad_M(IJM) \neq rad_M(0)$ for every proper ideal J of R, where JM is not in $rad_M(0)$. By Corollary 3.2, since $\mathcal{X}_{IM} \cap \mathcal{X}_{JM} \neq \emptyset$, it follows that $\mathcal{X}_{IM}$ is dense in $\mathcal{X}$. □

The following theorem gives a characterization for the module $M/rad_M(0)$ by using topological properties.

Theorem 3.4 *Let* M *be a faithful multiplication* R*-module. The following statements are equivalent:*

i) $rad_M(0)$ *is a prime submodule of* M.

ii) $Spec(M)$ *is irreducible.*

iii) Every submodule of $M/rad_M(0)$ *is essential.*

iv) Every open subset of $Spec(M)$ *is dense.*

Proof $(i) \Leftrightarrow (ii)$ By [5], it can be easily proved.

$(iii) \Rightarrow (iv)$ Let $\mathcal{X}_{IM}$ and $\mathcal{X}_{JM}$ be open subsets for any ideals I, J of R. Then $(JM + rad_M(0))/rad_M(0)$ and $(IM + rad_M(0))/rad_M(0)$ are submodules of $M/rad_M(0)$. Since $\bigcap_{i\in\Lambda}(I_iM) = \left(\bigcap_{i\in\Lambda} I_i\right)M$, we observe that

$rad_M\left(\bigcap_{i\in\Lambda}(I_iM)\right) = rad_M\left(\left(\bigcap_{i\in\Lambda} I_i\right)M\right) = \left(rad_R\left(\bigcap_{i\in\Lambda} I_i\right)\right)M$, then

$$\begin{aligned} rad_M(0) &\neq rad_M\left[(JM + rad_M(0)) \cap (IM + rad_M(0))\right] \\ &= rad_M\left[(J + rad_R(0))M) \cap (I + rad_R(0))M)\right] \\ &= rad_M\left[(J + rad_R(0)) \cap (I + rad_R(0))M\right] \\ &= rad_R\left[IJ + rad_R(0)\right]M \\ &= rad_M\left[IJM + rad_M(0)\right] \end{aligned}$$

and so $rad_M(IJM) \neq rad_M(0)$, which means that $\mathcal{X}_{IM}$ is dense.

$(iv) \Rightarrow (ii) \Rightarrow (iii)$ One can prove it by the above method. □

Theorem 3.5 *Let* M *be a finitely generated multiplication* R*-module and let* I_i *be a proper ideal of* R *for all* $i \in \Lambda$. *Then* $\bigcup_{i\in\Lambda} \mathcal{X}_{I_iM} = \mathcal{X}_{DM}$ *for any ideal* D *of* R *if and only if* $rad_M(DM) = rad_M\left(\left(\sum_{i\in\Lambda} I_i\right)M\right)$.

Proof Assume that $\bigcup_{i\in\Lambda} \mathcal{X}_{I_iM} = \mathcal{X}_{DM}$ for any ideal D of R.

Let P be a prime submodule of M such that $DM \subseteq P$. Then $P \notin \mathcal{X}_{DM}$ and so $P \notin \bigcup_{i\in\Lambda} \mathcal{X}_{I_iM}$, implying $P \notin \mathcal{X}_{I_iM}$ for all i. Then $I_iM \subseteq P$ for all i and so $\left(\sum_{i\in\Lambda} I_i\right) M \subseteq P$, which implies $rad_M\left(\left(\sum_{i\in\Lambda} I_i\right) M\right) \subseteq rad_M(DM)$.

Let $\left(\sum_{i\in\Lambda} I_i\right) M \subseteq P$, where P is a prime submodule of M. Then $I_iM \subseteq P$ for all i, implying $P \notin \mathcal{X}_{I_iM}$ for all i. Thus, $P \notin \bigcup_{i\in\Lambda} \mathcal{X}_{I_iM} = \mathcal{X}_{DM}$ and so $DM \subseteq P$. This means that $rad_M(DM) \subseteq rad_M\left(\left(\sum_{i\in\Lambda} I_i\right) M\right)$.

Assume that $rad_M(DM) = rad_M\left(\left(\sum_{i\in\Lambda} I_i\right) M\right)$.

Let $P \in \mathcal{X}_{DM}$. Then $DM \nsubseteq P$ and $rad_M\left(\left(\sum_{i\in\Lambda} I_i\right) M\right) \nsubseteq P$ and so $\left(\sum_{i\in\Lambda} I_i\right) M \nsubseteq P$, implying $I_iM \nsubseteq P$ for all i. Then $P \in \mathcal{X}_{I_iM}$ and so $P \in \bigcup_{i\in\Lambda} \mathcal{X}_{I_iM}$.

The converse can be proved with the above method, which completes the proof. □

The following corollary is a special case of Theorem 3.5.

Theorem 3.6 *Let M be a finitely generated multiplication R-module, I_i proper ideals of R for all $i \in \Lambda$ and D a finitely generated ideal of R. Then the following statements are equivalent:*

i) $\bigcup_{i\in\Lambda} \mathcal{X}_{I_iM} = \mathcal{X}_{DM}$.

ii) There is a finite subset Δ of Λ such that $\bigcup_{i\in\Delta} \mathcal{X}_{I_iM} = \mathcal{X}_{DM}$.

iii) There is a finite subset Δ of Λ such that $rad_M\left(\left(\sum_{i\in\Delta} I_i\right) M\right) = rad_M(DM)$.

Proof $(i) \Rightarrow (iii)$ Let $rad_M(DM) = rad_M\left(\left(\sum_{i\in\Lambda} I_i\right) M\right)$ and let D be an ideal finitely generated by the set $\{d_1, ..., d_t\}$. For each d_iM, there is a positive number n_i such that $d_i^{n_i}M \subseteq \left(\sum_{i\in\Lambda} I_i\right) M$ and so there is a finite subset Δ_i of Λ such that $d_i^{n_i}M \subseteq \left(\sum_{i\in\Lambda} I_i\right) M$. If $n = \max\{n_1, ...n_t\}$ and $\Delta = \bigcup_{i=1}^{n} \Delta_i$ then $rad_M\left(\left(\sum_{i\in\Delta} I_i\right) M\right) = rad_M(DM)$.

$(iii) \Rightarrow (ii)$ By Theorem 3.5.

$(ii) \Rightarrow (i)$ It is clear. □

The following corollary is a special case of Theorem 3.6.

Corollary 3.7 *Let M be a finitely generated multiplication R-module and let I_i be a proper ideal of R for all $i \in \Lambda$. Then the following statements are equivalent:*

i) $\bigcup_{i\in\Lambda} \mathcal{X}_{I_iM} = Spec(M)$.

ii) *There is a finite subset* Δ *of* Λ *such that* $\bigcup_{i\in\Delta} \mathcal{X}_{I_iM} = Spec(M)$.

iii) *There is a finite subset* Δ *of* Λ *such that* $\left(\sum_{i\in\Delta} I_i\right) M = M$.

Corollary 3.8 *Let* M *be a finitely generated multiplication* R*-module and let* D *be a finitely generated ideal of* R*. Then* $\mathcal{X}_{DM}$ *is quasicompact.*

Using topological properties, we are now ready to prove the following characterization for $rad_M(0)$.

Theorem 3.9 *Let* M *be a finitely generated multiplication* R*-module satisfying the* $\mathcal{T}$*-condition for every submodule. Then there are proper ideals* $I_1, ..., I_n$ *of* R *such that* $rad_M(0) = rad_M((I_1...I_n)M)$.

Proof Let $\mathcal{X} = Spec(M)$ be Noetherian topological space. By [9], $\mathcal{X}$ has only a finite number of distinct irreducible components U_i such that $\bigcup_{i=1}^{n} U_i = \mathcal{X}$. It is well known that any irreducible component in a topological space is closed and so for each i, there is an ideal I_i such that $U_i = V(I_iM)$. Then

$$\emptyset = \mathcal{X} \setminus \bigcup_{i=1}^{n} V(I_iM) = \bigcap_{i=1}^{n} (\mathcal{X} \setminus V(I_iM)) = \bigcap_{i=1}^{n} \mathcal{X}_{I_iM}.$$

Thus, by Theorem 3.1, $rad_M(0) = \bigcap_{i=1}^{n} rad_M(I_iM) = rad_M((I_1...I_n)M)$. □

By using Theorems 2.8 and 3.5, we close the paper with the following result.

Theorem 3.10 *Let* M *be a finitely generated multiplication* R*-module and let* I_i *be an ideal of* R*. Then* $\mathcal{X} = \bigcup_{i=1}^{n} \mathcal{X}_{I_iM}$*, where* $\mathcal{X}_{I_iM}$ *is irreducible, if and only if* $M = \left(\sum_{i=1}^{n} I_i\right) M$ *and* $\mathcal{N}_{I_iM}(0)$ *is a prime submodule of* M.

Acknowledgments

The authors would like to thank the referee for the valuable comments and suggestions on this paper.

References

[1] Ali MM, Smith DJ. Finite and infinite collections of multiplication modules. Contributions to Algebra and Geometry 2001; 42: 557-573.

[2] Alkan M, Tıraş Y. Projective modules and prime submodules. Czechoslovak Mathematical Journal 2006; 56: 601-611.

[3] Alkan M, Tıraş Y. On invertible and dense submodules. Communications in Algebra 2004; 32; 3911-3919.

[4] Ameri R. Some properties of Zariski topology of multiplication modules. Houston Journal of Mathematics 2010; 36: 337-344.

[5] Atiyah MF, MacDonald IG. Introduction to Commutative Algebra. London, UK: Addison-Wesley, 1969.

[6] Barnard A. Multiplication modules. Journal of Algebra 1981; 71: 174-178.

[7] Bourbaki N. Elements of Mathematics General Topology Part 1 and Part 2. Massachusetts, USA: Hermann and Addison-Wesley, 1966.

[8] El-Bast Z, Smith PF. Multiplication modules. Communications in Algebra 1988; 16: 755-779.

[9] Kunz E. Introduction to Commutative Algebra and Algebraic Geometry. Boston, US: Birkhauser, 1985.

[10] Lu CP. Modules with Noetherian spectrum. Communications in Algebra 2010; 38: 807-828.

[11] McCasland RL, Moore ME, Smith PF. On the spectrum of a module over a commutative ring. Communications in Algebra 1997; 25: 79-103.

[12] Öneş O, Alkan M. Zariski subspace topologies on ideals. Hacettepe Journal of Mathematics and Statistics 2018. doi: 10.15672/HJMS.2018.597

[13] Öneş O, Alkan M. A note on graded ring with prime spectrum. Advanced Studies in Contemporary Mathematics 2018; 28: 625-634.