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Studying new generalizations of Max-Min matrices with a novel approach

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Abstract: We consider new kinds of max and min matrices, $\left[a_{\max(i,j)}\right]_{i,j\geq 1}$ and $\left[a_{\min(i,j)}\right]_{i,j\geq 1}$, as generalizations of the classical max and min matrices. Moreover, their reciprocal analogues for a given sequence $\{a_n\}$ have been studied. We derive their LU and Cholesky decompositions and their inverse matrices as well as the LU-decompositions of their inverses. Some interesting corollaries will be presented.

Key words: LU-decomposition, inverse matrix, Lehmer matrix, min and max matrices

1. Introduction

There are many interesting and useful combinatorial matrices defined by a given sequence $\{a_n\}_{n\geq 0}$. One of them is known as the Hankel matrix and defined as follows:

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots \\ a_2 & a_3 & a_4 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

for more details see [17]. Considering some special number sequences instead of $\{a_n\}_{n\geq 0}$, there are many special matrices with nice algebraic properties. Some authors [8, 19] studied the Hankel matrix by considering the reciprocal sequence of $\{a_n\}_{n\geq 0}$ of the form

$$\begin{bmatrix} \frac{1}{a_0} & \frac{1}{a_1} & \frac{1}{a_2} & \cdots \\ \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} & \cdots \\ \frac{1}{a_2} & \frac{1}{a_3} & \frac{1}{a_4} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}.$$

They are called the Hilbert and Filbert matrices when $a_n = n+1$ and $a_n = F_{n+1}$, respectively, where F_n stands for the nth Fibonacci number. Kılıç and Prodinger [10] gave some parametric generalizations and variants of the Filbert matrix.

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In this paper, we define four new combinatorial matrices, which we called max and min matrices and their reciprocal analogues whose entries run in left-reversed and up-reversed L-shaped pattern, respectively. By a given sequence $\{a_n\}$, we define the matrices M_1, M_2, $\mathcal{M}_1$ and $\mathcal{M}_2$ as

$$(M_1)_{ij} = a_{\max(i,j)}, \qquad (M_2)_{ij} = \frac{1}{a_{\max(i,j)}}$$

and

$$(\mathcal{M}_1)_{ij} = a_{\min(i,j)}, \qquad (\mathcal{M}_2)_{ij} = \frac{1}{a_{\min(i,j)}}.$$

Clearly, the matrices M_1 and $\mathcal{M}_1$ have the forms

$$M_1 = \begin{bmatrix} a_1 & a_2 & a_3 & \cdots & a_n & \cdots \\ a_2 & a_2 & a_3 & \cdots & a_n & \cdots \\ a_3 & a_3 & a_3 & \cdots & a_n & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ a_n & a_n & a_n & \cdots & a_n & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

and

$$\mathcal{M}_1 = \begin{bmatrix} a_1 & a_1 & a_1 & \cdots & a_1 & \cdots \\ a_1 & a_2 & a_2 & \cdots & a_2 & \cdots \\ a_1 & a_2 & a_3 & \cdots & a_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ a_1 & a_2 & a_3 & \cdots & a_n & \ddots \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

It is worthwhile to note that if the sequence $\{a_n\}$ is increasing, then $a_{\max(i,j)} = \max(a_i, a_j)$ and $a_{\min(i,j)} = \min(a_i, a_j)$. Conversely, if the sequence $\{a_n\}$ is decreasing, then $a_{\max(i,j)} = \min(a_i, a_j)$ and $a_{\min(i,j)} = \max(a_i, a_j)$. Thus, our matrices are the generalizations of the classical max and min matrices.

For some particular sequences, some special cases of these matrices were studied in [3, 5, 6, 13, 20].

- Frank [6] studied the matrix
$$[\max(n+1-i, n+1-j)]_{1 \le i,j \le n},$$
which is called the Frank matrix.
- Choi [3] gave the Cholesky decomposition of the matrix
$$\left[\max\left(\frac{1}{i+1}, \frac{1}{j+1}\right)\right]_{i,j \ge 1},$$
which is called the loyal companion of the Hilbert matrix.

The above matrices are the special cases of the matrix M_1.

The following matrices are the particular cases of the matrix $\mathcal{M}_1$, which were studied before.

- Trench [20] found eigenvalues and eigenvectors of the matrices

$$[\min(i,j)]_{1\le i,j\le n} \quad \text{and} \quad [\min(2i-1,2j-1)]_{1\le i,j\le n}.$$

  Afterwards, Kovacec [13] presented a different proof for the same problem.

- Fonseca [5] studied general cases of the matrices considered in [13, 20] by defining the matrix $[\min(ai-b,aj-b)]_{1\le i,j\le n}$ for $a>0$ and $a\ne b$. Then he computed eigenvalues and eigenvectors of this matrix by computing its inverse. He also presented a result without proof in Remark 2.1. Our results will be given for the matrix M_1 would provide a proof for this remark.

Recently, Mattila and Haukkanen [14] studied more general matrix families. Let $T=\{a_1,a_2,\ldots,a_n\}$ be a finite multiset of real numbers, such that $a_1\le a_2\le\cdots\le a_n$. They considered the matrices $[\max(a_i,a_j)]_{1\le i,j\le n}$ and $[\min(a_i,a_j)]_{1\le i,j\le n}$ defined on the set T. They computed the determinants, inverses, Cholesky decompositions of these matrices and examined positive definiteness of them. They used the meet and join matrices, see [7], as a tool to obtain their results. Moreover, they indicated that it is difficult to verify their results by using only basic linear algebra methods.

We will study various properties of the matrices M_1, M_2, $\mathcal{M}_1$, and $\mathcal{M}_2$, defined by any sequence $\{a_n\}$, such as LU-decomposition, inverse, Cholesky decomposition. In Section 2, we focus on the matrices M_1 and M_2. We will only give the proofs of the results related with the matrix M_1. The others can be similarly done. In Section 3, we examine the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$. This section will show us how Lemma 2 is useful, which we will give at the end of this section, to derive new combinatorial identities.

One can derive many results on the above mentioned combinatorial matrices by applying our results to some particular sequences $\{a_n\}_{n\ge0}$. Additionally, our results provide alternative proofs for the results given in [14].

Finally, we give some further applications of our main results. For example, we shall give an idea about how we could obtain a sequential generalization of the Lehmer matrix and its reciprocal analogue.

Throughout the paper, we use the letters L, U, and, $\hat{L}$, $\hat{U}$ for the LU-decompositions of a given matrix and its inverse, respectively. We denote the (i,j)th entries of a given matrix M and its inverse M^{-1} by M_{ij} and M_{ij}^{-1}, respectively. Similarly calligraphic letters will be used for the results related with a matrix in written calligraphic font. Also we assume that $\{a_n\}$ is any sequence such that $a_i\ne0$ and $a_i\ne a_{i+1}$ for all $i\ge1$.

In general, for each section, the size of the matrix does not really matter except the results about inverse matrix, so that we may think about an infinite matrix M and restrict it whenever necessary to the first n rows resp. columns and use the notation M_n.

The matrix $D(a)=[D_{ij}]$ stands for a diagonal matrix constructed via the given sequence $\{a_n\}$, defined by

$$D_{ij}=\begin{cases} a_i & \text{if } i=j,\\ 0 & \text{otherwise.}\end{cases}$$

We have the following lemmas for later use.

Lemma 1 *Let $\{a_n\}$ be a real sequence. Then for all $i,j>0$, we have*

$$a_{\max(i,j)}a_{\min(i,j)}=a_ia_j.$$

Lemma 2 *Let $H=[H_{ij}]$ be a square matrix and suppose that its LU-decomposition, inverse, LU-decomposition of its inverse and Cholesky decomposition are known with the matrices $L=[L_{ij}]$, $U=[U_{ij}]$, $H^{-1}=\left[H_{ij}^{-1}\right]$, $\hat{L}=\left[\hat{L}_{ij}\right]$, $\hat{U}=\left[\hat{U}_{ij}\right]$ and $C=[C_{ij}]$, respectively. Assume that a new square matrix $\mathcal{H}=[\mathcal{H}_{ij}]$ is defined with the entries of the matrix H and terms of given nonzero sequences $\{s_n\}$ and $\{m_n\}$ such that $\mathcal{H}_{ij}=H_{ij}s_im_j$. Then we can determine the LU-decomposition, inverse, LU-decomposition of its inverse and Cholesky decomposition of the matrix $\mathcal{H}$ as shown*

$$\begin{aligned}
&\mathcal{L}_{ij}=L_{ij}\frac{s_i}{s_j} && \text{and} && \mathcal{U}_{ij}=U_{ij}s_im_j,\\
&\mathcal{L}_{ij}^{-1}=L_{ij}^{-1}\frac{s_i}{s_j} && \text{and} && \mathcal{U}_{ij}^{-1}=U_{ij}^{-1}\frac{1}{s_j}\frac{1}{m_i},\\
&\mathcal{H}_{ij}^{-1}=H_{ij}^{-1}\frac{1}{s_j}\frac{1}{m_i}, && && \\
&\hat{\mathcal{L}}_{ij}=\hat{L}_{ij}\frac{m_j}{m_i} && \text{and} && \hat{\mathcal{U}}_{ij}=\hat{U}_{ij}\frac{1}{s_j}\frac{1}{m_i},\\
&\hat{\mathcal{L}}_{ij}^{-1}=\hat{L}_{ij}^{-1}\frac{m_j}{m_i} && \text{and} && \hat{\mathcal{U}}_{ij}^{-1}=\hat{U}_{ij}^{-1}s_im_j
\end{aligned}$$

and when for all $i\geq 1$, $s_i=m_i$,

$$\mathcal{C}_{ij}=C_{ij}s_i.$$

Proof By our assumption for the matrix $\mathcal{H}$, first we can write

$$\mathcal{H}=D(s)\cdot H\cdot D(m).$$

Since the LU-decomposition of the matrix H is known, namely $H=L\cdot U$, we write

$$\mathcal{H}=D(s)\cdot L\cdot U\cdot D(m)=D(s)\cdot L\cdot D\left(\frac{1}{s}\right)\cdot D(s)\cdot U\cdot D(m).$$

Here we see that $D(s)\cdot L\cdot D\left(\frac{1}{s}\right)$ is a unite lower triangular matrix and $D(s)\cdot U\cdot D(m)$ is an upper triangular matrix. So

$$\mathcal{L}=D(s)\cdot L\cdot D\left(\frac{1}{s}\right)\quad\text{and } \mathcal{U}=D(s)\cdot U\cdot D(m),$$

which gives the LU-decomposition of $\mathcal{H}$. Moreover, we immediately derive

$$\mathcal{H}^{-1}=D\left(\frac{1}{m}\right)\cdot H^{-1}\cdot D\left(\frac{1}{s}\right).$$

For the Cholesky decomposition of $\mathcal{H}$, consider

$$\mathcal{H}=D(s)\cdot H\cdot D(s)=D(s)\cdot C\cdot C^T\cdot D(s)^T=(D(s)\cdot C)\cdot(D(s)\cdot C)^T,$$

as claimed. □

Lemma 2 allows us to derive many new matrix identities. For instance, the Pascal matrix $\left[\binom{i+j}{i}\right]_{i,j\geq 0}$ and its some variants have been studied by many authors, for more details see [4, 11, 18]. In [18], the LU-decomposition of the Pascal matrix was given. Since

$$(i+j)!=\binom{i+j}{i}\times i!\times j!,$$

by choosing $s_i = i!$ and $m_j = j!$ in Lemma 2, one can easily find the related results for the matrix $[(i+j)!]_{i,j\geq 0}$. For more identities, we refer to [12].

2. Max-matrices and their reciprocal analogues

In this section, we derive the LU-decompositions, inverses, Cholesky decompositions and LU-decompositions of the inverses of the matrices M_1 and M_2, respectively.

2.1. Max-Matrix M_1

We start with the LU-decomposition, $M_1 = L \cdot U$:

Theorem 1 *For $i, j \geq 1$,*

$$L_{ij} = \begin{cases} \dfrac{a_i}{a_j} & \text{if } i \geq j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$U_{ij} = \begin{cases} a_j & \text{if } i = 1, \\ \dfrac{a_j\left(a_{i-1} - a_i\right)}{a_{i-1}} & \text{if } j \geq i > 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we shall give the inverse matrices L^{-1} and U^{-1} by the following result.

Theorem 2 *For $i, j \geq 1$,*

$$L_{ij}^{-1} = \begin{cases} (-1)^{i+j} \dfrac{a_i}{a_j} & \text{if } 0 \leq i-j \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$U_{ij}^{-1} = \begin{cases} (-1)^{i+j} \dfrac{a_{j-1}}{a_i\left(a_{j-1} - a_j\right)} & \text{if } 0 \leq j-i \leq 1 \text{ and } j \neq 1, \\ \dfrac{1}{a_1} & \text{if } i = j = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Now we compute the inverse matrix $(M_1)_n^{-1}$ as follows.

Theorem 3 *For $1 \leq i, j \leq n$, $(M_1)_n^{-1}$ is the symmetric tridiagonal matrix defined by*

$$(M_1)_{ij}^{-1} = \begin{cases} \dfrac{1}{a_1 - a_2} & \text{if } i = j = 1, \\[2ex] \dfrac{a_{i-1} - a_{i+1}}{(a_{i+1} - a_i)(a_i - a_{i-1})} & \text{if } 1 \neq i = j \neq n, \\[2ex] \dfrac{a_{n-1}}{a_n (a_{n-1} - a_n)} & \text{if } i = j = n, \\[2ex] \dfrac{1}{a_i - a_{i-1}} & \text{if } i = j + 1. \end{cases}$$

For the Cholesky decomposition, $M_1 = C \cdot C^T$, we have the following result.

Theorem 4 *For $i, j \geq 1$, C is the lower triangular matrix defined by*

$$C_{ij} = \begin{cases} \dfrac{a_i}{\sqrt{a_1}} & \text{if } j = 1, \\ \dfrac{a_i}{a_j a_{j-1}} \sqrt{a_j a_{j-1} (a_{j-1} - a_j)} & \text{if } j > 1. \end{cases}$$

We will give the LU-decomposition of $(M_1)_n^{-1}$, that is $(M_1)_n^{-1} = \hat{L}_n \cdot \hat{U}_n$, and also the inverses of these factor matrices by the following results.

Theorem 5 *For $1 \leq i, j \leq n$,*

$$\hat{L}_{ij} = \begin{cases} (-1)^{i+j} & \text{if } 0 \leq i - j \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{U}_{ij} = \begin{cases} \dfrac{1}{a_n} & \text{if } i = j = n, \\[2ex] (-1)^{i+j} \dfrac{1}{(a_i - a_{i+1})} & \text{if } 0 \leq j - i \leq 1 \text{ and } i \neq n, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Theorem 6 *For $1 \leq i, j \leq n$,*

$$\hat{L}_{ij}^{-1} = \begin{cases} 1 & \text{if } i \geq j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{U}_{ij}^{-1} = \begin{cases} a_n & \text{if } j = n, \\ a_j - a_{j+1} & \text{if } i \leq j < n, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1 *If the sequence $\{a_n\}$ is positive and decreasing, then the matrix M_1 is a positive definite matrix, which can be easily seen from its LU-decomposition. On the other hand, the sequence $\{a_n\}$ is negative and increasing, then the matrix M_1 is a negative definite matrix.*

2.2. Proofs

Now we will present the proofs of the results given in the previous subsection.

In order to prove $M_1 = L \cdot U$, it is sufficient to show that

$$\sum_{d=1}^{\min(i,j)} L_{id} U_{dj} = a_{\max(i,j)}.$$

Consider

$$\begin{aligned}\sum_{d=1}^{\min(i,j)} L_{id} U_{dj} &= \frac{a_i}{a_1} a_j + \sum_{d=2}^{\min(i,j)} \frac{a_i}{a_d} \frac{a_j \left(a_{d-1} - a_d\right)}{a_{d-1}} \\ &= a_i a_j \left[\frac{1}{a_1} + \sum_{d=2}^{\min(i,j)} \left(\frac{1}{a_d} - \frac{1}{a_{d-1}} \right) \right] = \frac{a_i a_j}{a_{\min(i,j)}},\end{aligned}$$

which, by Lemma 1, equals $a_{\max(i,j)}$, as expected.

Define the matrix $T = [T_{ij}]$ with

$$T_{ij} = \begin{cases} 1 & \text{if } i \geq j, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that

$$T_{ij}^{-1} = \begin{cases} (-1)^{i+j} & \text{if } 0 \leq i - j \leq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the proofs related with $L^{-1}, U^{-1}, \hat{L}_n^{-1}$ and $\hat{U}_n^{-1}$ follow from Lemma 2.

In order to prove the LU-decomposition of $(M_1)_n^{-1}$, it is sufficient to show that $(M_1)_n = \hat{U}_n^{-1} \cdot \hat{L}_n^{-1}$. Consider

$$\sum_{d=\max(i,j)id}^{n-1} \hat{U}_{id}^{-1} \hat{L}_{dj}^{-1} = \sum_{d=\max(i,j)}^{n-1} \left(a_d - a_{d+1}\right) + a_n = a_{\max(i,j)},$$

as desired.

For the Cholesky decomposition, i.e. $M_1 = C \cdot C^T$, consider

$$\sum_{d=1}^{\min(i,j)} C_{id} C_{jd} = \frac{a_i a_j}{a_1} + \sum_{d=2}^{\min(i,j)} \frac{a_i a_j}{a_d a_{d-1}} \left(a_{d-1} - a_d\right) = a_{\max(i,j)},$$

which completes the proof.

Finally, in order to prove $M_1 \cdot M_1^{-1} = I$, we have three cases: $j = 1$, $1 < j < n$ and $j = n$. For these cases, consider the following equalities, respectively.

$$\sum_{d=1}^{n}(M_1)_{id}(M_1)_{d1}^{-1}=\frac{a_{\max(i,1)}}{a_1-a_2}+\frac{a_{\max(i,2)}}{a_2-a_1}=\delta_{i,1},$$

$$\sum_{d=1}^{n}(M_1)_{id}(M_1)_{dj}^{-1}=\frac{a_{\max(i,j-1)}}{a_j-a_{j-1}}+\frac{a_{\max(i,j)}\left(a_{j-1}-a_{j+1}\right)}{\left(a_{j+1}-a_j\right)\left(a_j-a_{j-1}\right)}+\frac{a_{\max(i,j+1)}}{a_{j+1}-a_j}=\delta_{i,j},$$

$$\sum_{d=1}^{n}(M_1)_{id}(M_1)_{dn}^{-1}=\frac{a_{\max(i,n-1)}}{a_n-a_{n-1}}+\frac{a_{n-1}a_{\max(i,n)}}{a_n\left(a_{n-1}-a_n\right)}=\delta_{i,n},$$

where $\delta_{i,j}$ is Kronecker delta. By all of them, the proofs are complete.

2.3. Reciprocal Max-matrix M_2

Similarly we shall give all results related with the matrix M_2 without proofs. All the proofs can be similarly done as in the previous subsection.

Theorem 7 *For* $i,j\geq 1$,

$$L_{ij}=\begin{cases}\dfrac{a_j}{a_i} & \text{if } i\geq j,\\ 0 & \text{otherwise}\end{cases}$$

and

$$U_{ij}=\begin{cases}\dfrac{1}{a_j} & \text{if } i=1,\\[2ex] \dfrac{(a_i-a_{i-1})}{a_ja_i} & \text{if } j\geq i>1,\\[2ex] 0 & \text{otherwise.}\end{cases}$$

Theorem 8 *For* $i,j\geq 1$,

$$L_{ij}^{-1}=\begin{cases}(-1)^{i+j}\dfrac{a_j}{a_i} & \text{if } 0\leq i-j\leq 1,\\ 0 & \text{otherwise}\end{cases}$$

and

$$U_{ij}^{-1}=\begin{cases}(-1)^{i+j}\dfrac{a_ia_j}{(a_j-a_{j-1})} & \text{if } 0\leq j-i\leq 1 \text{ and } j\neq 1,\\ a_1 & \text{if } i=j=1,\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 9 *For $1 \leq i, j \leq n$, $(M_2)_n^{-1}$ is the symmetric tridiagonal matrix defined by*

$$(M_2)_{ij}^{-1} = \begin{cases} \dfrac{a_1 a_2}{(a_2 - a_1)} & \text{if } i = j = 1, \\[2ex] \dfrac{a_i^2 (a_{i+1} - a_{i-1})}{(a_{i+1} - a_i)(a_i - a_{i-1})} & \text{if } 1 \neq i = j \neq n, \\[2ex] \dfrac{a_n^2}{(a_n - a_{n-1})} & \text{if } i = j = n, \\[2ex] \dfrac{a_i a_j}{(a_{i-1} - a_i)} & \text{if } i = j + 1. \end{cases}$$

Theorem 10 *For $i, j \geq 1$, C is the lower triangular matrix defined by*

$$C_{ij} = \begin{cases} \dfrac{\sqrt{a_1}}{a_i} & \text{if } j = 1, \\[2ex] \dfrac{\sqrt{a_j - a_{j-1}}}{a_i} & \text{if } j > 1. \end{cases}$$

Theorem 11 *For $1 \leq i, j \leq n$,*

$$\hat{L}_{ij} = \begin{cases} (-1)^{i+j} & \text{if } 0 \leq i - j \leq 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{U}_{ij} = \begin{cases} a_n & \text{if } i = j = n, \\[2ex] (-1)^{i+j} \dfrac{a_{i+1} a_i}{(a_{i+1} - a_i)} & \text{if } 0 \leq j - i \leq 1, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Theorem 12 *For $1 \leq i, j \leq n$,*

$$\hat{L}_{ij}^{-1} = \begin{cases} 1 & \text{if } i \geq j, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\hat{U}_{ij}^{-1} = \begin{cases} \dfrac{1}{a_n} & \text{if } j = n, \\[2ex] \dfrac{a_{j+1} - a_j}{a_{j+1} a_j} & \text{if } i \leq j < n, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

3. Min-matrices and their reciprocal analogues

In this section, we list the LU-decompositions, inverses, Cholesky decompositions and LU-decompositions of the inverse matrices of $\mathcal{M}_1$ and $\mathcal{M}_2$, respectively. We omit the results related with L^{-1}, U^{-1}, $\hat{L}^{-1}$ and $\hat{U}^{-1}$ here. They could be easily obtained as in the proof in Section 2.2.

Theorem 13 *For the matrix $\mathcal{M}_1$,*

$$L_{ij} = \begin{cases} 1 & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_{ij} = \begin{cases} a_1 & \text{if } i = 1, \\ a_i - a_{i-1} & \text{if } j \geq i > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\mathcal{M}_1)^{-1}_{ij} = \begin{cases} \dfrac{a_2}{a_1\left(a_2 - a_1\right)} & \text{if } i = j = 1, \\[2ex] \dfrac{\left(a_{i+1} - a_{i-1}\right)}{\left(a_{i+1} - a_i\right)\left(a_i - a_{i-1}\right)} & \text{if } 1 \neq i = j \neq n, \\[2ex] \dfrac{1}{\left(a_n - a_{n-1}\right)} & \text{if } i = j = n, \\[2ex] \dfrac{1}{\left(a_{i-1} - a_i\right)} & \text{if } i = j + 1, \end{cases}$$

$$C_{ij} = \begin{cases} \sqrt{a_1} & \text{if } j = 1, \\ \sqrt{a_j - a_{j-1}} & \text{if } j > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\hat{L}_{ij} = \begin{cases} (-1)^{i+j}\,\dfrac{a_j}{a_i} & \text{if } 0 \leq i - j \leq 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\hat{U}_{ij} = \begin{cases} (-1)^{i+j}\,\dfrac{a_{i+1}}{a_j\left(a_{i+1} - a_i\right)} & \text{if } 0 \leq j - i \leq 1 \text{ and } i \neq n, \\[2ex] \dfrac{1}{a_n} & \text{if } i = j = n, \\[2ex] 0 & \text{otherwise.} \end{cases}$$

Note that the inverse matrix $(\mathcal{M}_1)^{-1}$ is a symmetric tridiagonal matrix of order n.

Remark 2 *By the LU-decomposition of the matrix $\mathcal{M}_1$, it is seen that if a_1 is a positive real number and the sequence $\{a_n\}$ is increasing, then the matrix $\mathcal{M}_1$ is a positive definite matrix. Conversely, if a_1 is a negative real number and the sequence $\{a_n\}$ is decreasing, then the matrix $\mathcal{M}_1$ is a negative definite matrix.*

Theorem 14 *For the matrix $\mathcal{M}_2$, we have*

$$L_{ij} = \begin{cases} 1 & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases}$$

$$U_{ij} = \begin{cases} \dfrac{1}{a_1} & \text{if } i = 1, \\[2ex] \dfrac{a_{i-1} - a_i}{a_i a_{i-1}} & \text{if } j \geq i > 1, \\[2ex] 0 & \text{otherwise,} \end{cases}$$

$$(\mathcal{M}_2)^{-1}_{ij} = \begin{cases} \dfrac{a_1^2}{(a_1 - a_2)} & \textit{if } i = j = 1, \\[2ex] \dfrac{a_i^2 (a_{i-1} - a_{i+1})}{(a_{i+1} - a_i)(a_i - a_{i-1})} & \textit{if } 1 \neq i = j \neq n, \\[2ex] \dfrac{a_n a_{n-1}}{(a_{n-1} - a_n)} & \textit{if } i = j = n, \\[2ex] \dfrac{a_i a_j}{(a_i - a_{i-1})} & \textit{if } i = j + 1, \end{cases}$$

$$C_{ij} = \begin{cases} \dfrac{1}{\sqrt{a_1}} & \textit{if } j = 1, \\[2ex] \dfrac{1}{a_j a_{j-1}} \sqrt{a_j a_{j-1} (a_{j-1} - a_j)} & \textit{if } j > 1, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\hat{L}_{ij} = \begin{cases} (-1)^{i+j} \dfrac{a_i}{a_j} & \textit{if } 0 \leq i - j \leq 1, \\ 0 & \textit{otherwise}, \end{cases}$$

$$\hat{U}_{ij} = \begin{cases} (-1)^{i+j} \dfrac{a_i a_j}{a_j (a_i - a_{i+1})} & \textit{if } 0 \leq j - i \leq 1, \\ a_n & \textit{if } i = j = n, \\ 0 & \textit{otherwise}. \end{cases}$$

Proof By Lemma 1, we can write

$$a_{\min(i,j)} = \frac{a_i a_j}{a_{\max(i,j)}} \text{ and } \frac{1}{a_{\min(i,j)}} = \frac{a_{\max(i,j)}}{a_i a_j}.$$

So all claimed results follow by Lemma 2 and the results of Section 2. □

Note that the inverse matrix $(\mathcal{M}_2)^{-1}$ mentioned in Theorem 14 is a symmetric tridiagonal matrix of order n.

4. Applications

First, we present an application which is a prototype to derive some determinant identities.

Corollary 1 *Let T_1 and T_2 be the matrices defined by $[\max(i,j)]_{1 \leq i,j \leq n}$ and $[\min(i,j)]_{1 \leq i,j \leq n}$, respectively. Then*

$$\det T_1 = (-1)^{n-1} n \textit{ and } \det T_2 = 1.$$

Proof Let $\{a_n\}$ be the sequence of natural numbers, $a_n = n$, which is increasing. Thus, $a_{\max(i,j)} = \max(i,j)$ and $a_{\min(i,j)} = \min(i,j)$. Determinant of a matrix is equal to product of elements of the main diagonal entries of the triangular matrix U, which comes from its LU-decomposition. Thus, by the LU-decompositions of M_1

and $\mathcal{M}_1$, we obtain

$$\det T_1 = \prod_{d=2}^{n} \frac{(-1)\,d}{d-1} = (-1)^{n-1}\, n,$$

$$\det T_2 = \prod_{d=1}^{n} 1 = 1,$$

as claimed. □

Recall the well-known Lehmer matrix H (see [16]) defined by

$$H_{ij} = \frac{\min(i,j)}{\max(i,j)}.$$

By Lemma 1, one can write the (i,j)th entry of it as:

$$\frac{\min(i,j)}{\max(i,j)} = \frac{ij}{(\max(i,j))^2} = \frac{ij}{\max(i^2,j^2)}.$$

Using Lemma 2 and the results for the matrix M_2 by taking $a_n = n^2$, i.e. $a_{\max(i,j)} = \max\left(i^2,j^2\right)$, it is easily rediscovered the LU-decomposition, inverse and Cholesky decomposition of the Lehmer matrix. Also, the results of [1, 9] can be reobtained by using similar approach.

Moreover, our results give us an idea to find a sequential generalization of the Lehmer matrix. For example, we define the matrix $H = [H_{ij}]$ for any positive and strictly increasing sequence $\{a_n\}$ by

$$H_{ij} = \frac{\min(a_i,a_j)}{\max(a_i,a_j)} = \frac{a_i a_j}{\max\left(a_i^2,a_j^2\right)}.$$

Thus, by our general results, the LU-decomposition, inverse and Cholesky decomposition of the matrix H could be derived but we omit the details here due to the similarities with the following example. The interested reader could find a lattice-theoretic generalization of the Lehmer matrix in [2].

The following example will be a reciprocal-sequential generalization of the Lehmer matrix.

Corollary 2 *Let $\{a_n\}$ be a positive and strictly increasing sequence and $\mathcal{H} = [\mathcal{H}_{ij}]$ be the matrix defined by*

$$\mathcal{H}_{ij} = \frac{\max(a_i,a_j)}{\min(a_i,a_j)}.$$

Then

$$\mathcal{L}_{ij} = \begin{cases} \dfrac{a_i}{a_j} & \text{if } i \geq j, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{U}_{ij} = \begin{cases} \dfrac{a_j}{a_1} & \text{if } i = 1, \\ \dfrac{a_j\,(b_{i-1}-b_i)}{a_i b_{i-1}} & \text{if } j \geq i > 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$\mathcal{C}_{ij}=\begin{cases}\dfrac{a_i}{\sqrt{b_1}} & \textit{if } j=1,\\ \dfrac{a_i}{b_jb_{j-1}}\sqrt{b_jb_{j-1}\left(b_{j-1}-b_j\right)} & \textit{if } j>1,\\ 0 & \textit{otherwise,}\end{cases}$$

$$\mathcal{H}^{-1}_{ij}=\begin{cases}\dfrac{b_1}{\left(b_1-b_2\right)} & \textit{if } i=j=1,\\[2ex] \dfrac{b_i\left(b_{i-1}-b_{i+1}\right)}{\left(b_{i+1}-b_i\right)\left(b_i-b_{i-1}\right)} & \textit{if } 1\neq i=j\neq n,\\[2ex] \dfrac{b_{n-1}}{\left(b_{n-1}-b_n\right)} & \textit{if } i=j=n,\\[2ex] \dfrac{a_ia_j}{\left(b_i-b_{i-1}\right)} & \textit{if } i=j+1,\end{cases}$$

where $\mathcal{H}^{-1}$ is a symmetric tridiagonal matrix of order n and $b_i=a_i^2$.

Proof Since $\{a_n\}$ is a positive and strictly increasing, by Lemma 1, we have

$$\mathcal{H}_{ij}=\frac{\max\left(a_i,a_j\right)}{\min\left(a_i,a_j\right)}=\frac{a_ia_j}{b_{\min(i,j)}}.$$

Hence, the proof follows by Lemma 2 and the results of the matrix $\mathcal{M}_2$ for the sequence $\{b_n\}$. □

Note that when $a_n=n$, we get the reciprocal analogues of the usual Lehmer matrix. By using same approach, one can also derive related results for any positive and strictly decreasing sequence $\{a_n\}$.

Now we would like to give a useful note for the reader. There are some classes of matrix families, whose LU-decomposition, inverse, determinant etc. cannot be directly derived by our results. Nevertheless our results allow to guess their properties such as LU-decomposition, inverse with less effort. One of the examples of these kinds of matrix families is the matrix family obtained by deleting certain band entries starting from the upper right corner or the left down corner of the matrices M_1, M_2, M_1, or M_2. Then our results will give inspiration to obtain their properties. To show this, we shall give an example.

Corollary 3 *For positive integer r, define the matrix $\mathcal{F}=[\mathcal{F}_{ij}]$ with entries*

$$\mathcal{F}_{ij}=\begin{cases}a_{\max(i,j)} & \textit{if } i\geq j-r,\\ 0 & \textit{otherwise.}\end{cases}$$

Then for $i,j\geq 1$, the LU-decomposition of the matrix $\mathcal{F}$ is

$$L_{ij}=\begin{cases}\dfrac{a_i}{a_j} & \textit{if } i\geq j,\\ 0 & \textit{otherwise,}\end{cases}$$

and

$$U_{ij}=\begin{cases}a_j & \textit{if } i=1 \textit{ and } j\leq r+1,\\ a_j & \textit{if } j>r+1 \textit{ and } i=j-r,\\ \dfrac{a_j\left(a_{i-1}-a_i\right)}{a_{i-1}} & \textit{if } i+r-1>j\geq i>1,\\ 0 & \textit{otherwise.}\end{cases}$$

Clearly, for $n = 8$ and $r = 3$, it takes the form

$$\mathcal{F}_8 = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & 0 & 0 & 0 & 0 \\ a_2 & a_2 & a_3 & a_4 & a_5 & 0 & 0 & 0 \\ a_3 & a_3 & a_3 & a_4 & a_5 & a_6 & 0 & 0 \\ a_4 & a_4 & a_4 & a_4 & a_5 & a_6 & a_7 & 0 \\ a_5 & a_5 & a_5 & a_5 & a_5 & a_6 & a_7 & a_8 \\ a_6 & a_6 & a_6 & a_6 & a_6 & a_6 & a_7 & a_8 \\ a_7 & a_7 & a_7 & a_7 & a_7 & a_7 & a_7 & a_8 \\ a_8 & a_8 & a_8 & a_8 & a_8 & a_8 & a_8 & a_8 \end{bmatrix}.$$

The matrix $\mathcal{F}$ is obtained from the max-matrix by deleting the entries after rth superdiagonal (Note that similar example can be obtained for the matrix which is obtained by applying the same process to min-matrix).

Now we prove the claimed LU-decomposition of the matrix $\mathcal{F}$ just above.

Proof We should show that

$$\mathcal{F}_{ij} = \sum_{d=1}^{\min(i,j)} L_{id} U_{dj}.$$

The proof for the case $j \leq r+1$ can be similarly done as in Subsection 2.2. Now consider for $j > r+1$ and $i \geq j-r$,

$$\begin{aligned} \sum_{d=1}^{\min(i,j)} L_{id} U_{dj} &= \frac{a_i a_j}{a_{j-r}} + \sum_{d=j-r+1}^{\min(i,j)} L_{id} U_{dj} = \frac{a_i a_j}{a_{j-r}} + a_i a_j \sum_{d=j-r+1}^{\min(i,j)} \left(\frac{1}{a_d} - \frac{1}{a_{d-1}} \right) \\ &= \frac{a_i a_j}{a_{\min(i,j)}} = a_{\max(i,j)}. \end{aligned}$$

And the final cases $j > r+1$ and $i < j-r$ can be easily computed as 0, which completes the proof. □

The particular case $r = 1$ can be found in [6].

In general, we encounter a special family of the Hessenberg matrices for the case $r = 1$. By its LU-decomposition, we can compute their determinants. It would be valuable to note that Hessenberg matrices are very important combinatorial matrices. We could refer to a recent work [15] to see how Hessenberg matrices are useful matrices for deriving combinatorial identities involving integer partitions and multinomial coefficients.

One can also obtain similar results for the matrix which is derived by deleting the entries of max-matrix (or min-matrix) after rth subdiagonal. We left the details to the interested reader.

As a conclusion remark, our results cover the results for the matrices $[\max(a_i, a_j)]_{i,j>0}$ and $[\min(a_i, a_j)]_{i,j>0}$ (also their reciprocals analogues) when the sequence $\{a_n\}$ is increasing or decreasing. Unfortunately, if a sequence $\{c_n\}$ is neither increasing nor decreasing, such as unimodal sequences, then our results do not work for the matrices $[\max(c_i, c_j)]_{i,j>0}$ and $[\min(c_i, c_j)]_{i,j>0}$.

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