Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 2025 – 2031
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doi:10.3906/mat-1904-111

A description for the compactification of the orbit space

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Received: 18.04.2019 • **Accepted/Published Online:** 19.06.2019 • **Final Version:** 31.07.2019

Abstract: Let X be a locally compact and noncompact $G-$space with a compact group G. In this paper, we give some useful description of a compactification of the orbit space X/G when it is an orbit space of a $G-$compactification of X. As an application, we show that the closed bounded interval $[a,b]$ is homeomorphic to the space of maximal ideals with Stone topology of uniformly continuous even functions subring of $C^*(\mathbb{R})$.

Key words: Gelfand compactification, one-point compactification, orbit space, continuous and bounded functions ring

1. Introduction

By a topological transformation group, we mean a triple (X,G,θ), where G is a topological group, X is a Tychonoff space, and θ is a continuous action of G on X. That is to say, θ is a continuous mapping from $G\times X$ onto X such that the following conditions are fulfilled:

1. $\theta(e,x)=x$, for each $x\in X$ (e denotes the identity element in G)
2. $\theta(g,\theta(h,x))=\theta(gh,x)$, for each $g,h\in G$ and $x\in X$.

We shall write in generally gx for $\theta(g,x)$. If (X,G,θ) is a transformation group then X will be called a $G-$space. If X and Y are $G-$spaces, then a mapping $f:X\to Y$ is called equivariant whenever $f(gx)=gf(x)$ for all $g\in G$ and $x\in X$.

A compact $G-$space γX is called a $G-$compactification of X, if there is an equivariant dense embedding map $i:X\hookrightarrow\gamma X$. A Tychonoff space X may not have a $G-$compactification. For example, Megrelishvili [4] established a Tychonoff $G-$space admitting no compact Hausdorff extension. However, suppose that X is a locally compact $G-$space and $aX=X\cup\{\infty\}$ is the (Alexandroff) one-point compactification of X. Then it is clear that aX is a $G-$compactification of X defining $g\infty=\infty$ for all $g\in G$.

The following problems in the theory of $G-$spaces are well-known:

(1) the problem of existence of a $G-$compactification, say γX, of a Tychonoff $G-$space X such that $\alpha(X/G)=\gamma X/G$ for a given compactification, $\alpha(X/G)$, of the orbit space X/G.

(2) in case of existence of this compactification, the question is how $\alpha(X/G)$ is uniquely described with $C^*(X)$ (i.e. bounded continuous function ring of X.)

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2010 *AMS Mathematics Subject Classification:* 54D35, 54H15

Srivastava [6] proved that and $\beta(X/G) = \beta X/G$ for finite group G. (βX and $\beta(X/G)$ are Stone-Cech compactifications of X and X/G). However, by means of Gelfand's method, Karapınar [3] described the $\beta(X/G)$ as a space of some maximal ideals of a certain subring of $C^*(X)$.

In this paper, for locally compact space X with compact $G-$action, we answer the question (2) above. Specially, we describe the one-point compactification of the orbit space X/G by using the set of maximal ideals of a complete subring of $C^*(X)$. Among different methods for constructing compactifications, we use Gelfand's method.

2. Preliminaries

In this section, we shall state a few definitions and facts about transformation groups and Gelfand's method for compactifications. We refer the reader to [1, 5] for more details.

Definition 1 *([1]) A subspace A of a $G-$space X is called invariant, if*

$$\theta(G \times A) = A.$$

Definition 2 *([1]) If X is a $G-$space and $x \in X$, the subspace*

$$G(x) = \{gx : g \in G\}$$

is called the orbit of x. Let X/G denote the set of all orbits $G(x)$ of a $G-$space X and $\pi : X \to X/G$ denote the orbit map taking x to $G(x)$. Then X/G endowed with the quotient topology relative to π is called the orbit space of X.

Definition 3 *([1]) An action θ of a group G on a space X is called transitive, if $G(x) = X$ for all $x \in X$.*

Definition 4 *([5]) A topological space X is called locally compact if every point has a compact neighborhood.*

Theorem 5 *If X is a $G-$ space with G compact, then*

(1) X/G is Hausdorff.
(2) $\pi : X \to X/G$ is closed.
(3) $\pi : X \to X/G$ is proper (π^{-1}(compact) is compact).
(4) X is compact iff X/G is compact.
(5) X is locally compact iff X/G is locally compact.

Proof See [1, Theorem 3.1] □

Definition 6 *([5]) For a topological space X, a compactification aX of X is called a one- point compactification of X if $aX - X$ is a singleton*

Proposition 7 *A topological space X has a one-point compactifications iff X is locally compact and not compact.*

Proof See [5, Proposition 4.3.c] □

Proposition 8 *A noncompact, locally compact space X has a unique one- point compactification.*

Proof See [5, Proposition 4.3.f] □

Since a locally compact space is Tychonoff space, the following theorem and definitions about Tychonoff space is valid for locally compact space. We will denote continuous and bounded real-valued function rings by $C^*(X)$.

Definition 9 *([5]) Let X be a topological space. A subcollection $\mathcal{B}$ of subsets of X is called a closed base for X, if each closed subset of X can be written as an intersection of sets belonging to $\mathcal{B}$.*

Definition 10 *([5]) Let Ω be a complete (with respect to the sup norm metric) subring of $C^*(X)$ which contains all constant functions and $M_\Omega X$ denotes the set of all maximal ideals of Ω. For each $f \in \Omega$, define $S(f) = \{M \in M_\Omega X : f \in M\}$. It is easy to see that the family $\{S(f) : f \in \Omega\}$ is closed base for a topology on $M_\Omega X$ which is called the Stone topology.*

Theorem 11 *$M_\Omega X$ with the Stone topology is a compact and Haussdorff space.*

Proof See [5, Theorem 4.5.j] □

Definition 12 *([5]) A complete subring Ω of $C^*(X)$ with respect to the sup-norm metric is called regular, if it contains all constant functions and $Z(\Omega) = \{Z(f) : f \in \Omega\}$ is a closed base for X, where $Z(f)$ is the zero-set of f.*

If $x \in X$ and Ω is a regular subring of $C^*(X)$, then $M_x = \{f \in \Omega : f(x) = 0\} \in M_\Omega X$. (See [5, Theorem 4.5.l]) Thus, we can define a continuous function $\lambda : X \to M_\Omega X$ by $\lambda(x) = M_x$.

The proof of the following theorem is given in [2].

Theorem 13 *(Gelfand, [2]) If Ω is a regular subring of $C^*(X)$ for a space X, then $\lambda : X \to M_\Omega X$ is a dense embedding.*

Definition 14 *([5]) A compactification γX of a Tychonoff space X is called a Gelfand compactification, if for some regular subring Ω of $C^*(X)$, γX and $M_\Omega X$ are equivalent compactifications of X which is denoted by $\gamma X \equiv_X M_\Omega X$ (i.e. there exists a homeomorphism which is identitiy on X).*

For a given compactification γX of a Tychonoff space X, we denote by $\Omega_{\gamma X}$ the set of restricted functions to X of $C^*(\gamma X)$. The below theorem states that each compactification is a Gelfand compactification.

Theorem 15 *$\Omega_{\gamma X}$ is a regular subring of $C^*(X)$ and $\gamma X \equiv_X M_{\Omega_{\gamma X}} X$.*

Proof See [5, Theorem 4.5.o]. □

3. Main result

From now on, X will be a locally compact and noncompact $G-$space where G is a compact group and bX is a $G-$compactification of X. Then it is easy to see that the remainder $X^* = bX - X$ is an $G-$invariant subspace of bX.

Proposition 16 *bX/G is a compactification of X/G*

Proof Let $i : X \to bX$ denote dense embedding $G-$map. Then i induces a well-defined map by the equivariance of i

$$\bar{i} : X/G \to bX/G$$

$$G(x) \mapsto G(i(x)),$$

and the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{\quad i \quad} & bX \\ \Big\downarrow{\scriptstyle \pi} & & \Big\downarrow{\scriptstyle \pi'} \\ X/G & \xrightarrow{\quad \bar{i} \quad} & bX/G \end{array} .$$

If $\bar{i}(G(x)) = \bar{i}(G(y))$ namely $G(i(x)) = G(i(y))$, then there exists an element of $g \in G$ such that $i(y) = gi(x) = i(gx)$. Since i is one-to-one map, it follows that $y = gx$. Therefore, we conclude that $\bar{i}$ is also one-to-one map.

Since the canonical map $\pi' : bX \to bX/G$ is closed, it follows from the above commutative diagram that $\bar{i}$ is also closed.

Let $G(p) \in bX/G$. Since $i(X)$ is dense subset of bX, then there exists a net $(z_\lambda) = (i(x_\lambda))$ in $i(X)$ such that $z_\lambda \to p$. Let us consider the net $(G(x_\lambda))$ in the orbit space X/G. Then $\bar{i}(G(x_\lambda) = G(i(x_\lambda)) = G(z_\lambda) \to G(p)$ which implies the density of $\bar{i}(X/G)$ in bX/G. Thus, bX/G is a compactification of X/G. □

Now, we consider the Gelfand construction of bX. By Theorem 15 we have $bX = M_{\Omega_{bX}}X$ for the regular subring $\Omega_{bX} = \{f|_X : f \in C^*(bX)\}$.

Let Ω'_{bX} denote the set of all $f \in \Omega_{bX}$ such that if $G(x)$ is an orbit then the restriction of f to $G(x)$ is constant.

Lemma 17 *Ω'_{bX} is a complete subring of $C^*(X)$.*

Proof It is easy to see that Ω'_{bX} is a subring of $C^*(X)$ which contains all constant functions.

Let $(f_n|_X)_{n\in\mathbb{N}}$ be a Cauchy sequence in Ω'_{bX}. Since $C^*(bX)$ is complete ring, there exists $f \in C^*(bX)$ such that $lim f_n|_X = f|_X$. Since each $f_n|_X$ has constant value on orbits, $f_n(gx) = f_n(hx)$ for each $x \in X$ and $g, h \in G$. Therefore,

$$f(gx) = \lim f_n(gx) = \lim f_n(hx) = f(hx),$$

which implies completeness of Ω'_{bX}. Therefore, by the Theorem 11, $M_{\Omega'_{bX}}X$ is a compact Hausdorff space. □

Remark 18 *Observe that $Z(\Omega'_{bX})$ can be a closed base for X only in the case of trivial action of G. Indeed, every set of $Z(\Omega'_{bX})$ is a $G-$invariant subspace of X and hence, if $Z(\Omega'_{bX})$ is a closed base for X, every closed subset of invariant, in particular, every one-point set is invariant, that is the action of G is trivial.*

Lemma 19 *The rings Ω'_{bX} and $\Omega_{bX/G} = \{f|_{X/G} : f \in C^*(bX/G)\}$ are naturally isomorphic.*

Proof If $f \in \Omega'_{bX}$, then $f = \overline{f}|_X$ for some $\overline{f} \in C^*(bX)$ and $f|_{G(x)}$ is constant for each $x \in X$. Since X is dense subspace of bX, for each $p \in bX$ there exists a net (x_λ) in X such that $\lim x_\lambda = p$.

Furthermore, the nets $(f(gx_\lambda))$ and $(f(hx_\lambda))$ are equal for each $g, h \in G$. Thus,

$$\overline{f}(gp) = \lim f(gx_\lambda) = \lim f(hx_\lambda) = \overline{f}(hp)$$

which implies that $\overline{f}|_{G(p)}$ is indeed constant for each $p \in bX$. Therefore, there exists a unique $h_{\overline{f}} \in C^*(bX/G)$ such that $\overline{f} = h_{\overline{f}} \circ \pi'$, where π' is the orbit map (i.e. $h_{\overline{f}}(G(p)) = \overline{f}(p)$ for each $p \in bX$). Let h_f denote the restriction of $h_{\overline{f}}$ to the orbit space X/G.

The isomorphism between Ω'_{bX} and $\Omega_{bX/G}$ established by the well-defined map $\varphi : \Omega'_{bX} \to \Omega_{bX/G}$ given by $f \mapsto h_f$. This map obviously preserves the ring operations.

Similarly, if $f \in \Omega_{bX/G}$, then $f = \overline{f}|_{X/G}$ for some $\overline{f} \in C^*(bX/G)$ and clearly $(\overline{f} \circ \pi')|_X \in \Omega'_{bX}$. Thus, it is easy to see that φ is isomorphism because it has inverse ring homomorphism $\varphi^{-1} : \Omega_{bX/G} \to \Omega'_{bX}$ given by $f \mapsto (\overline{f} \circ \pi')|_X$. □

Theorem 20 *bX/G is homeomorphic to the compact Hausdorff space $M_{\Omega'_{bX}}X$.*

Proof Since bX/G is a compactification of X/G, we have $bX/G = M_{\Omega_{bX/G}}(X/G)$ from the Theorem 15.

On the other hand, the above ring isomorphism $\varphi : \Omega'_{bX} \to \Omega_{bX/G}$ induces the map, $M_\varphi : M_{\Omega'_{bX}}X \to M_{\Omega_{bX/G}}(X/G)$, defined by $M_\varphi(P) = \varphi(P) = \{h_f : f \in P\}$. If $f \in \Omega_{bX/G}$,

$$M_\varphi^{-1}(S(f)) = \{P : f \in \varphi(P)\} = \{P : (\overline{f} \circ \pi')|_X \in P\} = S((\overline{f} \circ \pi')|_X)$$

Then M_φ is a continuous map.

Similarly, the inverse ring isomorphism $\varphi^{-1} : \Omega_{bX/G} \to \Omega'_{bX}$ induces the map $M_{\varphi^{-1}} : M_{\Omega_{bX/G}}(X/G) \to M_{\Omega'_{bX}}X$, defined by $M_{\varphi^{-1}}(P) = \varphi^{-1}(P)$. If $f \in \Omega'_{bX}$,

$$M_{\varphi^{-1}}^{-1}(S(f)) = \{P : f \in \varphi^{-1}(P)\} = \{P : h_f \in P\} = S(h_f)$$

which implies the continuity of $M_{\varphi^{-1}}$ and it is easily checked that

$$M_\varphi M_{\varphi^{-1}}(P) = P \text{ for each } P \in M_{\Omega_{bX/G}}(X/G)$$

and,

$$M_{\varphi^{-1}} M_\varphi(P) = P \text{ for each } P \in M_{\Omega'_{bX}}X$$

□

Remark 21 *$K_G(X)$ denotes the category of $G-$ compactifications of X. Here $ObjK_G(X) =$ all $G-$ compactifications of X and if $\alpha X, \gamma X \in ObjK_G(X)$ then the morphism set;*

$$Hom(\alpha X, \gamma X) = \{f : f : \alpha X \to \gamma X \text{ equivariant map and } f(x) = x \text{ for all } x \in X\}.$$

Theorem 22 *The description of bX/G in Theorem 20 is functorially unique.*

Proof Let $bX, \alpha X \in K_G(X)$ and $bX/G = \alpha X/G$. Suppose that $Hom(bX, \alpha X) \neq \emptyset$ or $Hom(\alpha X, bX) \neq \emptyset$. Without loss of generality, we may assume that $Hom(bX, \alpha X) \neq \emptyset$. Consider a map $\psi \in Hom(bX, \alpha X)$. It is easy to see that ψ induces a ring homomorphism

$$\overline{\psi} : \Omega'_{\alpha X} \to \Omega'_{bX}, \ \overline{\psi}(f|_X) = (f \circ \psi)|_X.$$

Since $\psi(x) = x$ for each $x \in X$, we have $h_f = h_{f\circ\psi}$ for each $f \in \Omega'_{\alpha X}$.

Therefore, we have the following commutative diagram

$$\begin{array}{ccc} \Omega'_{\alpha X} & \xrightarrow{\ \overline{\psi}\ } & \Omega'_{bX} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ \Omega_{\alpha X/G} & \xrightarrow{\ \simeq\ } & \Omega_{bX/G} \end{array} .$$

Note that the vertical homomorphisms are ring isomorphisms by Lemma 19. It follows that $\overline{\psi}$ is a ring isomorphism as desired. □

4. Applications

Now, we focus on one-point compactification of the orbit space X/G. The below proposition states that the one-point compactification of the orbit space can be characterized by the $G-$compactifications such that the restricted action on X^* is transitive.

Proposition 23 *$bX/G = a(X/G)$ iff the restricted action of G on $X^* = bX - X$ is transitive.*

Proof If $bX/G = a(X/G)$, it is clear that the remainder $(X/G)^* = bX/G - X/G = a(X/G) - X/G$ is singleton. Since $(X/G)^* = X^*/G$, it follows the restricted action on X^* is transitive.

Conversely, assume that the restricted action of G on X^* is transitive. Then the orbit space X^*/G is singleton. Since the orbit space bX/G is disjoint union the orbit spaces X/G and X^*/G, we have $bX/G - X/G$ is singleton. Therefore, by Theorem 5 and Proposition 8, we obtain that $bX/G = a(X/G)$. □

Remark 24 *The one-point compactification $aX = X \cup \{\infty\}$ of a $G-$space X is a $G-$compactification with $g\infty = \infty$ for all $g \in G$. Since the restricted action of G on $X^* = \{\infty\}$ is transitive, we have $aX/G = a(X/G)$.*

Now consider a $G-$ compactification bX of X such that the restricted action of G on $X^* = bX - X$ is transitive.

Corollary 25 *The one-point compactification of the orbit space X/G is uniquely described with bX as stated in Theorem 20.*

Proof Consider the map $\psi : bX \to aX$ defined by $\psi(x) = x$ for each $x \in x$ and $\psi(p) = \infty$ for each $p \in X^*$. Since X is locally compact, X is open subset of bX and also aX. Moreover, since $\psi|_X = Id|_X$, ψ is continuous on X.

If $p \in X^*$ and $\psi(p)$ is contained in an open set W in aX, then p is contained the open set $H = bX - (X - W)$ in bX and clearly $H \subseteq \psi^{-1}(W)$. This implies the continuity of ψ. Since ψ is a clearly $G-$map, $\psi \in Hom(bX, aX)$. Then it follows from Theorem 22, Proposition 23, and Remark 24 that the one-point compactification $a(X/G)$ is uniquely described as the maximal ideals of the complete ring Ω'_{aX}. □

Example 26 *Consider the antipodal action of $G = \mathbb{Z}_2$ on $X = (-1, 1)$. Let $bX = [-1, 1]$ be two-point compactification of X. It is easy to see that bX is a G-compactification of X and the orbit space $bX/G = [0, 1]$. Furthermore, the orbit space of $X/G = [0, 1)$. Thus, one-point compactification of $a(X/G) = bX/G$. By Theorem 20, we have that bX/G is homeomorphic to the space $M_{\Omega'_{bX}}X = \{P : P \text{ is a maximal ideal of the ring } \Omega'_{bX}\}$ with Stone topology, where*

$$\Omega'_{bX} = \{f|_{(-1,1)} : f \in C^*([-1, 1]) \text{ and } f(-x) = f(x) \text{ for each } x \in [-1, 1]\}.$$

Since a continuous function $f : (-1, 1) \to \mathbb{R}$ can be extended continuously to $[-1, +1]$ iff f is uniformly continuous, then we conclude that any closed bounded interval $[a, b]$ is homeomorphic to the space of all maximal ideals of the ring of uniformly continuous, bounded and even functions on $\mathbb{R}$.

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