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Approximation by generalized Bernstein–Stancu operators

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Abstract: In this paper, we investigate approximation properties of the Stancu type generalization of the α-Bernstein operator. We obtain a recurrence relation for moments and the rate of convergence by means of moduli of continuity. Also, we present Voronovskaya and Grüss–Voronovskaya type asymptotic results for these operators. Finally, the study contains numerical considerations regarding the constructed operators based on Maple algorithms.

Key words: Stancu operators, α-Bernstein operators, moduli of continuity, rate of convergence, Voronovskaya type theorem

1. Introduction

It is very well known that the polynomial approximation of continuous functions has an important role in numerical analysis. The Lagrange interpolating polynomials have great practical interest in approximation theory of continuous functions, but they do not provide always uniform convergence of approximating sequences for any continuous function on a compact interval of the real axis, no matter how the nodes are chosen.

In 1905, Borel proposed a way to obtain an approximation polynomial of a function $f \in C[0,1]$ by using an interpolation polynomial having a form similar to the Lagrange ones and using the nodes $x_{n,k} = \dfrac{k}{n}, \quad k = \overline{0,n}$ and with an appropriate selection of the basic polynomials $p_{n,k}$.

In 1912, Bernstein had the wonderful idea to select

$$p_{n,k} = \binom{n}{k} x^k (1-x)^{n-k},$$

inspired by the binomial probability distribution. He considered the binomial probability distribution assuming that the discrete random variable has the value $f\left(\dfrac{k}{n}\right)$ with probability $p_{n,k}(x)$ and then he calculated the mean value.

In 1969, Stancu [34] wanted to choose the nodes in another different way, in order to obtain more flexibility. He considered the nodes such as when $n \to \infty$, the distance between two consecutive nodes and the distance between 0 and first node and also between the last node and 1 tending all to zero. Thus, Stancu introduced the

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following linear positive operators, which are known as Bernstein–Stancu polynomials in the literature:

$$B_n^{\alpha,\beta}(f,x)=\sum_{k=0}^{n} f\left(\frac{k+\alpha}{n+\beta}\right)\binom{n}{k}x^k(1-x)^{n-k}, \tag{1.1}$$

acting from $C[0,1]$ into $C[0,1]$, the space of all real valued continuous functions defined on $[0,1]$, where $n\in\mathbb{N}$, $f\in C[0,1]$, $x\in[0,1]$, and α,β are fixed real numbers such that $0\le\alpha\le\beta$.

The case $\alpha=\beta=0$ gives the classical Bernstein polynomials:

$$B_n(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}. \tag{1.2}$$

The authors of the present work wish to underline the fact that many mathematicians are confronted with the following problem: Stancu [33] first introduced in 1968 the Bernstein–Stancu operators depending only on the parameter α and having the following form:

$$S_n^{\alpha}=(f,x)=\sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^{[k,-\alpha]}(1-x)^{[n-k,-\alpha]},$$

expressed with the aid of the factorial powers of x and in which α depends only on the natural number n. After one year, in 1969 he introduced the second operators called Bernstein–Stancu, but this one is dependent on two parameters α and β and it is presented above in relation (1.1) (the paper is in Romanian and this could be an impediment for readers).

After the pioneer work of Stancu, these operators have been successfully used by other mathematicians to study properties of linear positive methods of approximation. The importance of Bernstein–Stancu operators has led researchers from all over the world in the last decades to discover and study numerous combinations and generalizations of Stancu operators of both kinds. For example, we can start, chronologically speaking, with Mastroianni and Occorsio's generalizations from 1978 [28] and Della Vechia's generalization from 1988 [15]. In 1991, Campiti [9] gave a general definition of the sequence of Stancu–Muhlbach operators associated with a nonnegative projection. The limit behavior of this sequence and its iterates were studied and a quantitative estimate of the convergence was established. Also, we can remember the Agratini linear combination from 1998 [6] and the Kageyama generalization from 1999 [21]. In 2007 Gavrea and Cleciu (Radu) continued in this direction with studies from papers [13, 14, 17]. Also, in 2007, Gonska et al. [18] proved convergence results for over-iterates of certain (generalized) Bernstein–Stancu operators. Using King's technique in modifying the positive linear operators, in [26] the authors obtained a new class starting with the Bernstein–Stancu type operators investigated in [13, 31]. We remark that recently the study of Bernstein–Stancu type operators was extended in new directions as we can see, for example, in the works of [8, 10, 11, 16, 20, 25, 29, 32].

The present work is organized as follows. In the first section, we give the definition of a new family of the generalized Bernstein–Stancu operators and certain elementary properties. In the second section, the main purpose is to study some important results concerning uniform convergence and estimates of the new linear positive operators, which are direct applications of the properties and formulas recalled in the first section. Finally, we present Voronovskaya and Grüss-Voronovskaya type asymptotic results for α-Bernstein–Stancu operators. Also, we prove a Grüss type inequality with second order moduli of smoothness.

2. Construction of the generalized Bernstein–Stancu operators and the approximation properties

The aim of this paper is to construct a generalization of the Bernstein–Stancu operators defined in (1.1). The motivation for this paper is the operator constructed in 2017 by Chen et al. in [12] and it is a new family of generalized Bernstein operators depending on a nonnegative real parameter α. The α-Bernstein operators and their generalizations were extensively studied in last two years by many researchers, as we can see in [1–4, 22–24].

The new α-Bernstein operator has the form

$$T_{n,\alpha}(f;x)=\sum_{i=0}^{n} f\left(\frac{i}{n}\right)p_{n,i}^{(\alpha)}(x), \tag{2.1}$$

for any function defined on $[0,1]$, each positive integer n, and any fixed real α.

Here, for $i=\overline{0,n}$ the α-Bernstein polynomial $p_{n,i}^{(\alpha)}(x)$ of degree n is defined by

$$\begin{cases} p_{1,0}^{(\alpha)}(x) & =1-x, \\ p_{n,i}^{(\alpha)}(x) & =\left[\binom{n-2}{i}(1-\alpha)x+\binom{n-2}{i-2}(1-\alpha)(1-x)\right. \\ & \quad \left.+\binom{n}{i}\alpha x(1-x)\right]x^{i-1}(1-x)^{n-i-1}, \\ p_{1,1}^{(\alpha)}(x) & =x, \end{cases} \tag{2.2}$$

where $n\geq 2,\ x\in\ [0,1]$, and the binomial coefficients are given by

$$\binom{k}{l}=\begin{cases} \dfrac{k!}{(k-l)!\cdot l!}, & \text{if} \quad 0\leq l\leq k, \\ 0, & \text{else.} \end{cases}$$

For $\alpha=1$, the α-Bernstein operator reduces to the classical Bernstein polynomial given in (1.2). Also, for $\alpha\in[0,1]$ the operators $T_{n,\alpha}$ are linear positive operators.

On the other hand, Chen et al. showed that the α-Bernstein operator defined by (2.1) has the following representation:

$$T_{n,\alpha}(f;x)=(1-\alpha)\sum_{i=0}^{n-1} g_i\binom{n-1}{i}x^i(1-x)^{n-i-1}+\alpha\sum_{i=0}^{n} f_i\binom{n}{i}x^i(1-x)^{n-i}, \tag{2.3}$$

where $f_i=f\left(\dfrac{i}{n}\right)$ and $g_i=\left(1-\frac{i}{n-1}\right)f_i+\frac{i}{n-1}f_{i+1}$ (see formula (5) in [12]).

In this work, inspired by representation (2.3), we introduce the α-Bernstein–Stancu operators, which are linear positive operators for $0\leq\alpha\leq 1$, as follows:

$$\begin{aligned} & T_{n,\alpha}^{\alpha^*,\beta^*}(f;x) \\ = \quad & (1-\alpha)\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left[\left(1-\frac{i}{n-1}\right)f\left(\frac{i+\alpha^*}{n+\beta^*}\right)\right. \\ + \quad & \left.\frac{i}{n-1}f\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)\right]+\alpha\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}f\left(\frac{i+\alpha^*}{n+\beta^*}\right), \end{aligned} \tag{2.4}$$

for $f \in C[0,1], n \in \mathbb{N}$, and α^*, β^* satisfying the condition $0 \le \alpha^* \le \beta^*$.

We note that if $\alpha = 1$, then $T_{n,\alpha}^{\alpha^*,\beta^*}$ gives the classical Bernstein–Stancu polynomials. If $\alpha = 1$ and $\alpha^* = \beta^* = 0$, $T_{n,\alpha}^{\alpha^*,\beta^*}$ reduces to the classical Bernstein polynomials. Also, the case $\alpha^* = \beta^* = 0$ gives the α-Bernstein operator.

First, let us give some useful results that will be needed in the proof of the main theorems.

For the Bernstein–Stancu operators given in (1.1), using the test functions $e_m(x) = x^m, \quad m = \overline{0,2}$, we have the following properties (see [34] or [7], p. 119):

$$\begin{cases} B_n^{\alpha^*,\beta^*}(1;x) &= 1, \\ B_n^{\alpha^*,\beta^*}(e_1;x) &= \dfrac{nx+\alpha^*}{n+\beta^*}, \\ B_n^{\alpha^*,\beta^*}(e_2;x) &= \dfrac{n^2x^2+nx(1-x)+2\alpha^*nx+\alpha^{*2}}{(n+\beta^*)^2}. \end{cases} \tag{2.5}$$

By direct calculation we can compute the moments of higher orders as three and four.

Lemma 2.1 *For the operators $B_n^{\alpha^*,\beta^*}$ given by (1.1) we have*

$$B_n^{\alpha^*,\beta^*}(e_3;x) = \frac{n(n-1)(n-2)x^3+3n(n-1)(1+\alpha^*)x^2+n\left(1+3\alpha^*+3\alpha^{*2}\right)x+\alpha^{*3}}{(n+\beta^*)^3} \tag{2.6}$$

and

$$B_n^{\alpha^*,\beta^*}(e_4;x) = \frac{1}{(n+\beta^*)^4}\left\{n(n-1)(n-2)(n-3)x^4+2n(n-1)(n-2)(3+2\alpha^*)x^3\right.$$
$$\left.+\,n(n-1)\left(7+12\alpha^*+6\alpha^{*2}\right)x^2+n(1+4\alpha^*+6\alpha^{*2}+4\alpha^{*3})x+\alpha^{*4}\right\}. \tag{2.7}$$

Assuming that the parameters α, α^*, and β^* have a fixed nonnegative value in each term of the sequence $\{T_{n,\alpha}^{\alpha^*,\beta^*}\}$, we shall establish an important recurrence formula, which will be useful for calculating the moments of $T_{n,\alpha}^{\alpha^*,\beta^*}$ operators.

Theorem 2.2 *For all $j \in \mathbb{N}\cup\{0\}$, $n \in \mathbb{N}$, $\alpha \in [0,1]$, and $0 \le \alpha^* \le \beta^*$, we have the following recurrence formula:*

$$T_{n,\alpha}^{\alpha^*,\beta^*}(e_{j+1};x) = \frac{x(1-x)}{n+\beta^*}\left(T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)\right)' + \frac{[\alpha^*+1+(n-1)x]}{n+\beta^*}T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)$$
$$+\frac{(1-\alpha)}{(n+\beta^*)(n-1)}\left(\frac{n-1+\beta^*}{n+\beta^*}\right)^j\left[(n-1+\beta^*)B_{n-1}^{\alpha^*,\beta^*}(e_{j+1};x)\right.$$
$$\left.-(n-1+\alpha^*)B_{n-1}^{\alpha^*,\beta^*}(e_j;x)\right]-\frac{\alpha(1-x)}{n+\beta^*}B_n^{\alpha^*,\beta^*}(e_j;x), \tag{2.8}$$

where $B_n^{\alpha,\beta}$ is the nth Bernstein–Stancu operator.

Proof Using (2.4), we can write

$$T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x) = (1-\alpha)\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j\right.$$

$$\left.+\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^j\right] + \alpha\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j.$$

By some calculations we get the derivative as follows:

$$\left(T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)\right)' = (1-\alpha)\sum_{i=0}^{n-1}\binom{n-1}{i}\left[ix^{i-1}(1-x)^{n-i-1} - (n-i-1)x^i(1-x)^{n-i-2}\right]$$

$$\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j + \frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^j\right]$$

$$+\alpha\sum_{i=0}^{n}\binom{n}{i}\left[ix^{i-1}(1-x)^{n-i} - (n-i)x^i(1-x)^{n-i-1}\right]\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j$$

$$= \frac{(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\,i\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j\right.$$

$$\left.+\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^j\right] - \frac{(n-1)(1-\alpha)}{(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}$$

$$\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j + \frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^j\right]$$

$$+\frac{\alpha}{x(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\,i\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j - \frac{\alpha n}{(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j$$

$$= \frac{(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\,i\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j\right]$$

$$+\frac{(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\,i\left[\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^j\right]$$

$$-\frac{(n-1)(1-\alpha)}{(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j\right.$$

$$\left.+\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^j\right] + \frac{\alpha}{x(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\,i\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j$$

$$-\frac{\alpha n}{(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j.$$

Considering that i could be written as $i+\alpha^*-\alpha^*$, we have

$$\left(T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)\right)' = \frac{(n+\beta^*)(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j+1}\right]$$

$$+\frac{(n+\beta^*)(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left[\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^{j+1}\right]$$

$$-\frac{\alpha^*(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j}\right]$$

$$-\frac{(1+\alpha^*)(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left[\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^{j}\right]$$

$$-\frac{(n-1)(1-\alpha)}{(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j}\right.$$

$$\left.+\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^{j}\right]+\frac{\alpha(n+\beta^*)}{x(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j+1}$$

$$-\frac{\alpha^*\alpha}{x(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j}-\frac{\alpha n}{(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j}.$$

Then

$$\left(T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)\right)'=\frac{(n+\beta^*)(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}$$

$$\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j+1}+\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^{j+1}\right]$$

$$-\frac{\alpha^*(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j}+\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^{j}\right]$$

$$-\frac{(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left[\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^{j}\right]$$

$$-\frac{(n-1)(1-\alpha)}{(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}$$

$$\left[\left(1-\frac{i}{n-1}\right)\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j}+\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^{j}\right]$$

$$+\frac{\alpha(n+\beta^*)}{x(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j+1}-\frac{\alpha^*\alpha}{x(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j}$$

$$-\frac{\alpha n}{(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j}.$$

Now, putting in evidence the moments of order j and $j+1$, we arrive at

$$\left(T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)\right)'=\frac{(n+\beta^*)}{x(1-x)}T_{n,\alpha}^{\alpha^*,\beta^*}(e_{j+1};x)-\frac{\alpha^*}{x(1-x)}T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)$$

$$-\frac{(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left[\frac{i}{n-1}\left(\frac{i+1+\alpha^*}{n+\beta^*}\right)^{j}\right]$$

$$-\frac{(n-1)}{(1-x)}T_{n,\alpha}^{\alpha_1,\beta^*}(e_j;x)-\frac{\alpha}{(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j.$$

Then, by using (2.4), we have

$$\left(T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)\right)'=\frac{(n+\beta^*)}{x(1-x)}T_{n,\alpha}^{\alpha^*,\beta^*}(e_{j+1};x)-\frac{[\alpha^*+(n-1)x]}{x(1-x)}T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)$$

$$-\frac{1}{x(1-x)}T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)+\frac{(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j$$

$$-\frac{(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\frac{i}{n-1}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j$$

$$+\frac{(\alpha-\alpha x)}{x(1-x)}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j.$$

Considering again $i=i+\alpha^*-\alpha^*$, we obtain

$$\left(T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)\right)'=\frac{(n+\beta^*)}{x(1-x)}T_{n,\alpha}^{\alpha^*,\beta^*}(e_{j+1};x)-\frac{[\alpha_1+1+(n-1)x]}{x(1-x)}T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)$$

$$+\frac{(1-\alpha)}{x(1-x)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j$$

$$-\frac{(1-\alpha)}{x(1-x)}\left(\frac{n+\beta^*}{n-1}\right)\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^{j+1}$$

$$+\frac{(1-\alpha)}{x(1-x)}\frac{\alpha^*}{(n-1)}\sum_{i=0}^{n-1}\binom{n-1}{i}x^i(1-x)^{n-i-1}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j$$

$$+\frac{\alpha}{x}\sum_{i=0}^{n}\binom{n}{i}x^i(1-x)^{n-i}\left(\frac{i+\alpha^*}{n+\beta^*}\right)^j$$

and finally

$$\left(T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)\right)'=\frac{(n+\beta^*)}{x(1-x)}T_{n,\alpha}^{\alpha^*,\beta^*}(e_{j+1};x)-\frac{[\alpha^*+1+(n-1)x]}{x(1-x)}T_{n,\alpha}^{\alpha^*,\beta^*}(e_j;x)$$

$$+\frac{(1-\alpha)}{x(1-x)}\left(1+\frac{\alpha^*}{n-1}\right)\left(\frac{n-1+\beta^*}{n+\beta^*}\right)^j B_{n-1}^{\alpha^*,\beta^*}(e_j;x)$$

$$-\frac{(1-\alpha)}{x(1-x)}\left(1+\frac{\beta^*}{n-1}\right)\left(\frac{n-1+\beta^*}{n+\beta^*}\right)^j B_{n-1}^{\alpha^*,\beta^*}(e_{j+1};x)+\frac{\alpha}{x}B_n^{\alpha^*,\beta^*}(e_j;x),$$

which completes the proof. □

By application of equations (2.5)–(2.7) in the recurrence formula given by Theorem 2.2 for the α-Bernstein–Stancu operator, for $k=0,1,2,3$ we obtain step by step the following:

Lemma 2.3 *For the operators* $T_{n,\alpha}^{\alpha^*,\beta^*}$ *given in (2.4), one has*

$$T_{n,\alpha}^{\alpha^*,\beta^*}(e_0;x)=1, \tag{2.9}$$

$$T_{n,\alpha}^{\alpha^*,\beta^*}(e_1;x)=\frac{nx+\alpha^*}{n+\beta^*}, \tag{2.10}$$

$$T_{n,\alpha}^{\alpha^*,\beta^*}(e_2;x)=\frac{\left[(n-2)(n+1)+2\alpha\right]x^2+\left(n+2+2n\alpha^*-2\alpha\right)x+\alpha^{*2}}{\left(n+\beta^*\right)^2}, \tag{2.11}$$

$$\begin{aligned}T_{n,\alpha}^{\alpha^*,\beta^*}(e_3;x)&=\frac{1}{(n+\beta^*)^3}\left\{(n-2)[(n-3)(n+2)+6\alpha]x^3\right.\\&+3\left[n^2(1+\alpha^*)+n(1-\alpha^*-2\alpha)+2(\alpha-1)(3+\alpha^*)\right]x^2\\&\left.+\left[n\left(1+3\alpha^*+3\alpha^{*2}\right)+6(1-\alpha)(1+\alpha^*)\right]x+\alpha^{*3}\right\},\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}T_{n,\alpha}^{\alpha^*,\beta^*}(e_4;x)&=\frac{1}{(n+\beta^*)^4}\left\{(n-3)(n-2)\left[(n-4)(n+3)+12\alpha\right]x^4\right.\\&+\left[2n^3(3+2\alpha^*)-6n^2(1+2\alpha+2\alpha^*)+4n\left(6\alpha\alpha^*-4\alpha^*+24\alpha-21\right)\right.\\&\left.+48\left(1-\alpha\right)\left(3+\alpha^*\right)\right]x^3\\&+\left[n^2\left(7+12\alpha^*+6\alpha^{*2}\right)+n(29-36\alpha-24\alpha\alpha^*-6\alpha^{*2}+12\alpha^*)\right.\\&\left.+2\left(\alpha-1\right)\left(43+36\alpha^*+6\alpha^{*2}\right)\right]x^2\\&\left.+\left[n\left(1+4\alpha^*+6\alpha^{*2}+4\alpha^{*3}\right)+2\left(1-\alpha\right)\left(7+12\alpha^*+6\alpha^{*2}\right)\right]x+\alpha^{*4}\right\}.\end{aligned} \tag{2.13}$$

Denoting the central moments with $\varphi_x^j(t):=(t-x)^j,\ j\in\mathbb{N}$ and using Lemma (2.3), we can write the following results.

Lemma 2.4 *For the new operators* $T_{n,\alpha}^{\alpha^*,\beta^*}$, *one has*

$$\begin{aligned}T_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^1;x\right)&=\frac{\alpha^*-\beta^*x}{n+\beta^*},\\T_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^2;x\right)&=\frac{x\left(1-x\right)\left(n+2-2\alpha\right)+\beta^*x\left(\beta^*x-2\alpha^*\right)+\alpha^{*2}}{\left(n+\beta^*\right)^2}.\end{aligned}$$

Lemma 2.5 *The following statements hold:*

i) $\lim\limits_{n\to\infty}nT_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^1;x\right)=\alpha^*-\beta^*x;$

ii) $\lim\limits_{n\to\infty}nT_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^2;x\right)=x(1-x).$

The expressions of central moments lead us to the fact that, for $\alpha \in [0,1]$, $0 \le \alpha^* \le \beta^*$, and $n \in \mathbb{N}$, we have the following upper bounds.

Lemma 2.6 *The following inequalities yield:*

i) $|T_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^1;x\right)| \le \mu_{n,\alpha}^{\alpha^*,\beta^*}$,

ii) $T_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^2;x\right) \le \xi_{n,\alpha}^{\alpha^*,\beta^*}$,

where

$$\mu_{n,\alpha}^{\alpha^*,\beta^*} = \frac{\beta^* + \alpha^*}{n+\beta^*}, \tag{2.14}$$

$$\xi_{n,\alpha}^{\alpha^*,\beta^*} = \frac{n+2-2\alpha}{4(n+\beta^*)^2} + \frac{\beta^{*2} + \alpha^{*2}}{(n+\beta^*)^2}. \tag{2.15}$$

3. Direct theorems

In this section we investigate the approximation properties of the operators $T_{n,\alpha}^{\alpha^*,\beta^*}$. We provide the uniform convergence property and we estimate the rate of convergence by using moduli of continuity.

Applying the classical Korovkin theorem to the sequence of linear positive operators $T_{n,\alpha}^{\alpha^*,\beta^*}$, from (2.9)–(2.11) we have the following convergence theorem.

Theorem 3.1 *Let $f \in C[0,1]$. Then, for any $\alpha \in [0,1]$, $0 \le \alpha^* \le \beta^*$, we have*

$$\lim_{n\to\infty} \left\| T_{n,\alpha}^{\alpha^*,\beta^*}(f) - f \right\|_{C[0,1]} = 0.$$

The usual modulus of continuity of first order for $f \in C[0,1]$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and is defined as

$$\omega(f;\delta) = \sup_{|y-x|\le\delta} |f(y) - f(x)|, \;\; x,y \in [0,1].$$

It is known that the modulus of continuity of f has the following properties:

$$\omega(f;\lambda\delta) \le (1+\lambda)\omega(f;\delta)$$

and

$$|f(t) - f(x)| \le \omega(f;\delta)\left(\frac{(t-x)^2}{\delta^2} + 1\right).$$

Also, the modulus of continuity of second order is defined as

$$\omega_2(f;\delta) := \sup\left\{ \left| f(x) - 2f\left(\frac{x+t}{2}\right) + f(t) \right| : x,t \in [0,1], |x-t| \le 2\delta \right\}$$

for $f \in C[0,1]$ and $\delta > 0$.

Our next results are the following local theorems, which provide some upper bounds for the approximation error in terms of moduli of continuity of first and second order.

Theorem 3.2 *If $f \in C[0,1]$, $\alpha \in [0,1]$, and $0 \leq \alpha^* \leq \beta^*$, then for all $x \in [0,1]$,*

$$\left| T_{n,\alpha}^{\alpha^*,\beta^*}(f;x) - f(x) \right| \leq 2\omega\left(f; \sqrt{\xi_{n,\alpha}^{\alpha^*,\beta^*}}\right),$$

where $\xi_{n,\alpha}^{\alpha^*,\beta^*}$ is defined in (2.15).

Proof We have

$$|T_{n,\alpha}^{\alpha^*,\beta^*}(f;x) - f(x)| \leq T_{n,\alpha}^{\alpha^*,\beta^*}(|f(t) - f(x)|;x) \leq \omega(f;\delta)\left(1 + \frac{T_{n,\alpha}^{\alpha^*,\beta^*}(\varphi_x^2;x)}{\delta^2}\right).$$

Using Lemma 2.6, we can write

$$|T_{n,\alpha}^{\alpha^*,\beta^*}(f;x) - f(x)| \leq \omega(f;\delta)\left(1 + \frac{1}{\delta^2}\,\xi_{n,\alpha}^{\alpha^*,\beta^*}\right).$$

Thus, if we choose $\delta = \sqrt{\xi_{n,\alpha}^{\alpha^*,\beta^*}}$, we have the desired result. □

Theorem 3.3 *If f is differentiable on $[0,1]$ with f' bounded on $[0,1]$, $\alpha \in [0,1]$, and $0 \leq \alpha^* \leq \beta^*$, then for all $x \in [0,1]$,*

$$|T_{n,\alpha}^{\alpha^*,\beta^*}(f;x) - f(x)| \leq \mu_{n,\alpha}^{\alpha^*,\beta^*}|f'(x)| + \frac{5}{4}\sqrt{\xi_{n,\alpha}^{\alpha^*,\beta^*}} \cdot \omega\left(f'; \sqrt{\xi_{n,\alpha}^{\alpha^*,\beta^*}}\right).$$

Proof Using [30, Theorem 2.3.8] for $r = 2$ and Lemma 2.6 we obtain

$$|T_{n,\alpha}^{\alpha^*,\beta^*}(f;x) - f(x)| \leq \mu_{n,\alpha}^{\alpha^*,\beta^*}|f'(x)| + \left(\frac{\delta}{4} + \frac{1}{\delta}\,\xi_{n,\alpha}^{\alpha^*,\beta^*}\right)\omega(f';\delta).$$

Making $\delta = \sqrt{\xi_{n,\alpha}^{\alpha^*,\beta^*}}$, we get the desired result. □

Theorem 3.4 *For $f \in C[0,1]$, $\alpha \in [0,1]$, and $0 \leq \alpha^* \leq \beta^*$ and $x \in [0,1]$, it follows that*

$$|T_{n,\alpha}^{\alpha^*,\beta^*}(f;x) - f(x)| \leq \frac{\mu_{n,\alpha}^{\alpha^*,\beta^*}}{\sqrt{\xi_{n,\alpha}^{\alpha^*,\beta^*}}}\,\omega\left(f; \sqrt{\xi_{n,\alpha}^{\alpha^*,\beta^*}}\right) + \frac{3}{2}\,\omega_2\left(f; \sqrt{\xi_{n,\alpha}^{\alpha^*,\beta^*}}\right).$$

Proof Applying [30, Corollary 2.2.1] for $s = 2$, we have

$$|T_{n,\alpha}^{\alpha^*,\beta^*}(f;x) - f(x)| \leq |T_{n,\alpha}^{\alpha^*,\beta^*}(\varphi_x^1;x)|\delta^{-1}\omega(f;\delta)$$

$$+ \left(T_{n,\alpha}^{\alpha^*,\beta^*}(e_0;x) + \frac{1}{2}\delta^{-2}\,T_{n,\alpha}^{\alpha^*,\beta^*}(\varphi_x^2;x)\right)\omega_2(f;\delta).$$

Using Lemma 2.6,

$$|T_{n,\alpha}^{\alpha^*,\beta^*}(f;x) - f(x)| \leq \mu_{n,\alpha}^{\alpha^*,\beta^*}\delta^{-1}\omega(f;\delta) + \left(1 + \frac{1}{2}\delta^{-2}\xi_{n,\alpha}^{\alpha^*,\beta^*}\right)\omega_2(f;\delta).$$

Making $\delta = \sqrt{\xi_{n,\alpha}^{\alpha^*,\beta^*}}$, we get the desired result. □

4. Graphical analysis

In order to show the relevance of operators $T_{n,\alpha}^{\alpha^*,\beta^*}$, in this section some numerical examples are given regarding the approximation properties by using Maple. Let $f:\mathbb{R}\to\mathbb{R},\ f(x)=\sin 2\pi x$.

Example 4.1 *For $\alpha^*=0.2,\ \beta^*=0.4,\ n=5$, and different values of the parameter α the convergence of the operators $T_{n,\alpha}^{\alpha^*,\beta^*}$ to the function f is illustrated in Figure 1.*

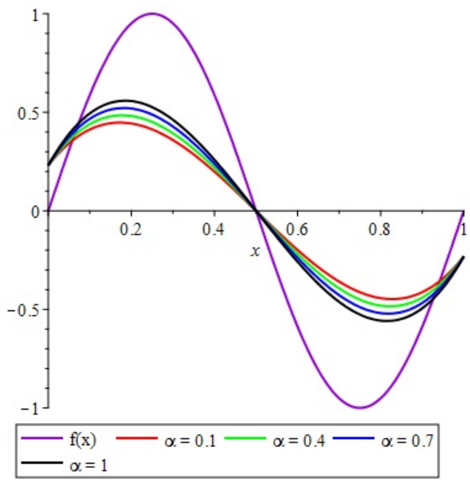

Figure 1. Approximation process by $T_{n,\alpha}^{\alpha^*,\beta^*}$.

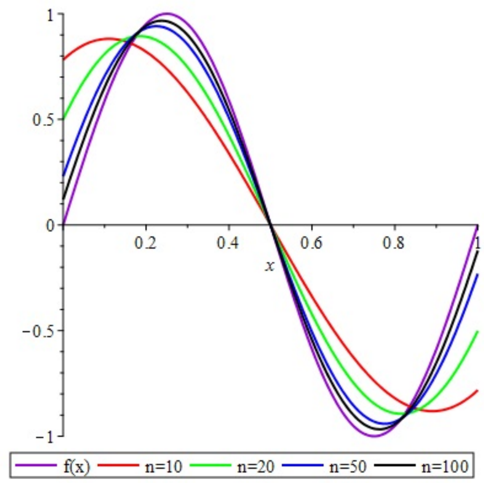

Figure 2. Approximation process by $T_{n,\alpha}^{\alpha^*,\beta^*}$.

We notice in Figure 1 that for a given natural number n, the larger the value of α is, the smaller the approximation error of operators $T_{n,\alpha}^{\alpha^*,\beta^*}$ is for $\alpha\in[0,1]$.

In Figure 2, for the given value $\alpha=0.5$, the bigger natural n is, the smaller the approximation error of operators $T_{n,\alpha}^{\alpha^*,\beta^*}$ is.

Example 4.2 *For $\alpha=0.5$ the convergence of the operators $T_{n,\alpha}^{\alpha^*,\beta^*}$ to the function f is illustrated in Figures 3, 4, and 5. We note that if n takes bigger values, the operators $T_{n,\alpha}^{\alpha^*,\beta^*}$ are going to the graph of the function f.*

5. Some Voronovskaya and Grüss–Voronovskaya type results

In this part, first, motivated by the method given by Mamedov (see [27] and [13]), we investigate a Voronovskaya type result for the α-Bernstein–Stancu operator $T_{n,\alpha}^{\alpha^*,\beta^*}$.

Theorem 5.1 *Suppose that $f\in C[0,1]$ and f has the second order derivative at $x\in[0,1]$. Then one has*

$$\lim_{n\to\infty}(n+\beta^*)\left[f(x)-T_{n,\alpha}^{\alpha^*,\beta^*}(f;x)\right]=-\frac{x(1-x)}{2}f''(x)+(\beta^*x-\alpha^*)f'(x).$$

Proof Let $\zeta:\mathbb{N}\to\mathbb{R}$ with

$$\lim_{n\to\infty}\zeta(n)=\infty$$

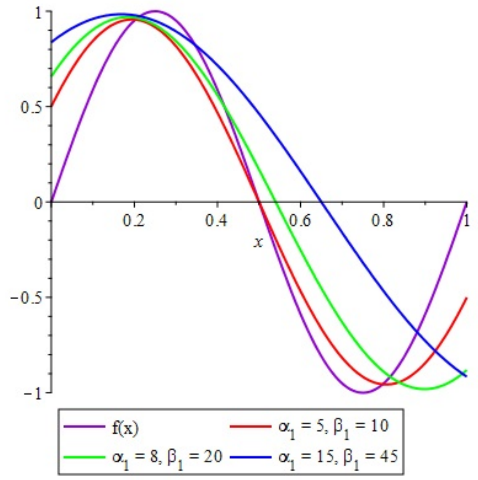

Figure 3. Approximation process by $T_{n,\alpha}^{\alpha^*,\beta^*}$ for $n = 50$.

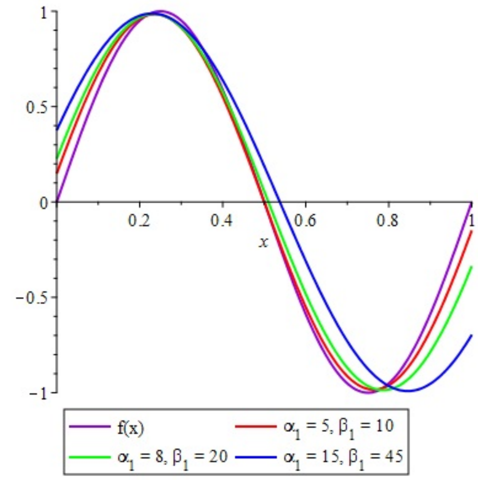

Figure 4. Approximation process by $T_{n,\alpha}^{\alpha^*,\beta^*}$ for $n = 200$.

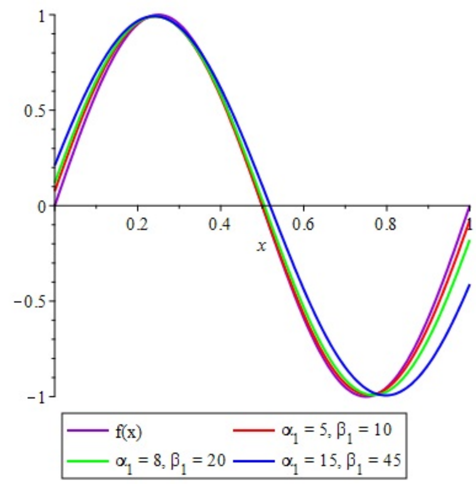

Figure 5. Approximation process by $T_{n,\alpha}^{\alpha^*,\beta^*}$ for $n = 400$.

such that

$$\lim_{n\to\infty} \zeta(n) \left[e_k(x) - T_{n,\alpha}^{\alpha^*,\beta^*}(e_k;x) \right] = \pi_k(x),$$

for $k \in \{1,2,3,4\}$.

In our case, $\zeta(n) = n + \beta^*$. Then we have

$$(n+\beta^*) \left[e_1(x) - T_{n,\alpha}^{\alpha^*,\beta^*}(e_1;x) \right] = \beta^* x - \alpha^*,$$

$$(n+\beta^*) \left[e_2(x) - T_{n,\alpha}^{\alpha^*,\beta^*}(e_2;x) \right] = \frac{\left[n(2\beta^*+1) + \left(\beta^{*2} + 2 - 2\alpha \right) \right] x^2}{n+\beta^*}$$

$$-\frac{\left[n(1+2\alpha^*) + 2(1-\alpha) \right] x}{n+\beta^*} - \frac{\alpha^{*2}}{n+\beta^*},$$

$$(n+\beta^*)\left[e_3(x)-T_{n,\alpha}^{\alpha^*,\beta^*}(e_3;x)\right]=\frac{\left[3n^2(\beta^*+1)+n\left(3\beta^{*2}+4-6\alpha\right)\right.}{(n+\beta^*)^2}$$

$$+\frac{\left.\left(\beta^{*3}+12\alpha-12\right)\right]x^3}{(n+\beta^*)^2}$$

$$-\frac{\left[3n^2(1+\alpha^*)+n(1-2\alpha-\alpha^*)+2(\alpha-1)(3+\alpha^*)\right]x^2}{(n+\beta^*)^2}$$

$$-\frac{\left[n\left(1+3\alpha^*+3\alpha^{*2}\right)+6(1-\alpha)(1+\alpha^*)\right]x}{(n+\beta^*)^2}-\frac{\alpha^{*3}}{(n+\beta^*)^2},$$

and

$$(n+\beta^*)\left[e_4(x)-T_{n,\alpha}^{\alpha^*,\beta^*}(e_4;x)\right]$$

$$=\frac{\left[2n^3(2\beta^*+3)+n^2\left(6\beta^{*2}-12\alpha+1\right)+2n\left(2\beta^{*3}+30\alpha-27\right)+\left(\beta^{*4}+72-72\alpha\right)\right]x^4}{(n+\beta^*)^3}$$

$$-\frac{\left[2n^3(3+2\alpha^*)-6n^2(1+2\alpha+2\alpha^*)+4n(6\alpha\alpha^*-4\alpha^*+24\alpha-21)\right.}{(n+\beta^*)^3}$$

$$+\frac{\left.48(1-\alpha)(3+\alpha^*)\right]x^3}{(n+\beta^*)^3}$$

$$-\frac{\left[n^2\left(7+12\alpha^*+6\alpha^{*2}\right)+n(29-36\alpha-24\alpha\alpha^*-6\alpha^{*2}+12\alpha^*)\right.}{(n+\beta^*)^3}$$

$$+\frac{\left.2(\alpha-1)\left(43+36\alpha^*+6\alpha^{*2}\right)\right]x^2}{(n+\beta^*)^3}$$

$$-\frac{\left[n\left(1+4\alpha^*+6\alpha^{*2}+4\alpha^{*3}\right)+2(1-\alpha)\left(7+12\alpha^*+6\alpha^{*2}\right)\right]x}{(n+\beta^*)^3}-\frac{\alpha^{*4}}{(n+\beta^*)^3}.$$

Thus, the following limits can be obtained:

$$\pi_1(x)=\lim_{n\to\infty}(n+\beta^*)\left[e_1(x)-T_{n,\alpha}^{\alpha^*,\beta^*}(e_1;x)\right]=\beta^*x-\alpha^*,$$

$$\pi_2(x)=\lim_{n\to\infty}(n+\beta^*)\left[e_2(x)-T_{n,\alpha}^{\alpha^*,\beta^*}(e_2;x)\right]=(2\beta^*+1)x^2-(1+2\alpha^*)x,$$

$$\pi_3(x)=\lim_{n\to\infty}(n+\beta^*)\left[e_3(x)-T_{n,\alpha}^{\alpha^*,\beta^*}(e_3;x)\right]=(3\beta^*+3)x^3-(3+3\alpha^*)x^2,$$

$$\pi_4(x)=\lim_{n\to\infty}(n+\beta^*)\left[e_4(x)-T_{n,\alpha}^{\alpha^*,\beta^*}(e_4;x)\right]=(4\beta^*+6)x^4-(6+4\alpha^*)x^3.$$

Using the identity

$$T_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^4;x\right)=-\left(x^4-T_{n,\alpha}^{\alpha^*,\beta^*}(e_4;x)\right)+4x\left(x^3-T_{n,\alpha}^{\alpha^*,\beta^*}(e_3;x)\right)$$

$$-6x^2\left(x^2 - T_{n,\alpha}^{\alpha^*,\beta^*}(e_2;x)\right) + 4x^3\left(x - T_{n,\alpha}^{\alpha^*,\beta^*}(e_1;x)\right),$$

it follows that

$$\lim_{n\to\infty}(n+\beta^*)\,T_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^4;x\right) = -\pi_4(x) + 4x\pi_3(x) - 6x^2\pi_2(x) + 4x^3\pi_1(x) = 0.$$

Then we have

$$\lim_{n\to\infty}\zeta(n)\left[f(x) - T_{n,\alpha}^{\alpha^*,\beta^*}(f;x)\right] = \left[f'(x) - xf''(x)\right]\pi_1(x) + \frac{\pi_2(x)}{2}f''(x),$$

which completes the proof. □

In the following, we will provide Grüss–Voronovskaya type theorems for α-Bernstein–Stancu operator $T_{n,\alpha}^{\alpha^*,\beta^*}$, inspired by the fact that in 1935 Grüss obtained an inequality that estimates the difference between the integral of a product and the product of integrals for two functions.

In 2011 Acu et al. in [5] and Gonska and Tachev in [19] obtained Grüss type inequalities and applied them for the Bernstein operators and other classical linear positive operators. We will do the same for the $T_{n,\alpha}^{\alpha^*,\beta^*}$ operators.

We denote by $T_{n,\alpha}^{\alpha^*,\beta^*}(f,g;x) = T_{n,\alpha}^{\alpha^*,\beta^*}(fg;x) - T_{n,\alpha}^{\alpha^*,\beta^*}(f;x)T_{n,\alpha}^{\alpha^*,\beta^*}(g;x)$ and for our operators we obtain:

Theorem 5.2 *Let $f,g\in C^2[0,1]$, $\alpha\in[0,1]$, and $0\le\alpha^*\le\beta^*$. Then, for each $x\in[0,1]$,*

$$\lim_{n\to\infty} n\cdot T_{n,\alpha}^{\alpha^*,\beta^*}(f,g;x) = x(1-x)f'(x)g'(x).$$

Proof Since

$$(fg)(x) = f(x)g(x), \quad (fg)'(x) = f'(x)g(x) + f(x)g'(x),$$

and

$$(fg)''(x) = f''(x)g(x) + 2f'(x)g'(x) + f(x)g''(x),$$

we can write

$$T_{n,\alpha}^{\alpha^*,\beta^*}(f,g;x) = T_{n,\alpha}^{\alpha^*,\beta^*}(fg;x) - T_{n,\alpha}^{\alpha^*,\beta^*}(f;x)\cdot T_{n,\alpha}^{\alpha^*,\beta^*}(g;x)$$

$$= \left\{T_{n,\alpha}^{\alpha^*,\beta^*}(fg;x) - f(x)g(x) - (fg)'(x)T_{n,\alpha}^{\alpha^*,\beta^*}(\varphi_x^1;x) - \frac{(fg)''(x)}{2!}T_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^2;x\right)\right\}$$

$$-g(x)\left\{T_{n,\alpha}^{\alpha^*,\beta^*}(f;x) - f(x) - f'(x)T_{n,\alpha}^{\alpha^*,\beta^*}(\varphi_x^1;x) - \frac{f''(x)}{2!}\,T_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^2;x\right)\right\}$$

$$-T_{n,\alpha}^{\alpha^*,\beta^*}(f;x)\left\{T_{n,\alpha}^{\alpha^*,\beta^*}(g;x) - g(x) - g'(x)T_{n,\alpha}^{\alpha^*,\beta^*}(\varphi_x^1;x) - \frac{g''(x)}{2!}T_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^2;x\right)\right\}$$

$$+\frac{1}{2!}T_{n,\alpha}^{\alpha^*,\beta^*}\left(\varphi_x^2;x\right)\left\{f(x)g''(x) + 2f'(x)g'(x) - g''(x)T_{n,\alpha}^{\alpha^*,\beta^*}(f;x)\right\}$$

$$+T_{n,\alpha}^{\alpha^*,\beta^*}(\varphi_x^1;x)\left\{f(x)g'(x) - g'(x)T_{n,\alpha}^{\alpha^*,\beta^*}(f;x)\right\}.$$

Therefore, by using Lemma 2.5, we get

$$\lim_{n\to\infty} n\cdot T_{n,\alpha}^{\alpha^*,\beta^*}(f,g;x) = \lim_{n\to\infty} n\left\{T_{n,\alpha}^{\alpha^*,\beta^*}(fg;x) - T_{n,\alpha}^{\alpha^*,\beta^*}(f;x)\cdot T_{n,\alpha}^{\alpha^*,\beta^*}(g;x)\right\}$$

$$= \lim_{n\to\infty} n\left[T_{n,\alpha}^{\alpha^*,\beta^*}(fg;x) - f(x)g(x)\right] - (\alpha^*-\beta^*x)(fg)'(x) - \frac{x(1-x)}{2}(fg)''(x)$$

$$-g(x)\left\{\lim_{n\to\infty} n\left[T_{n,\alpha}^{\alpha^*,\beta^*}(f;x) - f(x)\right] - (\alpha^*-\beta^*x)f'(x) - \frac{x(1-x)}{2}f''(x)\right\}$$

$$-\lim_{n\to\infty} T_{n,\alpha}^{\alpha^*,\beta^*}(f;x)\left\{\lim_{n\to\infty} n\left[T_{n,\alpha}^{\alpha^*,\beta^*}(g;x) - g(x)\right] - (\alpha^*-\beta^*x)g'(x) - \frac{x(1-x)}{2}g''(x)\right\}$$

$$+\frac{x(1-x)}{2}\left\{g''(x)\lim_{n\to\infty}\left[f(x) - T_{n,\alpha}^{\alpha^*,\beta^*}(f;x)\right] + 2f'(x)g'(x)\right\}$$

$$+(\alpha^*-\beta^*x)g'(x)\lim_{n\to\infty}\left[f(x) - T_{n,\alpha}^{\alpha^*,\beta^*}(f;x)\right].$$

Considering Theorem 3.1 and Theorem 5.1, we arrive at the desired result. □

Now, in light of the results obtained by Gonska and Tachev in [19], we have our final estimations as follows.

Theorem 5.3 *Let* $f,g\in C[0,1]$, $x\in[0,1]$, $\alpha\in[0,1]$, *and* $0\le\alpha^*\le\beta^*$. *Then*

$$|T_{n,\alpha}^{\alpha^*,\beta^*}(f,g;x)| \le \frac{3}{2}\cdot\sqrt{T(f,x)\cdot T(g,x)},$$

where $T(f,x) := \omega_2\left(f^2;\sqrt{\xi_{n,\alpha}^{\alpha^*,\beta^*}}\right) + 2||f||\cdot\omega_2\left(f;\sqrt{\xi_{n,\alpha}^{\alpha^*,\beta^*}}\right)$ *and* $T(g,x)$ *is defined analogously.*

Proof In [19, Theorem 1] we consider $H = T_{n,\alpha}^{\alpha^*,\beta^*}$. □

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