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On Hirano inverses in rings

Huanyin CHEN[1,*], **Marjan SHEIBANI**[2]
[1]Department of Mathematics, Hangzhou Normal University, Hangzhou, China
[2]Women's University of Semnan (Farzanegan), Semnan, Iran

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Abstract: We completely characterize a subclass of Drazin inverses by means of tripotents and nilpotents. We prove that an element a in a ring R has Hirano inverse if and only if $a^2 \in R$ has strongly Drazin inverse, if and only if $a - a^3$ is nilpotent. If $\frac{1}{2} \in R$, we prove that $a \in R$ has Hirano inverse if and only if there exists $p^3 = p \in comm^2(a)$ such that $a - p \in N(R)$, if and only if there exist two idempotents $e, f \in comm^2(a)$ such that $a + e - f \in N(R)$. Multiplicative and additive results for this generalized inverse are thereby obtained.

Key words: Drazin inverse, nilpotent, tripotent, multiplicative property, Jacobson's lemma

1. Introduction

Let R be an associative ring with an identity. An element a in R has Drazin inverse if there is a common solution to the equations $a^k = a^{k+1}x, x = xax, ax = xa$ for some $k \in \mathbb{N}$. As is well known, an element $a \in R$ has Drazin inverse if there exists $b \in R$ such that

$$a - a^2 b \in N(R), ab = ba \text{ and } b = bab.$$

The preceding b is unique, if such element exists. As usual, it will be denoted by a^D, and called the Drazin inverse of a.

The Drazin inverse is an important tool in ring theory and Banach algebra. It is very useful in matrix theory and in various applications in matrices (see [8–10]). The purpose of this paper is to introduce and study a new subclass of Drazin inverses. An element $a \in R$ has Hirano inverse if there exists $b \in R$ such that

$$a^2 - ab \in N(R), ab = ba \text{ and } b = bab.$$

The preceding b shall be unique, if such element exists. This is the dual of Drazin inverse in a ring. We observed such generalized inverses form a subclass of Drazin inverses which is characterized by tripotents. Here, $p \in R$ is a tripotent if $p^3 = p$ (see [1]).

Following [8], an element $a \in R$ has strongly Drazin inverse b if

$$a - ab \in N(R), ab = ba \text{ and } b = bab.$$

In Section 2, we prove that every Hirano inverse of an element is its Drazin inverse. An element $a \in R$ has Hirano inverse if and only if a^2 has strongly Drazin inverse.

*Correspondence: huanyinchen@aliyun.com
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In Section 3, we characterize Hirano inverses via tripotents and nilpotents. It is shown that an element $a \in R$ has Hirano inverse if and only if $a - a^3 \in N(R)$. If $\frac{1}{2} \in R$, we prove that $a \in R$ has Hirano inverse if and only if there exists $p^3 = p \in comm^2(a)$ such that $a - p \in N(R)$, if and only if there exist two idempotents $e, f \in comm^2(a)$ such that $a + e - f \in N(R)$.

Let R be a ring, and let $a, b, c \in R$ with $aba = aca$. In Section 4, we establish multiplicative property for Hirano inverses. We prove that ac has Hirano inverse if and only if ba has Hirano inverse. In the last section, we dwell on Jacobson's Lemma for such generalized inverses. We prove that $1 + ac$ has Hirano inverse if and only if $1 + ba$ has Hirano inverse.

Throughout, all rings are associative with an identity. We use $N(R)$ to denote the set of all nilpotent elements in R. The double commutant of $a \in R$ is defined by $comm^2(a) = \{x \in R \mid xy = yx \text{ if } ay = ya \text{ for } y \in R\}$. $\mathbb{N}$ stands for the set of all natural numbers.

2. Drazin inverse

The aim of this section is to investigate the relations among Drazin inverse, strongly Drazin inverse, and Hirano inverse. We have

Theorem 2.1 *Let R be a ring and $a \in R$. If a has Hirano inverse, then a has Drazin inverse.*

Proof Suppose that a has Hirano inverse b. Then $a^2 - ab \in N(R)$, $ab = ba$ and $bab = b$. Hence, $a^2 - a^2b^2 = a^2 - a(bab) = a^2 - ab \in N(R)$, and so

$$a^2(1 - a^2b^2) = (a^2 - a^2b^2)(1 - a^2b^2) \in N(R).$$

It follows that $a(a - a^2b) = a^2(1 - a^2b^2) \in N(R)$; hence, $(a - a^2b)^2 = a(a - a^2b)(1 - ab) \in N(R)$. Thus, $a - a^2b \in N(R)$. Moreover, we see that $ab = ba$ and $bab = b$. Therefore, a has Drazin inverse b, as desired. □

Corollary 2.2 *Let R be a ring and $a \in R$. Then a has at most one Hirano inverse in R, and if the Hirano inverse of a exists, it is exactly the Drazin inverse of a.*

Proof Let a has Hirano inverse x. In view of Theorem 2.1, a has Drazin inverse. By virtue of [8, Theorem 2.4], the Drazin inverse of a is unique. Thus, a has at most one Hirano inverse in R, as desired. □

Example 2.3 *Every 2×2 matrix over $\mathbb{Z}_2$ has Drazin inverse, but $a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_2)$ has no Hirano inverse.*

Proof Since $M_2(\mathbb{Z}_2)$ is a finite ring, every element in $M_2(\mathbb{Z}_2)$ has Drazin inverse. If $a = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_2)$ has Hirano inverse, it follows by Corollary 2.2 that its Hirano inverse is $a^D = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$. However, $a^2 - aa^D = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is not nilpotent. This gives a contradiction, and we are through. □

Theorem 2.4 *Let R be a ring and $a \in R$. Then a has Hirano inverse if and only if a^2 has strongly Drazin inverse.*

Proof $\Longrightarrow$ Since a has Hirano inverse, we have $b \in R$ such that $a^2 - ab \in N(R), ab = ba$ and $b = bab$. Thus, $a^2 - a^2b^2 = a^2 - a(bab) = a^2 - ab \in N(R), a^2b^2 = b^2a^2$ and $b^2a^2b^2 = b^2$. Therefore, $a^2 \in R$ has strongly Drazin inverse.

$\Longleftarrow$ Since a^2 has strongly Drazin inverse, we have $x \in R$ such that $a^2 - a^2x \in N(R), ax = xa$ and $x = xa^2x$. Set $b = ax$. Then $a^2 - ab = a^2 - a^2x \in N(R), ab = ba$ and $bab = (ax)a(ax) = a(xa^2x) = ax = b$. Hence, a has Hirano inverse. $\square$

Example 2.5 *Every element in $\mathbb{Z}_3$ has Hirano inverse, but $2 \in \mathbb{Z}_3$ has no strongly Drazin inverse.*

Proof One directly checks that every element in $\mathbb{Z}_3$ has Hirano inverse. If $2 \in \mathbb{Z}_3$ has strongly Drazin inverse, then $2 = 2(2^D)$ and $2^D = 2(2^D)^2$, whence $-1 = -2^D$, and so $2^D = 1$. This gives a contradiction. This shows that $2 \in \mathbb{Z}_3$ has no strongly Drazin inverse. $\square$

Example 2.6 *Let* $a = \begin{pmatrix} -2 & 3 & 2 \\ -2 & 3 & 2 \\ 1 & -1 & -1 \end{pmatrix} \in M_3(\mathbb{Z})$. *Then a has Hirano inverse, but it has no strongly Drazin inverse.*

Proof Clearly, $a = a^3$, and so a^2 has strongly Drazin inverse a^2; thus, a has Hirano inverse, by Theorem 2.4. On the other hand, the characteristic polynomial $\chi(a - a^2) = t^2(t+2) \not\equiv t^3 (mod\ N(R))$. In light of [3, Proposition 4.2], $a - a^2$ is not nilpotent. Therefore, a does not have strongly Drazin inverse by [8, Lemma 2.2]. $\square$

Thus, we have the relations of various type of inverses in a ring by the following:

$$\{\text{strongly Drazin inverses}\} \subsetneq \{\text{Hirano inverses}\} \subsetneq \{\text{Drazin inverses}\}.$$

Example 2.7 *Let $\mathbb{C}$ be the field of complex numbers.*

(1) *$2 \in \mathbb{C}$ is invertible, but 2 has no Hirano inverse in $\mathbb{C}$.*

(2) *$A \in M_n(\mathbb{C})$ has Hirano inverse if and only if there exists a decomposition $\mathbb{C}^n = U \oplus V$ such that U, V are A-invariant and $A^2 \mid_U, (I_n - A^2) \mid_V$ are both nilpotent if and only if eigenvalues of A^2 are only 0 or 1.*

Proof (1) Straightforward.

(2) In view of Theorem 2.4, $A \in M_n(\mathbb{C})$ has Hirano inverse if and only if $A^2 \in M_n(\mathbb{C})$ has strongly Drazin inverse if and only if A^2 is the sum of an idempotent and an nilpotent matrices that commute. Therefore, we are done by [8, Lemma 2.2]. $\square$

3. Characterizations

The purpose of this section is to characterize Hirano inverses by means of tripotents and nilpotents in a ring. The following result is crucial.

Theorem 3.1 *Let R be a ring, and let $a \in R$. Then the following are equivalent:*

(1) *$a \in R$ has Hirano inverse.*

(2) *There exists* $b \in comm^2(a)$ *such that* $b = b^2a, a^2 - ab \in N(R)$.

(3) $a - a^3 \in N(R)$.

Proof (1) $\Rightarrow$ (3) Since a has Hirano inverse, we have $b \in R$ such that

$$w := a^2 - ab \in N(R), ab = ba \text{ and } b = bab.$$

Hence, $a^2 = ab + w$ and $(ab)w = w(ab)$. Thus, $a^2 - a^4 = (ab + w) - (ab + w)^2 = (1 - 2ab - w)w \in N(R)$. Hence, $a(a - a^3) \in N(R)$, and so $(a - a^3)^2 = a(a - a^3)(1 - a^2) \in N(R)$. Accordingly, $a - a^3 \in N(R)$.

(3) $\Rightarrow$ (2) Suppose that $a - a^3 \in N(R)$. Then $a^2 - a^4 = a(a - a^3) \in N(R)$. In view of [2, Lemma 2.1], there exists an idempotent $e \in \mathbb{Z}[a^2]$ such that $w := a^2 - e \in N(R)$ and $ae = ea$. Set $c = (1 + w)^{-1}e$. Then $ac = ca$ and

$$\begin{array}{rcl} a^2c & = & a^2(1+w)^{-1}e \\ & = & (1+w)^{-1}e(a^2+1-e) \\ & = & (1+w)^{-1}e(1+w) \\ & = & e. \end{array}$$

Hence, $a^2 - a^2c = a^2 - e = w \in N(R)$. Moreover, $ca^2c = ce = c$. Set $b = ac$. Then $a^2 - ab \in N(R), ab = ba$ and $bab = (ac)a(ac) = (ca^2c)a = ca = b$. Since $b = a(1 + a^2 - e)^{-1}e$ and $e \in \mathbb{Z}[a^2]$, we easily check that $b \in comm^2(a)$, as required.

(2) $\Rightarrow$ (1) This is obvious, as $comm^2(a) \subseteq comm(a)$. □

Corollary 3.2 *Let* R *be a ring and* $a \in R$. *If* a *has Hirano inverse, then* a^D *has Hirano inverse.*

Proof Write $b = a^D$. Then $w := a^2 - ab \in N(R), ab = ba$ and $b = bab$. In view of Theorem 3.1, $a - a^3 \in N(R)$. One easily checks that

$$\begin{array}{rcl} b - b^3 & = & b^2a - b^2(bab) \\ & = & b^2a - b^3(a^2 - w) \\ & = & b^2a - b(b^2a)a - b^3w \\ & = & b^2a - (b^2a)ba - b^3w \\ & = & -b^3w \\ & \in & N(R). \end{array}$$

By using Theorem 3.1 again, b has Hirano inverse. This completes the proof. □

We are now ready to prove the following.

Theorem 3.3 *Let* R *be a ring with* $\frac{1}{2} \in R$, *and let* $a \in R$. *Then the following are equivalent:*

(1) $a \in R$ *has Hirano inverse.*

(2) *There exists* $p^3 = p \in comm^2(a)$ *such that* $a - p \in N(R)$.

(3) *There exist two idempotents* $e, f \in comm^2(a)$ *such that* $a + e - f \in N(R)$.

Proof (1) $\Rightarrow$ (3) In view of Theorem 3.1, $a - a^3 \in N(R)$. Set $b = \frac{a^2+a}{2}$ and $c = \frac{a^2-a}{2}$. Then

$$\begin{array}{l} b^2 - b = \frac{1}{4}(a^4 + 2a^3 - a^2 - 2a) = \frac{1}{4}(a+2)(a^3 - a), \\ c^2 - c = \frac{1}{4}(a^4 - 2a^3 - a^2 + 2a) = \frac{1}{4}(a-2)(a^3 - a). \end{array}$$

Thus, $b-b^2, c-c^2 \in N(R)$. In view of [2, Lemma 2.1], there exist two idempotents $e \in \mathbb{Z}[c]$ and $f \in \mathbb{Z}[b]$ such that $b-f, c-e \in N(R)$. One easily checks that $a = b-c$ and $b, c \in \mathbb{Z}[\frac{a}{2}]$. Therefore, $a-f+e = (b-f)-(c-e) \in N(R)$. We see that $e, f \in \mathbb{Z}[\frac{a}{2}] \subseteq comm^2(a)$, as desired.

(3) $\Rightarrow$ (2) Set $p = f-e$. Then $a-p \in N(R)$ and $p^3 = (f-e)^3 = f-e = p$, as $ef = fe$.

(2) $\Rightarrow$ (1) By hypothesis, there exists $p^3 = p \in comm^2(a)$ such that $w := a-p \in N(R)$. Thus, $a-a^3 = (p+w)-(p+w)^3 = w(1-3pw-3p^2-w^2) \in N(R)$. This completes the proof, by Theorem 3.1. $\square$

Corollary 3.4 *Let R be a ring with $\frac{1}{2} \in R$. Then $a \in R$ has Hirano inverse if and only if there exist commuting $b, c \in R$ such that $a = b-c$, where b, c have strongly Drazin inverses.*

Proof $\Longrightarrow$ In light of Theorem 3.3, there exist two idempotents $e, f \in comm^2(a)$ such that $w := a+e-f \in N(R)$. Hence, $a = f-(e-w)$ with $w \in \mathbb{Z}[\frac{a}{2}]$. Clearly, $f \in R$ has strongly Drazin inverse. Set $d = (1-w)^{-1}e$. Then we check that $(e-w)d = d(e-w)$, $(e-w)-(e-w)d = (1-w)^{-1}\big((e-w)(1-w)-(e-w)e\big) = -w \in N(R)$ and $d(e-w)d = (1-w)^{-1}(e-ew)(1-w)^{-1}e = (1-w)^{-1}e = d$. Thus, $e-w \in R$ has strongly Drazin inverse. Set $b = f$ and $c = e-w$. Then $a = b-c$ and $bc = cb$, as desired.

$\Longleftarrow$ Suppose that $a = b-c$, where b, c have strongly Drazin inverses and $bc = cb$. Then we have $x \in R$ such that $w := b-bx \in N(R), bx = xb$ and $x = xbx$. Thus, $b-b^2 = bx-b^2+w = b^2x^2-b^2+w = (w-b)^2-b^2+w = (w-2b+1)w \in N(R)$, as $w(w-2b+1) = (w-2b+1)w$. Likewise, $c-c^2 \in N(R)$. In view of [2, Lemma 2.1], there exist $e \in \mathbb{Z}[b]$ and $f \in \mathbb{Z}[c]$ such that $u = b-e, v = c-f \in N(R)$. Thus, $a = (e+u)-(f+v) = e-f+(u-v)$. As $bc = cb$, we see that e, f, u, v commute one another. It is easy to verify that $(e-f)^3 = e-f$, and so $a-a^3 \in N(R)$. In light of Theorem 3.1, $a \in R$ has Hirano inverse. This completes the proof. $\square$

As 2 is invertible in any Banach algebra, we now derive

Corollary 3.5 *Let A be a Banach algebra and let $a \in A$. Then the following are equivalent:*

(1) *a has Hirano inverse.*

(2) *There exists $p^3 = p \in comm^2(a)$ such that $a-p \in N(A)$.*

(3) *There exist two idempotents $e, f \in comm^2(a)$ such that $a+e-f \in N(A)$.*

(4) *There exist commuting $b, c \in R$ such that $a = b-c$, where b, c have strongly Drazin inverses.*

Recall that a ring R is strongly 2-nil-clean if every element in R is the sum of a tripotent and a nilpotent that commute (see [2]). We characterize strongly 2-nil-clean rings by using Hirano inverses.

Proposition 3.6 *A ring R is strongly 2-nil-clean if and only if every element in R has Hirano inverse.*

Proof $\Longrightarrow$ Let $a \in R$. In view of [2, Theorem 2.3], $a-a^3 \in N(R)$. Thus, a has Hirano inverse by Theorem 3.1.

$\Longleftarrow$ Let $a \in R$. In light of Theorem 3.1, $a-a^3 \in N(R)$. Therefore, we complete the proof, by [2, Theorem 2.3]. $\square$

Corollary 3.7 *A Banach algebra A is strongly 2-nil-clean if and only if every element in A is the difference of two commuting elements that have strongly Drazin inverses.*

Proof Combining Proposition 3.6 and Corollary 3.5, we obtain the result. $\square$

4. Multiplicative property

Let $a, b \in R$. Then ab has Drazin inverse if and only if ba has Drazin inverse. In this case, $(ba)^D = b((ab)^D)^2 a$. This was known as Cline's formula for Drazin inverses. It plays an important role in revealing the relationship between the Drazin inverse of a sum of two elements and the Drazin inverse of a block triangular matrix. In [14], Zeng and Zhong extended this result and proved that if $aba = aca$ then $1 + ac$ has Drazin inverse if and only if ba has Drazin inverse. The goal of this section is to generalize Cline's formula from Drazin inverses to Hirano inverses. We derive

Theorem 4.1 *Let R be a ring, and let $a, b, c \in R$. If $aba = aca$, Then ac has Hirano inverse if and only if ba has Hirano inverse.*

Proof $\Longrightarrow$ By induction on n we shall prove

$$(ba - (ba)^3)^{n+1} = b(ac - (ac)^3)^n (a - ababa).$$

For $n = 1$,

$$\begin{aligned} &((ba - (ba)^3))^2 \\ &= (ba - bababa)(ba - bababa) \\ &= b(a - ababa)ba(1 - baba) \\ &= b(aba - abababa)(1 - baba) \\ &= b(ac - acacac)(a - ababa) \\ &= b(ac - (ac)^3)(a - ababa). \end{aligned}$$

Suppose that the assertion is true for any $n \leq k (k \geq 2)$. Then,

$$\begin{aligned} &(ba - (ba)^3)^{k+1} \\ &= ((ba - (ba)^3))^k (ba - (ba)^3) \\ &= b(ac - (ac)^3)^{k-1}(a - ababa)(ba - bababa) \\ &= b(ac - (ac)^3)^{k-1}(aca - acacaca)(1 - baba) \\ &= b(ac - (ac)^3)^k (a - ababa). \end{aligned}$$

As ac has Hirano inverse, then by Theorem 3.1, $(ac - (ac)^3)^n = 0$ for some $n \in \mathbb{N}$, which implies that $ba - (ba)^3 \in N(R))$ by the preceding discussion. In light of Theorem 3.1, ba has Hirano inverse.

$\Longleftarrow$ This is symmetric. $\square$

Corollary 4.2 *Let R be a ring and $a, b \in R$. If ab has Hirano inverse, then so has ba.*

Proof It follows directly from Theorem 4.1, as $aba = aba$. $\square$

Theorem 4.3 *Let R be a ring, and let $a, b, c \in R$. If $aba = aca$, then $(ac)^k$ has Hirano inverse if and only if $(ba)^k$ has Hirano inverse.*

Proof $\Longrightarrow$ Let $k=1$, then by Theorem 4.1, ba has Hirano inverse. Now let $k\geq 2$. Clearly, $(ab)^k=\big(ac(ac)^{k-2}\big)(ab)$. As $(ab)\big(ac(ac)^{k-2}\big)=(ac)^k$ has Hirano inverse, it follows by Corollary 4.2 that $(ab)^k$ has Hirano inverse. Since $(ab)^k=a\big(b(ab)^{k-1}\big)$, then by Corollary 4.2, $(b(ab)^{k-1})a$ has Hirano inverse, so $(ba)^k$ has Hirano inverse.

$\Longleftarrow$ This is symmetric. $\square$

Theorem 4.4 *Let R be a ring, and let $a,b\in R$. If a,b have Hirano inverses and $ab=ba$. Then ab has Hirano inverse.*

Proof Since a,b have Hirano inverses, by Theorem 3.1, we have $a-a^3, b-b^3\in N(R)$. Note that

$$\begin{array}{rcl} ab-(ab)^3 & = & (a-a^3)(b-b^3)+ab^3+a^3b-a^3b^3-a^3b^3 \\ & = & (a-a^3)(b-b^3)+a^3(b-b^3)+b^3(a-a^3) \\ & \in & N(R), \end{array}$$

as $ab=ba$. By using Theorem 3.1 again, ab has Hirano inverse. By Theorem 2.1, ab has Drazin inverse, as asserted. $\square$

Corollary 4.5 *Let R be a ring. If $a\in R$ has Hirano inverse, then a^n has Hirano inverse.*

Proof By Theorem 4.4 and induction, we easily obtain the result. $\square$

We note that the converse of the preceding corollary is not true, as the following shows.

Example 4.6 *Let $\mathbb{Z}_5=\mathbb{Z}/5\mathbb{Z}$ be the ring of integers modulo 5. Then $-2\in\mathbb{Z}_5$ has no Hirano inverse, as $(-2)-(-2)^3=1\in\mathbb{Z}_5$ is not nilpotent, but $(-2)^2=-1\in\mathbb{Z}$ has Hirano inverse, as $(-1)-(-1)^3\in\mathbb{Z}_5$ is nilpotent.*

5. Additive results

Let $a,b\in R$. Jacobson's Lemma asserted that $1+ab\in R$ has Drazin inverse if and only if $1+ba\in R$ has Drazin inverse. More recently, Yan and Fang studied the link between basic operator properties of $1+ac$ and $1+ba$ when $aba=aca$ (see [11]). Mosic proved that the Drazin invertibility of $1+ac$ and $1+ba$ coincide under $aba=aca$ (see [6, Corollary 2.6]). We next generalize Jacobson's Lemma for Hirano inverses.

Theorem 5.1 *Let R be a ring, and let $a,b,c\in R$. If $aba=aca$, then $1+ac$ has Hirano inverse if and only if $1+ba$ has Hirano inverse.*

Proof $\Longrightarrow$ First by induction on n we prove that

$$((1+ba)-(1+ba)^3)^{n+1}=b((1+ac)-(1+ac)^3)^n(-2a-3aba-ababa).$$

For $n=1$,

$$\begin{array}{l} ((1+ba)-(1+ba)^3)^2 \\ =(-2ba-3baba-bababa)ba(-2-3ba-baba) \\ =b(-2aba-3ababa-abababa)(-2-3ba-baba) \\ =b(-2ac-3acac-acacac)(-2a-3aba-ababa) \\ =b((1+ac)-(1+ac)^3)(-2a-3aba-ababa). \end{array}$$

Let $k \geq 2$ and the assertion works for any $n \leq k$. Then

$$\begin{aligned}&((1+ba)-(1+ba)^3)^{k+1}\\&=((1+ba)-(1+ba)^3)^k((1+ba)-(1+ba)^3)\\&=b((1+ac)-(1+ac)^3)^k(-2a-3aba-ababa).\end{aligned}$$

As $1+ac$ has Hirano inverse, then by Theorem 3.1, $(1+ac)-(1+ac)^3 \in N(R)$. By the above equality, $(1+ba)-(1+ba)^3 \in N(R)$. In light of Theorem 3.1, $(1+ba)$ has Hirano inverse.

$\Longleftarrow$ This is symmetric. □

Corollary 5.2 *Let R be a ring, and let $a,b \in R$. Then $1+ab$ has Hirano inverse if and only if $1+ba$ has Hirano inverse.*

Proof The proof follows directly from the equality $aba = aba$ and Theorem 5.1. □

As is well known, $a \in R$ has strongly Drazin inverse if and only if so does $1-a \in R$ (see [8, Lemma 3.3]). However, Hirano inveses in a ring are not the case.

Example 5.3 *Let $a = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in M_2(\mathbb{Z}_3)$. Then a has Hirano inverse, but $1-a$ does not have.*

Proof It is easy to verify that $a - a^3 = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \in N(M_2(\mathbb{Z}_4))$, but $(1-a)-(1-a)^3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \notin N(M_2(\mathbb{Z}_4))$. Therefore, we are done by Theorem 3.1. □

Proposition 5.4 *Let R be a ring and $a,b \in R$. If a,b have Hirano inverses and $ab = ba = 0$, then $a+b$ has Hirano inverse.*

Proof Clearly, $ab^D = ab(b^D)^2 = 0$ and $ba^D = ba(a^D)^2 = 0$. Likewise, $a^Db = b^Da = 0$. Thus, we see that a, a^D, b, b^D commute. Thus, $(a^2 - aa^D)(b^2 - bb^D) = (b^2 - bb^D)(a^2 - aa^D)$. Also, we have

$$(a+b)(a^D + b^D)^2 = (aa^D + bb^D)(a^D + b^D) = a^D + b^D$$

and

$$(a+b)^2 - (a+b)(a^D + b^D) = (a^2 - aa^D) + (b^2 - bb^D) \in N(R).$$

Therefore, $a^D + b^D$ is the Hirano inverse of $a+b$, as asserted,. □

It is convenient at this stage to include the following.

Theorem 5.5 *Let R be a ring, and let $a,b \in R$. If ab has strongly inverse and $a^2 = b^2 = 0$, then $a+b$ has Hirano inverse.*

Proof Suppose that ab has strongly inverse and $a^2 = b^2 = 0$. Then ba has strongly inverse by [8, Theorem 3.1]. It is obvious that ab and ba have Drazin inverses. Let $x = a(ba)^D + b(ab)(ab)^D$. Then we easily check that $(a+b)x = x(a+b)$ and $(a+b)x^2 = x$. We easily check that

$$\begin{aligned}(a+b)^2 - (a+b)x &= ab + ba - (ab)(ab)^D - (ba)(ba)^D\\&= (ab - (ab)(ab)^D) + (ba - (ba)(ba)^D)\\&\in N(R),\end{aligned}$$

as $(ab-(ab)(ab)^D)(ba-(ba)(ba)^D)=(ba-(ba)(ba)^D)(ab-(ab)(ab)^D)\ =0$. Therefore, $a+b\in R$ has Hirano inverse. $\square$

Example 5.6 *Let* $a=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b=\begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\in M_2(\mathbb{Z}_3)$. *Then* $ab=\begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}\in M_2(\mathbb{Z})$ *has Hirano inverse,* $a^2=b^2=0$, *but* $a+b=\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}\in M_2(\mathbb{Z}_3)$ *has no Hirano inverse.*

Example 5.7 *Let* $a=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, b=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\in M_2(\mathbb{Z})$. *Then* $ab=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\in M_2(\mathbb{Z})$ *has strongly Drazin inverse,* $a^2=b^2=0$, *but* $a+b=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *has no strongly Drazin inverse. In this case,* $a+b$ *has Hirano inverse.*

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