


Trigonometric expressions for Gaussian ${}_2F_1$ -series

Wenchang CHU^{1,2*} 

¹School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou (Henan), P.R. China

²Department of Mathematics and Physics, University of Salento, Lecce, Italy

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Abstract: The classical Gaussian ${}_2F_1$ -series containing two free variables $\{x, y\}$ and two integer parameters $\{m, n\}$ are investigated by the linearization method. Several closed formulae are derived in terms of trigonometric functions. Some of them are lifted up, via a trigonometric integral approach, to identities of nonterminating ${}_3F_2$ -series.

Key words: Classical hypergeometric series, Pfaff–Saalsch summation theorem, linearization method, Gould–Hsu inversions, trigonometric integral approach

1. Introduction and motivation

Let \mathbb{Z} and \mathbb{N} be the sets of integers and natural numbers with $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$. For an indeterminate x and $n \in \mathbb{Z}$, define the rising and falling factorials by quotients of the Γ -function

$$(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)} \quad \text{and} \quad \langle x \rangle_n = \frac{\Gamma(1+x)}{\Gamma(1+x-n)}.$$

Following Bailey [1], the generalized hypergeometric series reads as

$${}_1+{}_pF_q \left[\begin{matrix} a_0, a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix} \middle| z \right] = \sum_{k=0}^{\infty} \frac{(a_0)_k (a_1)_k \cdots (a_p)_k}{k! (b_1)_k \cdots (b_q)_k} z^k.$$

The multiparameter forms of the shifted factorials and the Γ -function will be abbreviated respectively as

$$\left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right]_n = \frac{(\alpha)_n (\beta)_n \cdots (\gamma)_n}{(A)_n (B)_n \cdots (C)_n},$$

$$\Gamma \left[\begin{matrix} \alpha, \beta, \dots, \gamma \\ A, B, \dots, C \end{matrix} \right] = \frac{\Gamma(\alpha) \Gamma(\beta) \cdots \Gamma(\gamma)}{\Gamma(A) \Gamma(B) \cdots \Gamma(C)}.$$

The aim of this paper is to investigate, by the linearization method, the following Gaussian ${}_2F_1$ -series with two free complex variables $\{x, y\}$:

$$\Omega_{m,n}(x, y) := {}_2F_1 \left[\begin{matrix} x, m-x \\ n + \frac{1}{2} \end{matrix} \middle| y^2 \right] \quad \text{where } m, n \in \mathbb{Z}. \quad (1)$$

*Correspondence: chu.wenchang@unisalento.it

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The domain of convergence for the above series is $|y| \leq 1$ when $n \geq m$ and for $|y| < 1$ when $n < m$. This is motivated by the two classical formulae (see Gradshteyn and Ryzhik [15, §9.12])

$$\begin{aligned} \Omega_{0,0}(x, y) &= {}_2F_1 \left[x, \begin{matrix} -x \\ \frac{1}{2} \end{matrix} \middle| y^2 \right] = \cos(2x \arcsin y), \\ \Omega_{1,1}(x, y) &= {}_2F_1 \left[x, \begin{matrix} 1-x \\ \frac{3}{2} \end{matrix} \middle| y^2 \right] = \frac{\sin((2x-1) \arcsin y)}{(2x-1)y}. \end{aligned}$$

They have been employed by Chu [7, 12] and Chu and Zheng [13] to evaluate trigonometric sums and the Riemann zeta series weighted by harmonic numbers.

The rest of the paper will be organized as follows. In the next section, we shall first reduce, by the linearization method (cf. [3, 4, 8–10, 16–18]), the $\Omega_{m,n}$ -series for $m, n \in \mathbb{Z}$ to the $\Omega_{m',n'}$ -series for $m', n' \in \mathbb{N}_0$, which will be expressed, in turn, as the $\Omega_{n',n'}$ -series with the same $n' \in \mathbb{N}_0$. By means of Gould–Hsu [14] inversions, we shall establish an explicit formula for this last series in terms of the $\Omega_{0,0}$ -series. Then, in Section 3, the conclusive theorem is reached, which states that for any $m, n \in \mathbb{Z}$, the $\Omega_{m,n}(x, y)$ series can be evaluated by a linear combination of $\Omega_{0,0}(x', y)$ (with x being shifted to x' by integers) in the number of terms being a bivariate cubic polynomial of m and n . Several closed formulae for the $\Omega_{m,n}$ -series are presented as examples. Finally, in Section 4, the trigonometric integral approach will be illustrated that leads to further hypergeometric series identities by lifting up ${}_2F_1$ -series to ${}_3F_2$ -series.

2. Reduction formulae via linearization

By means of the linearization method (cf. [3, 4, 8–10, 16, 18]), we shall establish, in this section, reduction formulae for $\Omega_{m,n}(x, y)$ so that it can be evaluated by the initial one $\Omega_{0,0}(x, y)$.

2.1. $m, n \in \mathbb{Z}$

We start with the following linear representation lemma.

Lemma 1 (Linear representation) *For $\lambda \in \mathbb{N}_0$ and three indeterminates $\{A, B, C\}$, there exist constants $\{U_\lambda^i\}_{i=0}^\lambda$ such that*

$$(A+k)_\lambda = \sum_{i=0}^\lambda (B+k)_i (C+k)_{\lambda-i} U_\lambda^i \tag{2}$$

where U_λ^i is independent of the variable k and given by the following expression:

$$U_\lambda^i = (-1)^i \binom{\lambda}{i} \frac{(C-B+\lambda-2i)(A-C)_i (A-B)_{\lambda-i}}{(C-B-i)_{\lambda+1}}. \tag{3}$$

Proof Substituting (3) into (2), we can express the resulting binomial sum in terms of hypergeometric series and then evaluate it as follows:

$$\begin{aligned} & \sum_{i=0}^{\lambda} (-1)^i \binom{\lambda}{i} (B+k)_i (C+k)_{\lambda-i} \frac{(C-B+\lambda-2i)(A-C)_i (A-B)_{\lambda-i}}{(C-B-i)_{\lambda+1}} \\ &= \left[\begin{matrix} A-B, C+k \\ C-B \end{matrix} \right]_{\lambda} \times {}_5F_4 \left[\begin{matrix} B-C-\lambda, 1+\frac{B-C-\lambda}{2}, A-C, B+k, -\lambda \\ \frac{B-C-\lambda}{2}, 1+B-A-\lambda, 1-C-k-\lambda, 1+B-C \end{matrix} \middle| 1 \right] \\ &= \left[\begin{matrix} A-B, C+k \\ C-B \end{matrix} \right]_{\lambda} \times \left[\begin{matrix} 1+B-C-\lambda, 1-A-k-\lambda \\ 1+B-A-\lambda, 1-C-k-\lambda \end{matrix} \right]_{\lambda} = (A+k)_{\lambda}, \end{aligned}$$

where the last passage has been justified by Dougall’s summation formula (cf. Bailey [1, §4.3]) for the terminating very well-poised ${}_5F_4$ -series

$${}_5F_4 \left[\begin{matrix} a, 1+\frac{a}{2}, b, d, -n \\ \frac{a}{2}, 1+a-b, 1+a-d, 1+a+n \end{matrix} \middle| 1 \right] = \left[\begin{matrix} 1+a, 1+a-b-d \\ 1+a-b, 1+a-d \end{matrix} \right]_n.$$

Therefore, we have confirmed the linear relation (2) stated in the lemma. □

For $m, n \in \mathbb{Z}$ with $m < 0$ and/or $n < 0$, by specifying in Lemma 1

$$\lambda := \max(-m, -n) \quad \text{and} \quad \begin{cases} A \rightarrow n + \frac{1}{2} \\ B \rightarrow x \\ C \rightarrow m - x \end{cases}$$

we get from (2) the equality

$$\left(\frac{1}{2} + n + k\right)_{\lambda} = \sum_{i=0}^{\lambda} (x+k)_i (m-x+k)_{\lambda-i} \mathcal{U}_{\lambda}^i$$

with the connection coefficient \mathcal{U}_{λ}^i being given by

$$\mathcal{U}_{\lambda}^i = (-1)^i \binom{\lambda}{i} \frac{(\lambda - 2x + m - 2i) \left(\frac{1}{2} + x - m + n\right)_i \left(\frac{1}{2} - x + n\right)_{\lambda-i}}{(m - 2x - i)_{\lambda+1}}. \tag{4}$$

Now putting the last equality inside the $\Omega_{m,n}$ -series, we can manipulate, by interchanging the summation order, the following double series:

$$\begin{aligned} \Omega_{m,n}(x, y) &= \sum_{k \geq 0} y^{2k} \frac{(x)_k (m-x)_k}{k! \left(\frac{1}{2} + n\right)_k} \sum_{i=0}^{\lambda} \frac{(x+k)_i (m-x+k)_{\lambda-i}}{\left(\frac{1}{2} + n + k\right)_{\lambda}} \mathcal{U}_{\lambda}^i \\ &= \sum_{i=0}^{\lambda} \mathcal{U}_{\lambda}^i \frac{(x)_i (m-x)_{\lambda-i}}{\left(\frac{1}{2} + n\right)_{\lambda}} \sum_{k \geq 0} \frac{(x+i)_k (\lambda + m - x - i)_k}{k! \left(\frac{1}{2} + n + \lambda\right)_k} y^{2k}. \end{aligned}$$

Writing the last sum in terms of $\Omega_{m,n}$ -series, we find the reduction formula below.

Theorem 2 (Reduction formula) For $m, n \in \mathbb{Z}$ with $m < 0$ and/or $n < 0$, define the natural number λ by $\lambda := \max(-m, -n)$ and \mathcal{U}_{λ}^i by (4). Then the following formula holds:

$$\Omega_{m,n}(x, y) = \sum_{i=0}^{\lambda} \mathcal{U}_{\lambda}^i \frac{(x)_i (m-x)_{\lambda-i}}{\left(\frac{1}{2} + n\right)_{\lambda}} \Omega_{m+\lambda, n+\lambda}(x+i, y).$$

This theorem will be shown useful because it transforms $\Omega_{m,n}$ -series for $m < 0$ and/or $n < 0$ into $\Omega_{m',n'}$ -series with $m', n' \in \mathbb{N}_0$.

2.2. $m, n \in \mathbb{N}_0$

In Lemma 1, dividing by A^λ across equation (2) and then letting $A \rightarrow \infty$, we get the following limiting form.

Lemma 3 (Linear representation) For $\lambda \in \mathbb{N}_0$ and two indeterminates $\{B, C\}$, there exist constants $\{U_\lambda^i\}_{i=0}^\lambda$ such that

$$1 = \sum_{i=0}^\lambda (B+k)_i (C+k)_{\lambda-i} V_\lambda^i, \tag{5}$$

where V_λ^i is independent of the variable k and given by the following expression:

$$V_\lambda^i = (-1)^i \binom{\lambda}{i} \frac{C - B + \lambda - 2i}{(C - B - i)_{\lambda+1}}. \tag{6}$$

For $m, n \in \mathbb{N}_0$ with $m < n$, letting in Lemma 3

$$\lambda := n - m \quad \text{and} \quad \begin{cases} B \rightarrow x \\ C \rightarrow m - x \end{cases}$$

we get from (5) the equality

$$1 = \sum_{i=0}^{n-m} (x+k)_i (m-x+k)_{n-m-i} \mathcal{V}_{m,n}^i(x)$$

with the connection coefficient $\mathcal{V}_{m,n}^i(x)$ being given explicitly by

$$\mathcal{V}_{m,n}^i(x) = (-1)^i \binom{n-m}{i} \frac{n-2x-2i}{(m-2x-i)_{n-m+1}}. \tag{7}$$

By inserting the last relation inside the $\Omega_{m,n}$ -series, we get the double series

$$\begin{aligned} \Omega_{m,n}(x, y) &= \sum_{k \geq 0} y^{2k} \frac{(x)_k (m-x)_k}{k! (\frac{1}{2} + n)_k} \sum_{i=0}^{n-m} (x+k)_i (m-x+k)_{n-m-i} \mathcal{V}_{m,n}^i(x) \\ &= \sum_{i=0}^{n-m} (x)_i (m-x)_{n-m-i} \mathcal{V}_{m,n}^i(x) \sum_{k \geq 0} \frac{(x+i)_k (n-x-i)_k}{k! (\frac{1}{2} + n)_k} y^{2k}. \end{aligned}$$

Writing the last sum in terms of $\Omega_{m,n}$ -series, we find the following formula that expresses $\Omega_{m,n}$ -series in terms of $\Omega_{n,n}$.

Theorem 4 (Reduction formula) For $m, n \in \mathbb{N}_0$ with $m < n$, define the connection coefficient $\mathcal{V}_{m,n}^i$ by (7). Then the following formula holds:

$$\Omega_{m,n}(x, y) = \sum_{i=0}^{n-m} (x)_i (m-x)_{n-m-i} \mathcal{V}_{m,n}^i(x) \Omega_{n,n}(x+i, y).$$

When $m > n$, we can express the $\Omega_{m,n}$ -series by making use of the Pfaff–Euler transformation (cf. Bailey [1, §1.2]):

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} \middle| x \right] = (1-x)^{c-a-b} \times {}_2F_1 \left[\begin{matrix} c-a, c-b \\ c \end{matrix} \middle| x \right],$$

as another $\Omega_{m,n}$ -series,

$$\Omega_{m,n}(x, y) = (1-y^2)^{\frac{1}{2}-m+n} \Omega_{1+2n-m,n} \left(\frac{1}{2} + x - m + n, y \right). \tag{8}$$

Observing the fact that

$$1 + 2n - m \leq n \iff n < m,$$

we get, by applying Theorem 4 to the last series, another reduction formula.

Theorem 5 (Reduction formula) For $m, n \in \mathbb{N}_0$ with $m > n$, define the connection coefficient $\mathcal{V}_{m,n}^i$ by (7). Then the following formula holds:

$$\begin{aligned} \Omega_{m,n}(x, y) &= (1-y^2)^{\frac{1}{2}-m+n} \times \sum_{j=1}^{m-n} \left(\frac{1}{2} + x - m + n \right)_{j-1} \mathcal{V}_{1+2n-m,n}^{j-1} \left(\frac{1}{2} + x - m + n \right) \\ &\quad \times \left(\frac{1}{2} - x + n \right)_{m-n-j} \Omega_{n,n} \left(x - m + n + j - \frac{1}{2}, y \right). \end{aligned}$$

2.3. $m = n \in \mathbb{N}_0$

Finally, we are going to treat the remaining case $m = n$.

Lemma 6 (Linear representation) For $\lambda \in \mathbb{N}_0$ and three indeterminates $\{A, B, C\}$, there exist constants $\{W_\lambda^i\}_{i=0}^\lambda$ such that

$$(A+k)_\lambda = \sum_{i=0}^\lambda \langle B+k \rangle_i \langle C+k \rangle_{\lambda-i} W_\lambda^i, \tag{9}$$

where W_λ^i is independent of the variable k and given by the following expression:

$$W_\lambda^i = (-1)^i \binom{\lambda}{i} \frac{(A-C)_i (A-B+i)_{\lambda-i}}{(C-B)_\lambda}. \tag{10}$$

Proof Substituting (10) into (9), we can express the resulting binomial sum in terms of a hypergeometric series:

$$\begin{aligned} &\sum_{i=0}^\lambda (-1)^i \binom{\lambda}{i} \frac{(A-C)_i (A-B+i)_{\lambda-i}}{(C-B)_\lambda} \langle B+k \rangle_i \langle C+k \rangle_{\lambda-i} \\ &= \left[\begin{matrix} A-B, C+k \\ C-B \end{matrix} \right]_\lambda {}_3F_2 \left[\begin{matrix} -\lambda, A-C, -B-k \\ A-B, 1-C-k-\lambda \end{matrix} \middle| 1 \right]. \end{aligned}$$

In view of the summation formula due to Pfaff and Saalschutz (cf. Bailey [1, §2.2]) for terminating balanced ${}_3F_2$ -series

$${}_3F_2 \left[\begin{matrix} -n, a, b \\ c, 1+a+b-c-n \end{matrix} \middle| 1 \right] = \left[\begin{matrix} c-a, c-b \\ c, c-a-b \end{matrix} \right]_n$$

the linear relation (9) stated in the lemma is confirmed by simplifying the following factorial product:

$$\begin{bmatrix} A - B, C + k \\ C - B \end{bmatrix}_\lambda \times \begin{bmatrix} A + k, C - B \\ A - B, C + k \end{bmatrix}_\lambda = (A + k)_\lambda.$$

□

For $n \in \mathbb{N}_0$, by specifying in Lemma 6

$$\lambda \rightarrow n \quad \text{and} \quad \begin{cases} A \rightarrow x \\ B \rightarrow 0 \\ C \rightarrow -x - n \end{cases}$$

we get from (9) the equality

$$(x + k)_n = \sum_{i=0}^n \langle k \rangle_i (k - x - n)_{n-i} \mathcal{W}_n^i$$

with the connection coefficient \mathcal{W}_n^i being given by

$$\mathcal{W}_n^i = (-1)^{n-i} \binom{n}{i} \frac{x(2x+n)_i}{(x)_i(x+n)}.$$

Now putting the last equality inside the $\Omega_{0,0}$ -series, we can reformulate, by interchanging the summation order and then making the replacement $k = i + j$, the following double series:

$$\begin{aligned} \Omega_{0,0}(x+n, y) &= \sum_{k \geq 0} y^{2k} \frac{(x+n)_k (-x-n)_k}{k! (\frac{1}{2})_k} \sum_{i=0}^n \mathcal{W}_n^i \frac{\langle k \rangle_i (k-x-n)_{n-i}}{(x+k)_n} \\ &= \sum_{i=0}^n \mathcal{W}_n^i \sum_{k \geq 0} \frac{\langle k \rangle_i (x)_k (-x-n)_{k+n-i}}{k! (x)_n (\frac{1}{2})_k} y^{2k} \\ &= \sum_{i=0}^n (-1)^n \mathcal{W}_n^i \frac{(1+x)_i (x+n)}{(\frac{1}{2})_i (x+i)} y^{2i} \sum_{j \geq 0} \frac{(x+i)_j (-x)_j}{j! (\frac{1}{2}+i)_j} y^{2j}. \end{aligned}$$

Writing the last sum in terms of $\Omega_{i,i}$ -series, we find the recurrence relation below.

Proposition 7 (Recurrence relation: $n \in \mathbb{N}_0$)

$$\Omega_{0,0}(x+n, y) = \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{(2x+n)_i}{(\frac{1}{2})_i} y^{2i} \Omega_{i,i}(x+i, y).$$

In order to find an explicit expression for $\Omega_{n,n}(x, y)$, we record here the well-known pair of inverse series relations discovered in 1973 by Gould and Hsu [14], which has wide applications to terminating series identities (see [2, 5, 6], for example).

Let $\{a_i, b_i\}$ be any two complex sequences such that the ϕ -polynomials defined by

$$\phi(x; 0) \equiv 1 \quad \text{and} \quad \phi(x; n) = \prod_{i=0}^{n-1} (a_i + xb_i) \quad \text{for } n \in \mathbb{N} \tag{11}$$

differ from zero for $x, n \in \mathbb{N}_0$. Then there hold the inverse series relations

$$f(n) = \sum_{i=0}^n (-1)^i \binom{n}{i} \phi(i; n) g(i), \tag{12}$$

$$g(n) = \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{a_i + ib_i}{\phi(n; i+1)} f(i). \tag{13}$$

Now rewrite first the binomial relation in Proposition 7 as

$$(2x)_n \Omega_{0,0}(x+n, y) = \sum_{i=0}^n (-1)^i \binom{n}{i} (2x+i)_n \frac{(2x)_i}{(\frac{1}{2})_i} y^{2i} \Omega_{i,i}(x+i, y).$$

This matches perfectly to (12) under the following specifications:

$$\left. \begin{matrix} a_k = 2x + k \\ b_k = 1 \end{matrix} \right\} \text{ and } \left\{ \begin{matrix} f(n) = (2x)_n \Omega_{0,0}(x+n, y) \\ g(n) = \frac{(2x)_n}{(\frac{1}{2})_n} y^{2n} \Omega_{n,n}(x+n, y). \end{matrix} \right.$$

Then the dual relation corresponding to (13) reads as

$$\frac{(2x)_n}{(\frac{1}{2})_n} y^{2n} \Omega_{n,n}(x+n, y) = \sum_{i=0}^n (-1)^i \binom{n}{i} \frac{2x+2i}{(2x+n)_{i+1}} (2x)_i \Omega_{0,0}(x+i, y).$$

Under the replacements $x \rightarrow x - n$ and $i \rightarrow n - k$, we can highlight the resulting expression in the following theorem.

Theorem 8 (Recurrence formula: $n \in \mathbb{N}_0$)

$$\Omega_{n,n}(x, y) = \frac{(\frac{1}{2})_n}{y^{2n}} \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} \frac{2x-2k}{(2x-n-k)_{n+1}} \Omega_{0,0}(x-k, y).$$

3. Conclusive theorem and examples

Based on the reduction formulae derived in the last section, we can evaluate, for any $m, n \in \mathbb{Z}$, the $\Omega_{m,n}$ series in terms of $\Omega_{0,0}$ series by carrying out the following procedure:

- **Step-A:** If $m, n \in \mathbb{Z}$ with $m < 0$ and/or $n < 0$, apply Theorem 2 to express $\Omega_{m,n}$ in terms of $\Omega_{m',n'}$ with both $m' \geq 0$ and $n' \geq 0$.
- **Step-B:** If $m, n \in \mathbb{N}_0$ with $m \neq n$, apply Theorems 4 and 5 to express $\Omega_{m,n}$ in terms of $\Omega_{n,n}$ with $n \geq 0$.
- **Step-C:** If $m, n \in \mathbb{N}_0$ with $m = n$, apply Theorem 8 to express $\Omega_{n,n}$ in terms of $\Omega_{0,0}$ explicitly.

Summing up, we have shown the following conclusive theorem.

Theorem 9 (Conclusion) For any $m, n \in \mathbb{Z}$, the $\Omega_{m,n}(x, y)$ series can be evaluated by a linear combination of $\Omega_{0,0}(x', y)$ (with x being shifted to x' by half integers and the coefficients being rational functions of x and y) in the number of terms being a bivariate cubic polynomial of m and n .

According to the procedure described at the beginning of this section, we have devised appropriate Mathematica commands to compute closed expressions for $\Omega_{m,n}(x, y)$ series. Our results suggest that, in general for $m, n \in \mathbb{Z}$, the series $\Omega_{m,n}$ is just a linear combination of two trigonometric functions, $\cos(2x \arcsin y)$ and $\sin(2x \arcsin y)$, but with coefficients being functions that are “rational and algebraic” with respect to x and y , respectively. In view of transformation (8), these remarkable formulae are recorded in pairs, where the variable x is shifted to $x + \frac{m}{2}$ in $\Omega_{m,n}(x, y)$ for symmetry.

The first two simplest pairs are included for integrity, even though they are well known (see [15, §9.12] and [19, §7.3.1]), for example). Because the initial evaluation $\Omega_{0,0}(x, y)$ is crucial, we present, in order to make the paper self-contained, an elementary proof for its closed formula in the Appendix after the references.

Corollary 10 ($\Omega_{0,0}(x, y)$ and $\Omega_{1,0}(x + \frac{1}{2}, y)$)

$${}_2F_1 \left[x, \frac{-x}{\frac{1}{2}} \mid y^2 \right] = \cos(2x \arcsin y), \tag{\Omega_{0,0}}$$

$${}_2F_1 \left[\frac{1}{2} + x, \frac{\frac{1}{2} - x}{\frac{1}{2}} \mid y^2 \right] = \frac{\cos(2x \arcsin y)}{\sqrt{1 - y^2}}. \tag{\Omega_{1,0}}$$

Corollary 11 ($\Omega_{1,1}(x + \frac{1}{2}, y)$ and $\Omega_{2,1}(x + 1, y)$)

$${}_2F_1 \left[\frac{1}{2} + x, \frac{\frac{1}{2} - x}{\frac{3}{2}} \mid y^2 \right] = \frac{\sin(2x \arcsin y)}{2xy}, \tag{\Omega_{1,1}}$$

$${}_2F_1 \left[1 + x, \frac{1 - x}{\frac{3}{2}} \mid y^2 \right] = \frac{\sin(2x \arcsin y)}{2xy\sqrt{1 - y^2}}. \tag{\Omega_{2,1}}$$

Corollary 12 ($\Omega_{-1,0}(x - \frac{1}{2}, y)$ and $\Omega_{2,0}(x + 1, y)$)

$${}_2F_1 \left[x - \frac{1}{2}, \frac{-\frac{1}{2} - x}{\frac{1}{2}} \mid y^2 \right] = \Lambda(x, y), \tag{\Omega_{-1,0}}$$

$${}_2F_1 \left[1 + x, \frac{1 - x}{\frac{1}{2}} \mid y^2 \right] = \frac{\Lambda(x, y)}{\sqrt{(1 - y^2)^3}}; \tag{\Omega_{2,0}}$$

$$\text{where } \Lambda(x, y) = \frac{y}{2x} \sin(2x \arcsin y) + \sqrt{1 - y^2} \cos(2x \arcsin y).$$

Corollary 13 ($\Omega_{-1,-1}(x - \frac{1}{2}, y)$ and $\Omega_{0,-1}(x, y)$)

$${}_2F_1 \left[x - \frac{1}{2}, \frac{-\frac{1}{2} - x}{-\frac{1}{2}} \mid y^2 \right] = \Lambda(x, y), \tag{\Omega_{-1,-1}}$$

$${}_2F_1 \left[x, \frac{-x}{-\frac{1}{2}} \mid y^2 \right] = \frac{\Lambda(x, y)}{\sqrt{1 - y^2}}; \tag{\Omega_{0,-1}}$$

$$\text{where } \Lambda(x, y) = \sqrt{1 - y^2} \cos(2x \arcsin y) + 2xy \sin(2x \arcsin y).$$

Corollary 14 ($\Omega_{0,1}(x, y)$ and $\Omega_{3,1}(x + \frac{3}{2}, y)$)

$${}_2F_1 \left[x, \quad -x \mid y^2 \right] = \Lambda(x, y), \tag{\Omega_{0,1}}$$

$${}_2F_1 \left[\frac{3}{2} + x, \quad \frac{3}{2} - x \mid y^2 \right] = \frac{\Lambda(x, y)}{\sqrt{(1 - y^2)^3}}; \tag{\Omega_{3,1}}$$

$$\text{where } \Lambda(x, y) = \frac{y \cos(2x \arcsin y) - 2x \sqrt{1 - y^2} \sin(2x \arcsin y)}{(1 - 4x^2)y\sqrt{(1 - y^2)^3}}.$$

Corollary 15 ($\Omega_{2,2}(x + 1, y)$ and $\Omega_{3,2}(x + \frac{3}{2}, y)$)

$${}_2F_1 \left[1 + x, \quad 1 - \frac{x}{2} \mid y^2 \right] = \Lambda(x, y), \tag{\Omega_{2,2}}$$

$${}_2F_1 \left[\frac{3}{2} + x, \quad \frac{3}{2} - \frac{x}{2} \mid y^2 \right] = \frac{\Lambda(x, y)}{\sqrt{1 - y^2}}; \tag{\Omega_{3,2}}$$

$$\text{where } \Lambda(x, y) = \frac{6xy \cos(2x \arcsin y) - 3\sqrt{1 - y^2} \sin(2x \arcsin y)}{2xy^3(1 - 4x^2)}.$$

Corollary 16 ($\Omega_{1,2}(x + \frac{1}{2}, y)$ and $\Omega_{4,2}(x + 2, y)$)

$${}_2F_1 \left[\frac{1}{2} + x, \quad \frac{1}{2} - \frac{x}{2} \mid y^2 \right] = \Lambda(x, y), \tag{\Omega_{1,2}}$$

$${}_2F_1 \left[2 + x, \quad 2 - \frac{x}{2} \mid y^2 \right] = \frac{\Lambda(x, y)}{\sqrt{(1 - y^2)^3}}; \tag{\Omega_{4,2}}$$

$$\text{where } \Lambda(x, y) = \frac{3\sqrt{1 - y^2} \cos(2x \arcsin y)}{4(1 - x^2)y^2} - \frac{(3 - 6y^2) \sin(2x \arcsin y)}{8x(1 - x^2)y^3}.$$

Corollary 17 ($\Omega_{-2,-1}(x - 1, y)$ and $\Omega_{1,-1}(x + \frac{1}{2}, y)$)

$${}_2F_1 \left[x - 1, \quad -x - 1 \mid y^2 \right] = \Lambda(x, y), \tag{\Omega_{-2,-1}}$$

$${}_2F_1 \left[\frac{1}{2} + x, \quad \frac{1}{2} - x \mid y^2 \right] = \frac{\Lambda(x, y)}{\sqrt{(1 - y^2)^3}}; \tag{\Omega_{1,-1}}$$

$$\text{where } \Lambda(x, y) = (1 - 2y^2) \cos(2x \arcsin y) + 2xy\sqrt{1 - y^2} \sin(2x \arcsin y).$$

Corollary 18 ($\Omega_{3,3}(x + \frac{3}{2}, y)$ and $\Omega_{4,3}(x + 2, y)$)

$${}_2F_1 \left[\frac{3}{2} + x, \quad \frac{3}{2} - \frac{x}{2} \mid y^2 \right] = \Lambda(x, y), \tag{\Omega_{3,3}}$$

$${}_2F_1 \left[2 + x, \quad 2 - \frac{x}{2} \mid y^2 \right] = \frac{\Lambda(x, y)}{\sqrt{1 - y^2}}; \tag{\Omega_{4,3}}$$

$$\text{where } \Lambda(x, y) = \frac{15(3 - 2y^2 - 4x^2y^2) \sin(2x \arcsin y)}{8xy^5(1 - x^2)(1 - 4x^2)} - \frac{45\sqrt{1 - y^2} \cos(2x \arcsin y)}{4y^4(1 - x^2)(1 - 4x^2)}.$$

Corollary 19 ($\Omega_{-2,0}(x - 1, y)$ and $\Omega_{3,0}(x + \frac{3}{2}, y)$)

$${}_2F_1 \left[x - 1, \quad -x - 1 \mid y^2 \right] = \Lambda(x, y), \tag{\Omega_{-2,0}}$$

$${}_2F_1 \left[\frac{3}{2} + x, \quad \frac{3}{2} - x \mid y^2 \right] = \frac{\Lambda(x, y)}{\sqrt{(1 - y^2)^5}}; \tag{\Omega_{3,0}}$$

$$\text{where } \Lambda(x, y) = \frac{(1 - 4x^2 + 2y^2 + 4x^2y^2) \cos(2x \arcsin y)}{1 - 4x^2} - \frac{6xy\sqrt{1 - y^2} \sin(2x \arcsin y)}{1 - 4x^2}.$$

Corollary 20 ($\Omega_{-1,1}(x - \frac{1}{2}, y)$ and $\Omega_{4,1}(x + 2, y)$)

$${}_2F_1 \left[x - \frac{1}{2}, \quad -x - \frac{1}{2} \mid y^2 \right] = \Lambda(x, y), \tag{\Omega_{-1,1}}$$

$${}_2F_1 \left[2 + x, \quad 2 - x \mid y^2 \right] = \frac{\Lambda(x, y)}{\sqrt{(1 - y^2)^5}}; \tag{\Omega_{4,1}}$$

$$\text{where } \Lambda(x, y) = \frac{(1 - 4x^2 + 2y^2 + 4x^2y^2) \sin(2x \arcsin y)}{8xy(1 - x^2)} + \frac{3\sqrt{1 - y^2} \cos(2x \arcsin y)}{4(1 - x^2)}.$$

Corollary 21 ($\Omega_{-4,-2}(x - 2, y)$ and $\Omega_{1,-2}(x + \frac{1}{2}, y)$)

$${}_2F_1 \left[x - 2, \quad -x - 2 \mid y^2 \right] = \Lambda(x, y), \tag{\Omega_{-4,-2}}$$

$${}_2F_1 \left[\frac{1}{2} + x, \quad \frac{1}{2} - x \mid y^2 \right] = \frac{\Lambda(x, y)}{\sqrt{(1 - y^2)^5}}; \tag{\Omega_{1,-2}}$$

$$\text{where } \Lambda(x, y) = 2xy(1 - 2y^2)\sqrt{1 - y^2} \sin(2x \arcsin y) + \frac{(3 - 8y^2 + 8y^4 - 4x^2y^2 + 4x^2y^4) \cos(2x \arcsin y)}{3}.$$

Corollary 22 ($\Omega_{4,4}(x + 2, y)$ and $\Omega_{5,4}(x + \frac{5}{2}, y)$)

$${}_2F_1 \left[2 + x, \quad 2 - x \mid y^2 \right] = \Lambda(x, y), \tag{\Omega_{4,4}}$$

$${}_2F_1 \left[\frac{5}{2} + x, \quad \frac{5}{2} - x \mid y^2 \right] = \frac{\Lambda(x, y)}{\sqrt{1 - y^2}}; \tag{\Omega_{5,4}}$$

$$\text{where } \Lambda(x, y) = \frac{105xy(15 - 11y^2 - 4x^2y^2) \cos(2x \arcsin y)}{4xy^7(1 - x^2)(1 - 4x^2)(9 - 4x^2)} - \frac{315\sqrt{1 - y^2}(5 - 2y^2 - 8x^2y^2) \sin(2x \arcsin y)}{8xy^7(1 - x^2)(1 - 4x^2)(9 - 4x^2)}.$$

4. Trigonometric integral approach

In a personal communication to Richard Askey in 1977, Bill Gosper discovered the following nonterminating series identities.

Proposition 23 (Nonterminating series evaluation)

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} + 3x, \frac{1}{2} - 3x, \lambda \\ \frac{1}{2}, 3\lambda \end{matrix} \middle| \frac{3}{4} \right] = \Gamma \left[\begin{matrix} \frac{1}{3} + \lambda, \frac{2}{3} + \lambda \\ \frac{1}{2} + \lambda + x, \frac{1}{2} + \lambda - x \end{matrix} \right] \frac{2 \cos(\pi x)}{\sqrt{3}}.$$

Proposition 24 (Nonterminating series evaluation)

$${}_3F_2 \left[\begin{matrix} 1 + 3x, 1 - 3x, \lambda \\ \frac{3}{2}, 3\lambda - 1 \end{matrix} \middle| \frac{3}{4} \right] = \Gamma \left[\begin{matrix} \lambda + \frac{1}{3}, \lambda - \frac{1}{3} \\ \lambda + x, \lambda - x \end{matrix} \right] \frac{2 \sin(\pi x)}{3\sqrt{3x}}.$$

New proofs by integrating trigonometric products were found by the author [11], who also derived two further hypergeometric series identities.

Proposition 25 (Nonterminating series evaluation) Define

$$\psi(x, \lambda) := \Gamma \left[\begin{matrix} \frac{1}{3} + \lambda, \frac{2}{3} + \lambda \\ \frac{1}{3} + \lambda + x, \frac{2}{3} + \lambda - x \end{matrix} \right] \frac{\cos(\pi x - \frac{\pi}{6})}{\sqrt{3}}.$$

Then

$${}_3F_2 \left[\begin{matrix} 3x, -3x, \lambda \\ \frac{1}{2}, 3\lambda \end{matrix} \middle| \frac{3}{4} \right] = \psi(x, \lambda) + \psi(-x, \lambda).$$

Proposition 26 (Nonterminating series evaluation) Define

$$\psi(x, \lambda) := \Gamma \left[\begin{matrix} \lambda + \frac{1}{3}, \lambda - \frac{1}{3} \\ \lambda + x + \frac{1}{6}, \lambda - x - \frac{1}{6} \end{matrix} \right] \frac{\sin(\pi x + \frac{\pi}{6})}{6x\sqrt{3}}.$$

Then

$${}_3F_2 \left[\begin{matrix} \frac{1}{2} + 3x, \frac{1}{2} - 3x, \lambda \\ \frac{3}{2}, 3\lambda - 1 \end{matrix} \middle| \frac{3}{4} \right] = \psi(x, \lambda) + \psi(-x, \lambda).$$

For the three variables $\{y, z, \theta\}$ related by $z = \frac{4}{3}y^2$ and $y = \sin \theta$, we have the following beta integral:

$$\begin{aligned} \int_0^1 z^{A-1}(1-z)^{B-1}y^{2k}dz &= \left(\frac{3}{4}\right)^k \int_0^1 z^{k+A-1}(1-z)^{B-1}dz \\ &= \left(\frac{3}{4}\right)^k \frac{(A)_k}{(A+B)_k} \Gamma \left[\begin{matrix} A, B \\ A+B \end{matrix} \right]. \end{aligned}$$

Then it is not hard to establish the following expression, which can be obtained as a particular case of Rainville [20, Theorem 38(§56)]:

$$\begin{aligned} {}_3F_2 \left[\begin{matrix} 3x, m - 3x, A \\ n + \frac{1}{2}, A + B \end{matrix} \middle| \frac{3}{4} \right] &= \frac{\Gamma(A+B)}{\Gamma(A)\Gamma(B)} \int_0^1 z^{A-1}(1-z)^{B-1} \Omega_{m,n}(3x, y) dz \\ &= \frac{2^{2A+1}}{3^{A+B-1}} \frac{\Gamma(A+B)}{\Gamma(A)\Gamma(B)} \int_0^{\frac{\pi}{3}} \sin^{2A-1} \theta (3 - 4 \sin^2 \theta)^{B-1} \Omega_{m,n}(3x, y) \cos \theta d\theta. \end{aligned} \tag{14}$$

By making use of the trigonometric identity

$$\sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta$$

and two known integrals (cf. Gradshteyn and Ryzhik [15, Entries 3.631.1 & 3.631.8])

$$\int_0^\pi \sin^{a-1} \varphi \sin(c\varphi) d\varphi = \frac{2^{1-a} \pi \sin \frac{\pi c}{2}}{aB\left(\frac{1+a+c}{2}, \frac{1+a-c}{2}\right)}, \tag{15}$$

$$\int_0^\pi \sin^{a-1} \varphi \cos(c\varphi) d\varphi = \frac{2^{1-a} \pi \cos \frac{\pi c}{2}}{aB\left(\frac{1+a+c}{2}, \frac{1+a-c}{2}\right)}, \tag{16}$$

we can lift some identities from ${}_2F_1$ -series to ${}_3F_2$ -series. For instance, according to the closed formula of $\Omega_{0,-1}(3x, y)$, we can first reformulate the function

$$\begin{aligned} \Omega_{0,-1}(3x, \sin \theta) \cos \theta &= \cos \theta \cos(6\theta x) + 6x \sin \theta \sin(6\theta x) \\ &= \frac{1-6x}{2} \cos(6\theta x + \theta) + \frac{1+6x}{2} \cos(6\theta x - \theta). \end{aligned}$$

Then we can express, by choosing $A = \lambda$, $B = 2\lambda$ and making the change of variable $\theta = \varphi/3$, the corresponding integral (14) as follows:

$$\begin{aligned} &\int_0^{\frac{\pi}{3}} \sin^{2\lambda-1} \theta (3 - 4 \sin^2 \theta)^{2\lambda-1} \Omega_{0,-1}(3x, \sin \theta) \cos \theta d\theta \\ &= \int_0^\pi \sin^{2\lambda-1} \varphi \left\{ \frac{1-6x}{6} \cos(2\varphi x + \frac{\varphi}{3}) + \frac{1+6x}{6} \cos(2\varphi x - \frac{\varphi}{3}) \right\} d\varphi. \end{aligned}$$

Evaluating the last integral by (16) and then appealing to the triplicate relation formula (cf. Rainville [20, §20])

$$\Gamma(3\lambda) = \frac{3^{3\lambda}}{2\pi\sqrt{3}} \Gamma(\lambda) \Gamma(\lambda + \frac{1}{3}) \Gamma(\lambda + \frac{2}{3}),$$

we find the following hypergeometric ${}_3F_2$ -series identity.

Proposition 27 (Nonterminating series from $\Omega_{0,-1}(3x, y)$) Define

$$\psi(x, \lambda) := \Gamma \left[\begin{matrix} \lambda + \frac{1}{3}, \lambda + \frac{2}{3} \\ \frac{1}{3} + \lambda + x, \frac{2}{3} + \lambda - x \end{matrix} \right] \frac{(1+6x) \cos(\pi x - \frac{\pi}{6})}{\sqrt{3}}.$$

Then

$${}_3F_2 \left[\begin{matrix} 3x, -3x, \lambda \\ -\frac{1}{2}, 3\lambda \end{matrix} \middle| \frac{3}{4} \right] = \psi(x, \lambda) + \psi(-x, \lambda).$$

By applying the same approach, we can also derive the identities below.

Proposition 28 (Nonterminating series from $\Omega_{0,1}(3x, y)$) Define

$$\psi(x, \lambda) := \frac{\Gamma(\lambda + \frac{1}{3}) \Gamma(\lambda - \frac{1}{3})}{(1-36x^2)\sqrt{3}} \left\{ \frac{(1-6x) \sin(\pi x + \frac{\pi}{3})}{\Gamma(\lambda+x+\frac{1}{3}) \Gamma(\lambda-x-\frac{1}{3})} - \frac{6x \sin(\pi x)}{\Gamma(\lambda+x) \Gamma(\lambda-x)} \right\}.$$

Then

$${}_3F_2 \left[\begin{matrix} 3x, -3x, \lambda \\ \frac{3}{2}, 3\lambda - 1 \end{matrix} \middle| \frac{3}{4} \right] = \psi(x, \lambda) + \psi(-x, \lambda).$$

Proposition 29 (Nonterminating series from $\Omega_{-1,0}(3x - \frac{1}{2}, y)$) Define

$$\psi(x, \lambda) := \frac{\Gamma(\lambda + \frac{1}{3})\Gamma(\lambda + \frac{2}{3})}{12\sqrt{3}} \left\{ \frac{6 \cos(\pi x)}{\Gamma(\frac{1}{2} + \lambda + x)\Gamma(\frac{1}{2} + \lambda - x)} + \frac{(1 + 6x) \cos(\pi x - \frac{\pi}{3})}{x\Gamma(\frac{1}{6} + \lambda + x)\Gamma(\frac{5}{6} + \lambda - x)} \right\}.$$

Then

$${}_3F_2 \left[\begin{matrix} 3x - \frac{1}{2}, -3x - \frac{1}{2}, \lambda \\ \frac{1}{2}, 3\lambda \end{matrix} \middle| \frac{3}{4} \right] = \psi(x, \lambda) + \psi(-x, \lambda).$$

Proposition 30 (Nonterminating series from $\Omega_{-1,-1}(3x - \frac{1}{2}, y)$) Define

$$\psi(x, \lambda) := \frac{\Gamma(\lambda + \frac{1}{3})\Gamma(\lambda + \frac{2}{3})}{2\sqrt{3}} \left\{ \frac{\cos(\pi x)}{\Gamma(\frac{1}{2} + \lambda + x)\Gamma(\frac{1}{2} + \lambda - x)} + \frac{(1 + 6x) \cos(\pi x - \frac{\pi}{3})}{\Gamma(\frac{1}{6} + \lambda + x)\Gamma(\frac{5}{6} + \lambda - x)} \right\}.$$

Then

$${}_3F_2 \left[\begin{matrix} 3x - \frac{1}{2}, -3x - \frac{1}{2}, \lambda \\ -\frac{1}{2}, 3\lambda \end{matrix} \middle| \frac{3}{4} \right] = \psi(x, \lambda) + \psi(-x, \lambda).$$

Proposition 31 (Nonterminating series from $\Omega_{2,2}(3x + 1, y)$) Define

$$\psi(x, \lambda) := \frac{2\Gamma(\lambda + \frac{1}{3})\Gamma(\lambda + \frac{2}{3})}{\lambda(1 - 36x^2)\sqrt{3}} \left\{ \frac{(6x - 1) \sin(\pi x + \frac{\pi}{3})}{x\Gamma(\lambda + x + \frac{1}{3})\Gamma(\lambda - x - \frac{1}{3})} - \frac{\sin(\pi x)}{x\Gamma(\lambda + x)\Gamma(\lambda - x)} \right\}.$$

Then

$${}_3F_2 \left[\begin{matrix} 1 + 3x, 1 - 3x, 1 + \lambda \\ \frac{5}{2}, 3\lambda \end{matrix} \middle| \frac{3}{4} \right] = \psi(x, \lambda) + \psi(-x, \lambda).$$

Proposition 32 (Nonterminating series from $\Omega_{3,2}(3x + \frac{3}{2}, y)$) Define

$$\psi(x, \lambda) := \Gamma \left[\begin{matrix} \lambda + \frac{1}{3}, \lambda + \frac{2}{3} \\ \lambda + x + \frac{1}{6}, \lambda - x - \frac{1}{6} \end{matrix} \right] \frac{4(6x - 1) \sin(\pi x + \frac{\pi}{6})}{\lambda x(1 - 36x^2)\sqrt{3}}.$$

Then

$${}_3F_2 \left[\begin{matrix} \frac{3}{2} + 3x, \frac{3}{2} - 3x, 1 + \lambda \\ \frac{5}{2}, 3\lambda \end{matrix} \middle| \frac{3}{4} \right] = \psi(x, \lambda) + \psi(-x, \lambda).$$

Proposition 33 (Nonterminating series from $\Omega_{4,3}(3x + 2, y)$) Define

$$\psi(x, \lambda) := \frac{20\Gamma(\lambda + \frac{1}{3})\Gamma(\lambda + \frac{2}{3})}{(\lambda + 1)(1 - 9x^2)(1 - 36x^2)\sqrt{3}} \left\{ \frac{2(1 - 9x^2) \sin(\pi x)}{x\Gamma(\lambda + x)\Gamma(\lambda - x)} + \frac{(1 - 9x + 18x^2) \sin(\pi x + \frac{\pi}{3})}{x\Gamma(\lambda + x + \frac{1}{3})\Gamma(\lambda - x - \frac{1}{3})} \right\}.$$

Then

$${}_3F_2 \left[\begin{matrix} 2 + 3x, 2 - 3x, 2 + \lambda \\ \frac{7}{2}, 1 + 3\lambda \end{matrix} \middle| \frac{3}{4} \right] = \psi(x, \lambda) + \psi(-x, \lambda).$$

Theoretically, for all the remaining ${}_2F_1$ -series displayed in the preceding section, we can derive the corresponding ${}_3F_2$ -series identities, except for those with $m - n > 1$, where the extra factor $\sqrt{(1 - y^2)^\nu} = \cos^\nu \theta$ with $\nu > 1$ appearing in the integrand as a denominator causes substantial difficulty in the integral evaluation.

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Appendix.

Proof [Proof of the formula for $\Omega_{0,0}(x, y)$.] Recalling the Pfaff–Euler transformation (cf. Bailey [1, §2.4]),

$${}_2F_1 \left[\begin{matrix} a, b \\ c \end{matrix} \middle| x \right] = (1-x)^{-b} \times {}_2F_1 \left[\begin{matrix} c-a, b \\ c \end{matrix} \middle| \frac{x}{x-1} \right],$$

we can deduce the binomial series expression

$$\begin{aligned} \Omega_{0,0}(x, y) &= {}_2F_1 \left[\begin{matrix} x, -x \\ \frac{1}{2} \end{matrix} \middle| y^2 \right] = (1-y^2)^x {}_2F_1 \left[\begin{matrix} -x, \frac{1}{2}-x \\ \frac{1}{2} \end{matrix} \middle| \frac{-y^2}{1-y^2} \right] \\ &= (1-y^2)^x \sum_{k \geq 0} \frac{(-2x)_{2k}}{(2k)!} \left(\frac{-y^2}{1-y^2} \right)^k \\ &= (1-y^2)^x \sum_{k \geq 0} \binom{2x}{2k} \frac{(-1)^k y^{2k}}{(1-y^2)^k} \\ &= \frac{1}{2} \left\{ \left(\sqrt{1-y^2} + y\sqrt{-1} \right)^{2x} + \left(\sqrt{1-y^2} - y\sqrt{-1} \right)^{2x} \right\}. \end{aligned}$$

Letting $y = \sin \theta$, we can simplify $\Omega_{0,0}(x, y)$ further:

$$\Omega_{0,0}(x, y) = \frac{1}{2} \left\{ e^{2xi\theta} + e^{-2xi\theta} \right\} = \cos(2x\theta) = \cos(2x \arcsin y).$$

□