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Cremona transformations of plane configurations of 6 points

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Abstract: We analyze how a set of 6 points of $\mathbb{RP}^2$ in general position changes under quadratic Cremona transformations based at triples of points of the given six. As an application, we give an alternative approach to determining the deformation types (i.e. icosahedral, bipartite, tripartite and hexagonal) of 36 real Schläfli double sixes on any nonsingular real cubic surface performed by Segre.

1. Introduction

1.1. Motivation and the principal result

We call a set of six distinct points of $\mathbb{P}^2$ in general position a *typical* 6*-point configuration*; "generality" here means that no triple of points is collinear and all six are not coconic. Typical 6-point configurations and related to them Schläfli double sixes are classical objects. In our recent work [5], we intended to revisit certain aspects of these topics, where we were motivated by a deformation classification of such 6-point configurations in $\mathbb{RP}^2$, by searching for a relation between Segre's deformation classification of real Schläfli double sixes and Mazurovskiĭ's deformation classification of six skew lines in $\mathbb{RP}^3$.

The general study of plane Cremona transformations which are birational transformations of a projective plane to itself was first originated in [1, 2]. The special examples of plane Cremona transformations are the *quadratic Cremona transformations*, Cr_{ijk}, given as follows: blowing up projective plane $\mathbb{P}^2$ at three points p_i, p_j, p_k and blowing down the proper transformations of three lines passing through each pair of these points. Such transformations play a fundamental role in the theory of Cremona transformations since by Noether's Factorization Theorem, any Cremona transformation can be factorized into a finite sequence of quadratic Cremona transformations.

In this paper, we revisit other related results that are based on a study of the behavior of typical 6-point configurations $\mathcal{P}$ under *internal quadratic Cremona transformations*. The latter are quadratic Cremona transformations generated by triples of points of $\mathcal{P}$. We study such transformations in a real setting.

The deformation classification of typical 6-point configurations in $\mathbb{RP}^2$ was given by Segre [7], §62, who proved that there are four classes: I, II, III and IV. In fact, he used the term *primary* instead of typical. He pointed representatives of typical 6-point configurations $\{p_1, \dots, p_6\}$ for each of them as shown in Figure 1. We enhance these representative configurations to decorated adjacency graphs introduced in Subsection 2.2. According to the shape of the graphs, we call these 6-point configurations *hexagonal, bipartite, tripartite*, and

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icosahedral and denote by QC_i^6, $i = 1, 2, 3, 6$ their deformation classes, respectively. By deformation, we mean a path in the space of typical 6-point configurations (for more details, see Subsection 2.1).

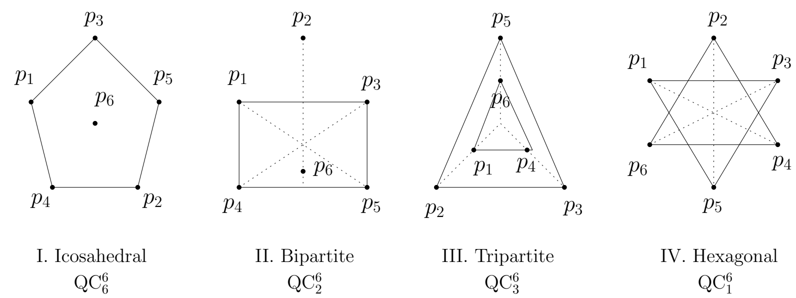


Figure 1. Four deformation classes of typical 6-point configurations.

On each of the four kinds of typical 6-point configurations $\mathcal{P} = \{p_1, \dots, p_6\}$, Segre (see [7], §61) built a simple full topological invariant in a form of plane pentagrams that depend up to homeomorphism. Namely, let us denote by L_{i6}, $i = 1, \dots, 5$ the line passing through points p_i and p_6 and numerate the lines in the pencil of lines passing through point p_6. Then the five intersection points $p_i = Q_6 \cap L_{i6}$, $i = 1, \dots 5$ follow on the conic Q_6 (passing all points of $\mathcal{P}$ except for p_6) in some (cyclic) order $p_{\sigma(1)}, \dots, p_{\sigma(5)}$, where $\sigma \in S_5$ is a permutation. The diagram of $\mathcal{P}$ is obtained from a regular pentagon whose vertices are cyclically numerated by $1, \dots, 5$ and connected by a broken line $\sigma(1) \dots \sigma(5)\sigma(1)$ (see Figure 2). This pentagram does not depend on both the numeration of points of $\mathcal{P}$ and the choice of p_6.

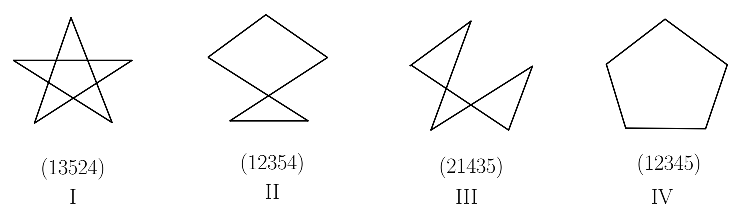


Figure 2. Pentagrams associated to the four deformation classes.

We scheme the modification of typical 6-point configurations in $\mathbb{RP}^2$ under internal quadratic Cremona transformations as a well-defined weighed directed graph, where vertices represent their deformation classes QC_i^6, edges represent internal quadratic Cremona transformations which take a 6-point configuration $\mathcal{P}$ of type QC_i^6 to the one of type QC_j^6 and weights of the edges represent the number of triples p_m, p_n, p_r of points of $\mathcal{P}$ so that $\mathrm{Cr}_{mnr}(\mathcal{P}) \in \mathrm{QC}_j^6$. We call this graph the *Cremona-Segre transformation* graph.

Our principal result is the following statement, which is proved in Subsection 3.2.

Theorem 1.1 *Cremona-Segre transformation graph is like indicated in Figure 3.*

Corollary 1.2 *Any typical* 6*-point configuration in* $\mathbb{RP}^2$ *is obtained from a hexagonal (and tripartite) one by an internal quadratic Cremona transformation. In contrast, icosahedral typical* 6*-point configurations are never obtained from an icosahedral (or a bipartite) one by such a transformation.*

1.2. Complementary 6-point configurations

It is well known that blowing up $\mathbb{P}^2$ at six points of a typical 6-point configuration $\mathcal{P} = \{p_1, \dots, p_6\}$ is a del Pezzo surface $X_{\mathcal{P}}$ that can be realized by anticanonical embedding as a nonsingular cubic surface in $\mathbb{P}^3$. This

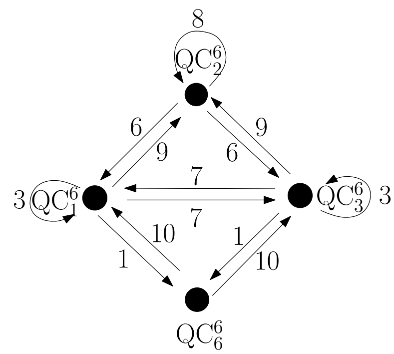


Figure 3. Cremona-Segre transformation graph.

surface contains the marking, $E_{\mathcal{P}} = \{E_1, \ldots, E_6\}$, formed by the exceptional divisors of blowing up at p_i, as well as its complementary marking, $\widetilde{Q}_{\mathcal{P}} = \{\widetilde{Q}_1, \ldots, \widetilde{Q}_6\}$, in which $\widetilde{Q}_i$ is the proper transformation of conic $Q_i \subset \mathbb{P}^2$ passing through the five points of $\mathcal{P}$ other than p_i. Blowing down six lines of $\widetilde{Q}_{\mathcal{P}}$ we obtain a typical 6-point configuration, $\widetilde{\mathcal{P}}$, in $\widetilde{\mathbb{P}}^2$. In the real setting, for $\mathcal{P} \subset \mathbb{RP}^2$, $X_{\mathcal{P}}$ is a real cubic M-surface (i.e. the real locus $\mathbb{R}X_{\mathcal{P}} \subset X_{\mathcal{P}}$ is homeomorphic to $\mathbb{RP}^2 \# 6\mathbb{RP}^2$), and $\widetilde{\mathcal{P}} \subset \mathbb{R}\widetilde{\mathbb{P}}^2$.

Remark 1.3 The configurations $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ belong to different planes. However, $\mathrm{QC}^6(\mathbb{P}^2)/PGL(3,\mathbb{R}) = \mathrm{QC}^6(\widetilde{\mathbb{P}}^2)/PGL(3,\mathbb{R})$ is a canonical identification, where $\mathrm{QC}^6(\mathbb{P}^2), \mathrm{QC}^6(\widetilde{\mathbb{P}}^2)$ are respectively the sets of typical 6-point configurations in $\mathbb{P}^2$, $\widetilde{\mathbb{P}}^2$, and the corresponding deformation classes in $\mathrm{QC}^6(\mathbb{P}^2)$ and $\mathrm{QC}^6(\widetilde{\mathbb{P}}^2)$ are identified with the connected components of the corresponding quotient.

The planes $\mathbb{P}^2$ and $\widetilde{\mathbb{P}}^2$ obtained by blowing down $E_{\mathcal{P}}$ and $\widetilde{Q}_{\mathcal{P}}$, respectively are called the *complementary planes*, and the configurations $\mathcal{P} \subset \mathbb{P}^2$ and $\widetilde{\mathcal{P}} \subset \widetilde{\mathbb{P}}^2$ obtained as the result of blowing down $E_{\mathcal{P}}$ and $\widetilde{E}_{\mathcal{P}}$, respectively are called the *complementary* 6*-point configurations*.

As a first application of Theorem 1.1, we can see the following statement that is proved in Section 3.4.

Theorem 1.4 *Assume that* $\mathcal{P}$, $\widetilde{\mathcal{P}}$ *are complementary typical* 6*-point configurations. Then they belong to the same deformation class (i.e.* QC_1^6, *or* QC_2^6, *or* QC_3^6, *or* QC_6^6*).*

1.3. Real Schläfli double sixes

A Schläfli double six $(\mathcal{L}, \mathcal{L}')$ is a pair of sextuple of skew lines, $\mathcal{L} = \{L_1, \ldots, L_6\}$, $\mathcal{L}' = \{L'_1, \ldots, L'_6\}$ in $\mathbb{P}^3$ such that L_i and L'_j intersect at a point if $i \neq j$ and are disjoint if $i = j$. It is well known that any Schläfli double six determines a unique nonsingular cubic surface. By a *real Schläfli double six*, we mean a Schläfli double six consisting of 12 real lines. Only nonsingular real cubic M-surfaces may contain real Schläfli double sixes.

The classification of real Schläfli double sixes $(\mathcal{L}, \mathcal{L}')$ up to continuous deformations of the real nonsingular cubic surface defined by them was given by Segre [7] who found 4 types of them. One way to distinguish the 4 classes is to blow down six lines of $\mathcal{L}$ (or $\mathcal{L}'$) on the cubic surface, which gives a typical 6-point configuration, $\mathcal{P}_{\mathcal{L}}$ (or $\mathcal{P}_{\mathcal{L}'}$) in $\mathbb{RP}^2$. It is claimed in Theorem 1.4 that $\mathcal{P}_{\mathcal{L}}$ and $\mathcal{P}_{\mathcal{L}'}$ have the same deformation types (that is, they both belong to QC_i^6, $i = 1, 2, 3, 6$), and so we can speak of the four types of real Schläfli double sixes corresponding to 4 types of typical 6-point configurations as shown on Figure 5.

Schläfli [6] observed that any nonsingular cubic surface contains 36 Schläfli double sixes. In the case of real cubic M-surfaces, all of them are real. As a second application of Theorem 1.1, we determine the types of 36 real Schläfli double sixes by using the analysis of internal quadratic Cremona transformations. The following statement is proved in Section 3.5.

Theorem 1.5 *Among 36 real Schläfli double sixes, 10 are hexagonal, 15 are bipartite, 10 are tripartite, and the remaining one is icosahedral.*

In fact, a similar combinatorial description was given by Segre [7] in the context of studying elliptic lines (for definition and more details, see Subsection 2.5 below): 1 double six of the 1st kind that contains 12 elliptic lines (for us, it corresponds to QC^6_6): 15 double sixes of the 2nd kind that contain 4 elliptic lines consisting of 2 pairs of incident lines without common points (for us, they correspond to QC^6_2); 20 double sixes of the 3rd kind that contains 6 elliptic lines consisting of a pair of triple of skew lines such that each line from one of two sets is incident with each of three lines of the other one. According to the type of the pair of triple of lines, he divided double sixes of the 3rd kind into two types: the 1st and 2nd type. Ten of 20 double sixes of the 3rd kind are of the 1st type (for us, they correspond to QC^6_3), and the other 10 are of the 2nd type (for us, they correspond to QC^6_1).

1.4. Structure of the paper

In Section 2 we recall from [4] the concepts of deformation of 6-point configurations and the monodromy groups of typical 6-point configurations. Then, we describe the action of these groups on the triples of points among the given six (Proposition 2.2). This result is used to prove our main theorem. In Section 3, we show how a typical 6-point configuration in $\mathbb{RP}^2$ is changing under the internal quadratic Cremona transformations. We give a simple explicit composition of such Cremona transformations taking a 6-point configuration to its complemantary one. The results are applied to prove Theorems 1.4 and 1.5.

2. Preliminaries

2.1. Deformation classes of 6-point configurations

For us, a *deformation* of a 6-point configuration (i.e. it is a projective configuration of 6 distinct points in plane) is a continuous family $\mathcal{P}_t$, $t \in [0,1]$, formed by 6-point configurations. We call it *L-deformation* if $\mathcal{P}_t$ are simple 6-point configurations that are 6-point configurations in which no triple of points is collinear and *Q-deformation* (or just deformation) if $\mathcal{P}_t$ are typical ones.

We know from [3] that there are 4 deformation classes of simple 6-point configurations shown in Figure 4. In this figure, we sketched configurations $\mathcal{P}$ together with some edges (line segments in $\mathbb{RP}^2$) joining pairs of points if and only if they have no intersections with the lines connecting pairwise the remaining four points. The graph, $\Gamma_{\mathcal{P}}$, that we obtain for a given configuration $\mathcal{P}$ will be called the *adjacency graph* of $\mathcal{P}$.

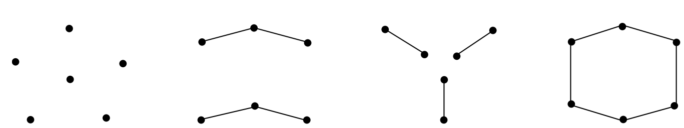


Figure 4. Four kinds of simple 6-point configurations with adjacency graphs.

As we observed in [4], two classifications are coincided: typical 6-point configurations can be connected by an L-deformation if and only if they can be connected by a Q-deformation.

2.2. Coloring graphs $\Gamma_{\mathcal{P}}$ for typical 6-point configurations

Given a typical 6-point configuration $\mathcal{P}$, we say that its point $p \in \mathcal{P}$ is *dominant* (*subdominant*) if it lies outside (respectively, inside) conic Q_p that passes through the remaining 5 points of $\mathcal{P}$. Here, by points inside (outside of) Q_p we mean points lying in the component of $\mathbb{RP}^2 \smallsetminus Q_p$ homeomorphic to a disc, (respectively, in the other component). We color the vertices of adjacency graph $\Gamma_{\mathcal{P}}$ the dominant points of $\mathcal{P}$ in black and subdominant ones in white, see Figure 5 for the result.

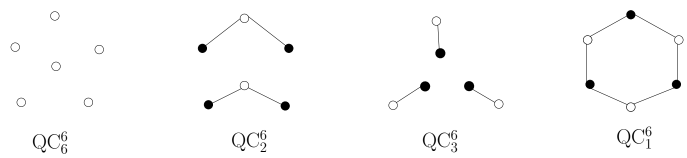


Figure 5. Decorated adjacency graphs.

Remark 2.1 Let $\mathcal{P}' = \{p_1, \ldots, p_5\}$ be a set of 5 points in $\mathbb{RP}^2$ containing no collinear triples of points, and let assume that the numeration of the points of $\mathcal{P}'$ is cyclic as shown in Figure 6. Take a point $p_6 \in \mathbb{RP}^2$ different than points of $\mathcal{P}'$. We showed in [4] that the new configuration $\mathcal{P} = \mathcal{P}' \cup \{p_6\}$ belongs to deformation class QC^6_i, $i = 1, 2, 3, 6$, if the point p_6 lies in the region labeled by i as shown in this figure. Notice that for region 1, the sixth point can lie either outside or inside the conic. In the former possibility, this point will be dominant whereas it will be subdominant for the latter one.

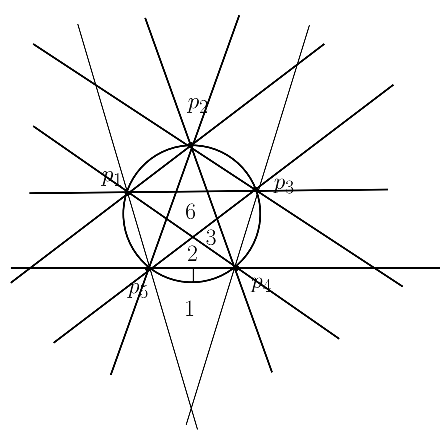


Figure 6. Labels $i = 1, 2, 3, 6$ of regions represent the corresponding class QC^6_i of $\mathcal{P} = \{p_1, ..., p_5, p_6\}$, where p_6 lies in region i.

2.3. The monodromy group of 6-point configurations

By the *L-deformation monodromy group* of a simple 6-point configuration $\mathcal{P}$ in $\mathbb{RP}^2$ we mean the subgroup, $\mathrm{Aut}_L(\mathcal{P})$, of the permutation group $S(\mathcal{P})$ realized by L-deformations (using some fixed numeration of points of $\mathcal{P}$, we can and will identify $S(\mathcal{P})$ with the symmetric group S_6). For a typical 6-point configuration $\mathcal{P}$, we similarly define the *(Q-deformation) monodromy group* $\mathrm{Aut}(\mathcal{P}) \subset S(\mathcal{P}) \cong S_6$ formed by the permutations realized by Q-deformations.

The L-deformation monodromy groups for simple 6-point configurations were given in [3]. We showed in [4], §2.1 and §2.6, monodromy groups for typical 6-point configurations are the same except for the hexagonal ones. That is, for configurations $\mathcal{P}$ from deformation classes QC_2^6, QC_3^6, and QC_6^6, groups $\mathrm{Aut}(\mathcal{P})$ are respectively $\mathbb{Z}/4$, $\mathbb{D}_3$, and the icosahedral group. However, for $\mathcal{P} \in \mathrm{QC}_1^6$, $\mathrm{Aut}(\mathcal{P})$ is not $\mathbb{D}_6$, but its subgroup $\mathbb{D}_3$ that consists of the permutations of $\mathbb{D}_6$ preserving the colors of vertices of the corresponding graph in Figure 5. In fact, monodromy groups for the latter ones were described by Segre [7] as well (§62 of his book) without proof. However, there was a mistake for the case QC_2^6. In this case, he claimed that monodromy group is $\mathbb{Z}_2$.

2.4. $\mathrm{Aut}(\mathcal{P})$-action on triples

For a given typical 6-point configuration $\mathcal{P} = \{p_1, \dots, p_6\}$, let $T_{\mathcal{P}}$ be the set of $\binom{6}{3}$ triples of points of $\mathcal{P}$. Consider the action of monodromy group $\mathrm{Aut}(\mathcal{P})$ on $T_{\mathcal{P}}$ defined by $\sigma(\{p_i, p_j, p_k\}) = \{p_{\sigma(i)}, p_{\sigma(j)}, p_{\sigma(k)}\}$, where $\sigma \in \mathrm{Aut}(\mathcal{P})$ and $p_i, p_j, p_k \in \mathcal{P}$. We denote the orbit of a triple $\{p_i, p_j, p_k\}$ of points by $[\{p_i, p_j, p_k\}]$ or (abuse of the notation) by $[ijk]$.

The following statement describes the action of the monodromy group on the set of triples of points among the given six.

Proposition 2.2 *Let $\mathcal{P}$ be a typical 6-point configuration. Then, the action of $\mathrm{Aut}(\mathcal{P})$ on $T_{\mathcal{P}}$*

(a) has six orbits if $\mathcal{P}$ is either hexagonal or tripartite: two consist of six elements, two consist of three elements and each of the remaining ones consists of one element.

(b) has six orbits if $\mathcal{P}$ is bipartite: four consist of four elements, each of the remaining ones consists of two elements.

(c) has two orbits if $\mathcal{P}$ is icosahedral: each consists of ten elements.

Proof Let $\mathcal{P} \in \mathrm{QC}_i^6$, $i = 1, 2, 3, 6$. For each i, assume that the numeration $p_1, \dots, p_5$ of points of $\mathcal{P}$ other than p_6 is cyclic such that point p_6 is inside the region i as shown in Figure 6. In addition, without loss of generality we can assume that point p_6 is subdominant when $\mathcal{P}$ is hexagonal (i.e. $\mathcal{P} \in \mathrm{QC}_1^6$). We label a point $p \in \mathcal{P}$ at some level with an index which represents a numeration of points of $\mathcal{P}$ up to the action of $Aut(\mathcal{P})$. The numeration that we start stands with the labeling at the first level. For a certain choice of numerations, we get Figure 7, where there are 6 levels for $\mathcal{P} \in \mathrm{QC}_i^6$, $i = 1, 3, 6$ and 4 levels for $\mathcal{P} \in \mathrm{QC}_2^6$.

We start to consider a triple of points at the first level. The corresponding triples at the other levels in this figure represent the elements of orbits of the initial triple. For example, let $\mathcal{P}$ be hexagonal, and let us consider the triple of points $p_1, p_2, p_3 \in \mathcal{P}$. Its corresponding triples from the first level to the sixth one are 123, 156, 345, 156, 345, and 123, respectively. Thus, $[123] = \{123, 156, 345\}$. Now, let us consider triple of points $p_1, p_2, p_4 \in \mathcal{P}$. In this case, we get $[124] = \{124, 256, 346, 146, 245, 236\}$. By proceeding the same way for any other triple of points of $\mathcal{P}$, we can find all distinct orbits. For the cases $\mathcal{P} \in \mathrm{QC}_i^6$, $i = 2, 3$, the idea is the same. Now, suppose that $\mathcal{P} \in \mathrm{QC}_6^6$. We start with the triple of points $p_2, p_3, p_6 \in \mathcal{P}$.

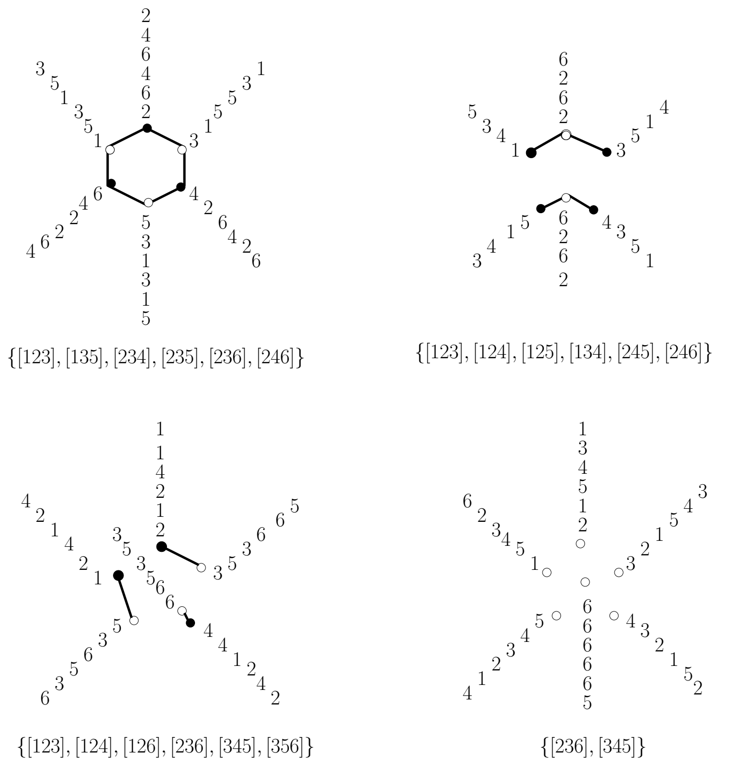


Figure 7. Numerations of six points of $\mathcal{P}$ at levels induced by its monodromy group by starting the fixed one.

Its corresponding triples are 236, 126, 156, 456, 346, and 135. However, the corresponding triples of $p_1, p_2, p_4 \in \mathcal{P}$ are 124, 135, 245, 134, 235, and 126. Two orbits of 236 and 124 contain common elements, so $[236] = \{236, 126, 156, 456, 346, 135, 124, 245, 134, 235\} = [124]$. In the same manner, for triple of points p_3, p_4, p_5 we get $[345] = \{345, 234, 123, 125, 145, 146, 356, 246, 136, 256\} = [146]$. To show why two orbits [236] and [345] are not equal, consider the icosahedron whose 12 vertices are the preimage of points of $\mathcal{P}$ under the double covering $\phi : S^2 \to \mathbb{RP}^2$. Notice that icosahedral group preserves faces of this icosahedron, which correspond to the triples in [236]. However, there are also triples in [345] which do not correspond to the faces. □

2.5. Elliptic and hyperbolic lines

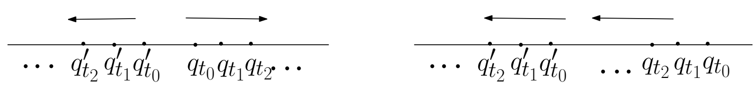
According to Segre [7], a real line on a nonsingular real cubic surface X can be divided in two types: hyperbolic and elliptic. Namely, let L be a real line on X, and consider the one parameter family of planes $\pi_t \subset \mathbb{P}^3$, $t \in \mathbb{P}^1$, containing this line. Then, $X \cap \pi_t = L \cup \Omega_t$, where Ω_t are residual conics. For each $t \in \mathbb{P}^1$, $\quad \Omega_t \cap L = \{q_t, q_t^{'}\}$. The pencil $\{\Omega_t : t \in \mathbb{P}^1\}$ of residual conics defines a double covering $\varphi : L \to \mathbb{P}^1$ such that $\varphi : \{q_t, q_t^{'}\} \to t$. Two branch points of φ are called the *Segre points*. A real line on X is called *hyperbolic* if the Segre points are real, and called *elliptic* if they are complex conjugates to each other.

Recall that blowing up $\mathbb{P}^2$ at six points of a given typical 6-point configuration $\mathcal{P} = \{p_1, \ldots, p_6\}$ in $\mathbb{RP}^2$, we get a nonsingular cubic surface with 27 real lines: 6 exceptional divisors E_i over blown up points p_i, the proper transformations A_{ij} of 15 lines L_{ij} joining two of the blown up points p_i, p_j, and the proper transformations $\widetilde{Q}_i$ of 6 conics Q_i through all but one of the blown up points p_i.

Segre (see [7], §31) showed that for icosahedral typical 6-point configurations, $E_1, \dots, E_6$ and $\widetilde{Q}_1, \dots, \widetilde{Q}_6$ are all elliptic lines. The following statement describes the type (elliptic or hyperbolic) of line A_{ij} for any typical 6-point configuration.

Proposition 2.3 *Let $\mathcal{P} = \{p_1, \dots, p_6\}$ be a typical 6-point configuration, and let A_{ij} be as introduced above. Then, A_{ij} is hyperbolic if the subconfiguration $\mathcal{P} \setminus \{p_i, p_j\}$ are in convex position in $\mathbb{RP}^2 \setminus L_{ij}$, and elliptic otherwise.*

Proof The residual pencil Ω_t associated to line A_{ij} is the pencil of conics passing through the four points other than p_i and p_j. Consider all possible mutual positions (up to homeomorphism) of L_{ij} and this pencil. For each of the possibilities, the intersection points of $L_{ij} \cap \Omega_t$ lie on line L_{ij} like indicated in Figure 8a, b if the four points are in convex position or not, respectively. The former one guarantees that A_{ij} is hyperbolic, and the latter guarantees that A_{ij} is elliptic. $\square$



(a) Hyperbolic (b) Elliptic

Figure 8. The points q_t, q_t' of the intersection $\Omega_t \cap L_{ij}$.

3. Cremona transformations of 6-configurations

3.1. Method of real Cremona transformations

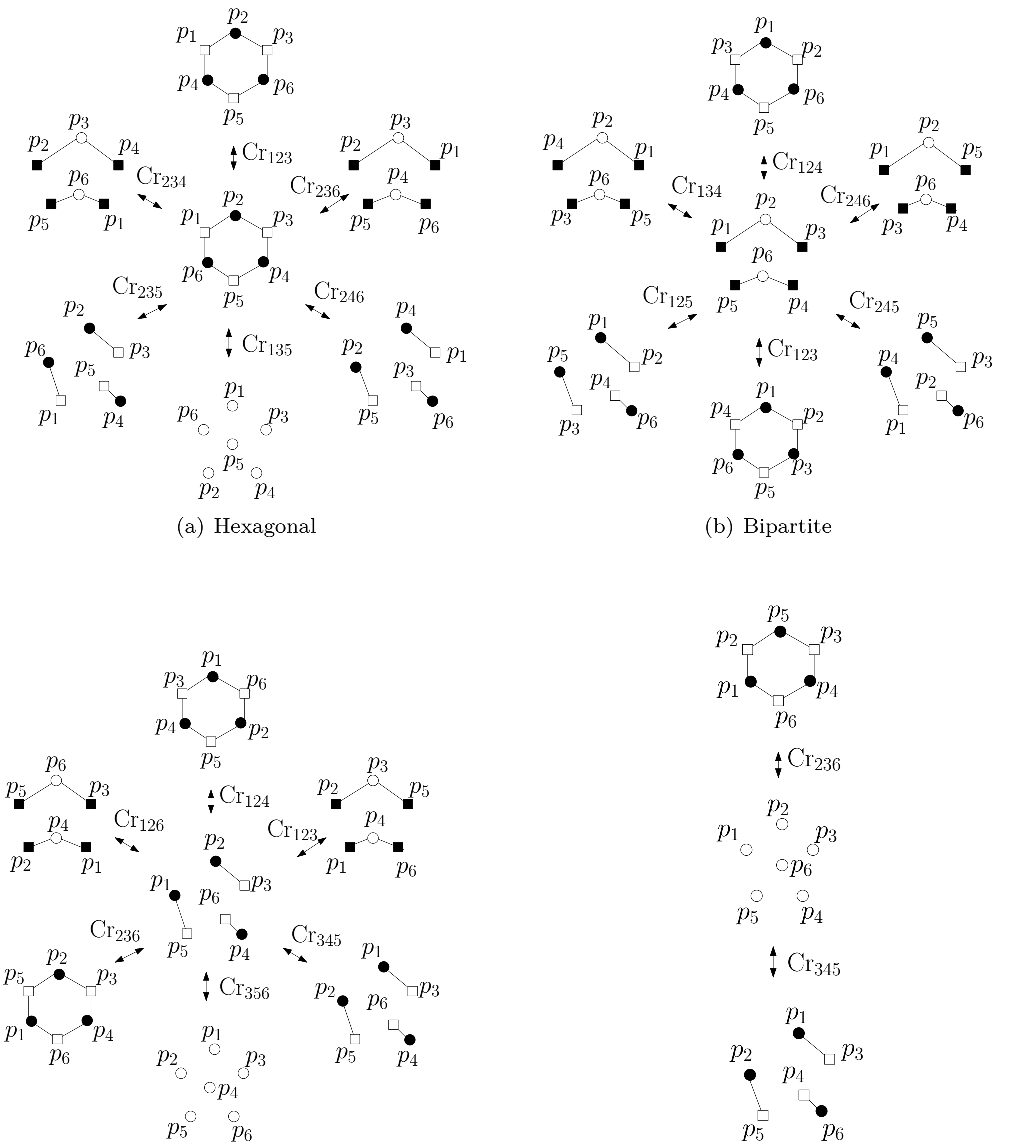
An *internal real quadratic Cremona transformation*, $\mathrm{Cr}_{ijk} : \mathbb{RP}^2 \to \mathbb{RP}^2$, based at a triple of points $\{p_i, p_j, p_k\} \subset \mathcal{P}$ transforms a typical 6-point configuration $\mathcal{P} = \{p_1, \dots, p_6\}$ to another typical 6-point configuration $\mathcal{P}_{ijk} = \mathrm{Cr}_{ijk}(\mathcal{P})$ formed by the three points different from p_i, p_j, p_k, and the three images of L_{ij}, L_{jk}, L_{ki}, which are denoted by p_{ij}, p_{jk}, p_{ki}, or (abuse of the notation if it does not lead to a confusion) by p_k, p_i, p_j, respectively.

By the definition of the monodromy group of a given configuration, we immediately get the following observation.

Proposition 3.1 *Let $\mathcal{P}$ be a typical 6-point configuration. The deformation class of $\mathcal{P}_{ijk}$ is invariant with respect to the action of $Aut(\mathcal{P})$ on the set $T_{\mathcal{P}}$ of triples, .*

Theorem 3.2 *Let $\mathcal{P} \in \mathrm{QC}^6_i$, $i = 1, 2, 3, 6$. For each i, assume that the numeration $p_1, \dots, p_5$ of points of $\mathcal{P}$ other than p_6 is cyclic such that the point p_6 lies in the region i as shown in Figure 6. In addition, if $\mathcal{P}$ is hexagonal (i.e. $\mathcal{P} \in \mathrm{QC}^6_1$), we assume that p_6 is dominant. Then, the modifications of $\mathcal{P}$ under internal quadratic Cremona transformations up to the action of $Aut(\mathcal{P})$ on the set $T_{\mathcal{P}}$ of triples are like indicated in Figure 9.*

Proof By Propositions 2.2 and 3.1, we know that for $\mathcal{P} \in \mathrm{QC}^6_i, i = 1, 2, 3$, the action of $Aut(\mathcal{P})$ on $T_{\mathcal{P}}$ have six orbits while it has two orbits for icosahedral ones, and that the deformation type of $\mathcal{P}_{klm} = \mathrm{Cr}_{klm}(\mathcal{P})$ does not depend on the representative triples p_k, p_l, p_m chosen from the orbits. We obtain Figure 9 as a result of



(a) Hexagonal (b) Bipartite

(c) Tripartite (d) Icosahedral

Figure 9. Modifications of 6-point configurations under the Cremona transformations, where □ and ◦ represent the corresponding hyperbolic and elliptic lines, respectively. The colors black and white show the dominant and subdominant points, respectively.

case-by-case consideration how a typical 6-point configuration (at the center) from each of the 4 classes changes under internal quadratic Cremona transformations by considering concrete choice of triple of points from the orbits.

For the description of ellipticity and hyperbolicity of six real lines E_i over blown up points p_i for each of classes QC_1^6 and QC_3^6, we start with the icosahedral typical 6-point configuration $\mathcal{P}$ in Figure 9d. We already showed that its images under Cr_{236} and Cr_{345} are hexagonal and tripartite, respectively. Recall that the images of points $p_i, p_j, p_k \in \mathcal{P}$ under Cremona transformation Cr_{ijk} are respectively points p_i, p_j, p_k which are the blowing down of the proper transformations of lines L_{jk}, L_{ik}, L_{ij}. By Proposition 2.3, lines L_{36}, L_{26}, and L_{23}

correspond to hyperbolic lines, so exceptional curves E_2, E_3, E_6 of blowing up at p_2, p_3, p_6 that are the images of these lines under Cr_{236} are also hyperbolic. However, the exceptional curves E_1, E_4, E_5 are elliptic as before. Thus, by starting icosahedral 6-point configuration we described types of six real lines over blown up points for hexagonal ones. Similarly, we can see that for the tripartite 6-point configuration $\mathrm{Cr}_{345}(\mathcal{P})$, exceptional curves E_3, E_4, E_5 over p_3, p_4, p_5 are hyperbolic while E_1, E_2, E_6 are elliptic. To describe types of six real lines over blown up points for bipartite ones, we can start with heptagonal or tripartite 6-point configurations for which we have already determined points corresponding to elliptic and hyperbolic lines. Choose the starting configuration and use the same idea for this case.

□

Corollary 3.3 *The kinds (i.e. dominant and subdominant) of three points p_i, p_j, p_k of a typical 6-point configuration are preserved under Cr_{ijk}.*

3.2. Proof of Theorem 1.1

Let $\mathcal{P}$ be a typical 6-point configuration in QC_i^6, $i = 1, 2, 3, 6$. Denote by M_{ij} the number of the internal quadratic Cremona transformations which takes $\mathcal{P}$ to a 6-point configuration of type QC_j^6. We start to assume that $\mathcal{P}$ is hexagonal (that is, $\mathcal{P} \in \mathrm{QC}_1^6$) and use Theorem 3.2 and Proposition 2.2 to count representative triple of points p_m, p_n, p_r of $\mathcal{P}$ so that $\mathrm{Cr}_{mnr}(\mathcal{P}) = \mathcal{P}_{mnr}$ belongs to QC_j^6, $j = 1, 2, 3, 6$. From Figure 9, we see that $\mathcal{P}_{mnr}$ is also hexagonal only if $\{n, m, r\} = \{1, 2, 3\}$. From Figure 7, we extract that orbit [123] consists of three elements. Thus, $M_{11} = 3$. Configuration $\mathcal{P}_{mnr}$ belongs to QC_2^6 if $\{m, n, r\}$ is equal to either $\{2, 3, 6\}$ or $\{2, 3, 4\}$. From Figure 7, we see that orbits [236] and [234] consist of six and three elements, respectively. Thus, $M_{12} = 6 + 3 = 9$. By proceeding the same way, we can find $M_{13} = 7$ and $M_{16} = 1$. For the cases $\mathcal{P} \in \mathrm{QC}_i^6$, $i = 2, 3, 6$, the idea is the same.

3.3. Modifications of lines and conics

The following result shows the modification of lines and conics under elementary quadratic Cremona transformations.

Proposition 3.4 *Let $\mathcal{P} = \{p_1, \ldots, p_6\}$ be a typical 6-point configuration and let L_{ij} and Q_i be a line joining points $p_i, p_j \in \mathcal{P}$ and a conic passing through all points of $\mathcal{P}$ except for p_i, respectively. Then:*

(a) the image of line L_{ij} under Cr_{ijk} for some $k \in \{1, \ldots, 6\} \setminus \{i, j\}$ is a point. However, its image under Cr_{klm} for some $k, l, m \in \{1, \ldots, 6\} \setminus \{i, j\}$ is a conic passing through the five points p_i, p_j, p_k, p_l, p_m of $\mathcal{P}_{klm}$.

(b) the image of conic Q_i under Cr_{jkl} for some $j, k, l \in \{1, \ldots, 6\} \setminus \{i\}$ is a line joining two points $p_n, p_m \in \mathcal{P}$, where $n, m \in \{1, \ldots, 6\} \setminus \{i, j, k, l\}$. However, its image under Cr_{ijk} for some $j, k \in \{1, \ldots, 6\} \setminus \{i\}$ is a conic passing through all points of $\mathcal{P}_{ijk}$ except for p_i.

Proof Its proof is a straightforward analysis using the model $[x : y : z] \mapsto [yz : xz : xy]$ of a quadratic Cremona transformation. □

Corollary 3.5 *Let $\mathcal{P} = \{p_1, \dots, p_6\}$ be a typical 6-point configuration and $E_{\mathcal{P}}$, $\widetilde{Q}_{\mathcal{P}}$ be the sets as introduced in Section 1.2. Then, the modification of lines of $E_{\mathcal{P}}$ and $\widetilde{Q}_{\mathcal{P}}$ under $\mathrm{Cr}_{123} \circ \mathrm{Cr}_{456} \circ \mathrm{Cr}_{123}$ are like indicated in Figure 10, in which $A_{ij} \subset X_{\mathcal{P}}$ for $i \neq j$ are the proper transformations of lines L_{ij}.*

$$
\begin{array}{ccccccc}
 & \mathrm{Cr}_{123} & & \mathrm{Cr}_{456} & & \mathrm{Cr}_{123} & \\
E_1 & \longleftrightarrow & A_{23} & \longleftrightarrow & \widetilde{Q}_1 & \longleftrightarrow & \widetilde{Q}_1 \\
E_2 & \longleftrightarrow & A_{13} & \longleftrightarrow & \widetilde{Q}_2 & \longleftrightarrow & \widetilde{Q}_2 \\
E_3 & \longleftrightarrow & A_{12} & \longleftrightarrow & \widetilde{Q}_3 & \longleftrightarrow & \widetilde{Q}_3 \\
E_4 & \longleftrightarrow & E_4 & \longleftrightarrow & A_{56} & \longleftrightarrow & \widetilde{Q}_4 \\
E_5 & \longleftrightarrow & E_5 & \longleftrightarrow & A_{46} & \longleftrightarrow & \widetilde{Q}_5 \\
E_6 & \longleftrightarrow & E_6 & \longleftrightarrow & A_{45} & \longleftrightarrow & \widetilde{Q}_6
\end{array}
$$

Figure 10. The modification of two markings $\{E_1, \dots, E_6\}$, $\{\widetilde{Q}_1, \dots, \widetilde{Q}_6\}$ on a cubic surface which form a real Schläfli double six.

The following statement provides a simple explicit series of internal quadratic Cremona transformations whose composition transforms a given typical 6-configuration in its complementary one.

Lemma 3.6 *If $\mathcal{P} \subset \mathbb{P}^2$ and $\widetilde{\mathcal{P}} \subset \widetilde{\mathbb{P}}^2$ are complementary typical 6-point configurations, then the composition $\mathrm{Cr}_{123} \circ \mathrm{Cr}_{456} \circ \mathrm{Cr}_{123}$ transforms the plane $\mathbb{P}^2$ into the plane $\widetilde{\mathbb{P}}^2$, and sends $\mathcal{P}$ to $\widetilde{\mathcal{P}}$.*

Proof Let $\mathcal{P} = \{p_1, \dots, p_6\}$ and $\widetilde{\mathcal{P}} = \{\widetilde{p}_1, \dots, \widetilde{p}_6\}$ be complementary configurations to each other. Blowing up $\mathbb{P}^2$ and $\widetilde{\mathbb{P}}^2$ at six points of $\mathcal{P}$ and $\widetilde{\mathcal{P}}$, we obtain cubic surfaces $X_{\mathcal{P}}$ and $X_{\widetilde{\mathcal{P}}}$ together with a sextuple of skew lines $E_{\mathcal{P}} = \{E_1, \dots, E_6\}$ and $E_{\widetilde{\mathcal{P}}} = \{\widetilde{E}_1, \dots, \widetilde{E}_6\}$ formed by the exceptional divisors over blown up points, respectively. According to the definition of complementary configurations (see Section 1.2), $X_{\mathcal{P}}$ is identical to $X_{\widetilde{\mathcal{P}}}$. In addition, there is a natural identification between $E_{\widetilde{\mathcal{P}}}$ and $\widetilde{Q}_{\mathcal{P}} = \{\widetilde{Q}_1, \dots, \widetilde{Q}_6\}$, where $\widetilde{Q}_i$, $i = 1, \dots, 6$, are represented in the $\mathbb{P}^2$ by conics Q_i passing through the points of $\mathcal{P}$ other than p_i. By Figure 10, we see that the map $\mathrm{Cr}_{123} \circ \mathrm{Cr}_{456} \circ \mathrm{Cr}_{123}$ sends $E_{\mathcal{P}}$ to $\widetilde{Q}_{\mathcal{P}}$, or vice versa. This completes the proof.

□

Proposition 3.7 *The deformation type of a typical 6-point configuration is preserved under the composite function $\mathrm{Cr}_{ijk}\,\mathrm{Cr}_{mnr}\,\mathrm{Cr}_{ijk}$, where $i, j, k, m, n, r \in \{1, \dots, 6\}$ are all distinct.*

Proof Let $\mathcal{P} \in \mathrm{QC}_i^6$, $i = 1, 2, 3, 6$. For each i, assume that the numeration $p_1, \dots, p_5$ of points of $\mathcal{P}$ other than p_6 is cyclic such that the point p_6 is inside the region i as shown in Figure 6. In addition, without loss of generality we can assume that point p_1 is subdominant when $\mathcal{P}$ is hexagonal. Using Figure 9, we obtain Figure 11, where one concludes that deformation type of $\mathcal{P}$ is not changing under $\mathrm{Cr}_{ijk}\,\mathrm{Cr}_{mnr}\,\mathrm{Cr}_{ijk}$.

□

3.4. Proof of Theorem 1.4

Let $\mathcal{P} \subset \mathbb{P}^2$ and $\widetilde{\mathcal{P}} \subset \widetilde{\mathbb{P}}^2$ be complementary 6-point configurations to each other. Due to Lemma 3.6, the map $\mathrm{Cr}_{123} \circ \mathrm{Cr}_{456} \circ \mathrm{Cr}_{123}$ sends $\mathcal{P}$ to $\widetilde{\mathcal{P}}$. Using Proposition 3.7, we conclude that $\mathcal{P}$ and $\widetilde{\mathcal{P}}$ have the same deformation class.

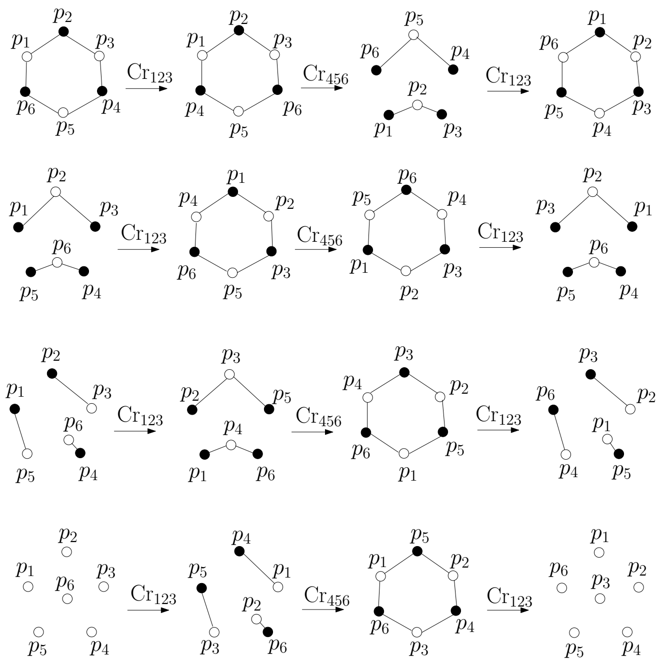


Figure 11. The modification of a 6-point configuration from each of classes QC_i^6, $i = 1,2,3,6$, under $\mathrm{Cr}_{123} \circ \mathrm{Cr}_{456} \circ \mathrm{Cr}_{123}$.

3.5. Proof of Theorem 1.5

Let X be a nonsingular real cubic M-surface and let n_i be the number of real Schläfli double sixes $(\mathcal{L}, \widetilde{\mathcal{L}})$ on X, which correspond to QC_i^6, $i = 1,2,3,6$, that is to say, the typical 6-point configuration obtained by blowing down six lines of $\mathcal{L}$ (or $\widetilde{\mathcal{L}}$) belongs to QC_i^6. Any nonsingular real cubic M-surface has 36 real Schläfli double sixes, so we have immediately $n_1 + n_2 + n_3 + n_6 = 36$.

According to Cremona-Segre transformation graph given in Theorem 1.1, we know the numbers of internal quadratic Cremona transformations between real Schläfli double sixes of types QC_i^6 and the ones of type QC_j^6 on a fixed cubic surface. Thus, we have $9n_1 = 6n_2$, $7n_1 = 7n_3$, $n_1 = 10n_6$, and $6n_2 = 9n_3$. Since $n_1 + n_2 + n_3 + n_6 = 36$ we get

$$n_1 + \frac{3}{2}n_1 + n_1 + \frac{1}{10}n_1 = 36$$

$$\frac{36}{10}n_1 = 36$$

$$n_1 = 10$$

Therefore, $n_2 = 15$, $n_3 = 10$ and $n_6 = 1$.

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