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Basicity of a system of exponents with a piecewise linear phase in Morrey-type spaces

Bilal BILALOV[1,*], **Fidan SEYIDOVA**[2]
[1]Department of Non-Harmonic Analysis, Institute of Mathematics and Mechanics, NAS of Azerbaijan, Baku, Azerbaijan
[2]Ganja State University, Ganja, Azerbaijan

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Abstract: In this paper a perturbed system of exponents with a piecewise linear phase depending on two real parameters is considered. The sufficient conditions for these parameters are found, under which the considered system of exponents is complete, minimal, or it forms a basis for a Morrey-type space.

Key words: Exponential system, basicity, Morrey space

1. Introduction

Basis properties of the following exponential system are studied in this work

$$E_{\beta;\mu} \equiv \left\{ e^{i(nt+\lambda_n(t))} \right\}_{n\in Z}, \tag{1.1}$$

where $\lambda_n(t) = -(\beta t + \mu\, signt)\ sign\ n\,;\ \beta, \mu \in R$ are real parameters, and Z is the set of integers. This system is a modification of the following perturbed exponential system

$$e_\beta \equiv \left\{ e^{i(n+\beta sign\, n)t} \right\}_{n\in Z},$$

which has been considered by many mathematicians. The study of basis properties of e_β (such as completeness, minimality, and basicity) has a long history. It dates back to the works by Paley and Wiener [33] and Levinson [23, 24]. Basicity (Riesz basicity) criterion for the system e_β in $L_2(-\pi,\pi)$ with respect to the real parameter $\beta \in R$ follows from the results obtained by Levinson [23, 24] and Kadets [22], and this criterion is the inequality $|\beta| < \frac{1}{4}$. Basicity criterion for the system e_β in the Lebesgue spaces $L_p(-\pi,\pi)$, $1 < p < +\infty$, with respect to the parameter β has been obtained later by Sedletski [36] and Moiseev [25]. Basis properties of e_β are closely related to the similar properties of perturbed sine systems

$$\{\sin(n-\beta)t\}_{n\in N}, \tag{1.2}$$

and cosine systems

$$1 \cup \{\cos(n-\beta)t\}_{n\in N}, \tag{1.3}$$

*Correspondence: b_bilalov@mail.ru
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$$\{\cos(n-\beta)t\}_{n\in Z_+}\,(Z_+=\{0\}\cup N)\,, \tag{1.4}$$

in corresponding Banach spaces of functions on $[0,\pi]$. These systems arise when solving partial differential equations of mixed (or elliptic) type using the Fourier method in special domains. To justify the formally constructed solution, it is very important to study the basis properties of these systems in appropriate spaces of functions (see, e.g., [26, 34]). Many authors have studied the basis properties of systems in various functional spaces (mainly Lebesgue spaces and their weighted versions; see, e.g., [2–7, 12–14, 27–29, 31, 37, 38]). The works which consider the approximation properties of the systems (1.1)–(1.4) can be divided into two groups. The first one includes the works which used the methods of the theory of entire functions (see, e.g., [22–25, 33, 36]), and the second group consists of those which used the methods of boundary value problems for analytic functions (see, e.g., [10, 27, 31, 32, 34]). The latter idea originated from Bitsadze [15], later to be successfully used in [25, 27, 34]. Further development of this approach, used in establishing basis properties of perturbed trigonometric systems and power systems, was by Bilalov [2, 3, 7, 11, 14]. Similar problems are studied in [32, 38].

In the context of applications to some problems of mechanics and mathematical physics, recently there has been great interest in the nonstandart spaces of functions. As examples of this kind of spaces, we can mention Lebesgue space with the variable summability index, Morrey space, Companato space, etc. The theory of differential equations and its relationship with the harmonic analysis requires the study of many cornerstone issues of analysis in these spaces. A lot of classical facts about harmonic analysis have been extended to these spaces (for detailed information about these matters see Xianling and Dun [39], Zorko [40], Morrey [30], Cruz-Uribe and Fiorenza [16], Adams [1], etc.). Along with this, of course you have to study approximation matters in suchlike spaces. Approximation matters have been (and are being) relatively well studied in generalized Lebesgue spaces by Sharapudinov [37], Israfilov [20, 21], Bilalov and Huseynov [12, 13], etc. (see e.g., [19, 31]). The situation is different in the case of Morrey-type spaces. Only recently the approximation matters began to be studied in these spaces, and many problems in this field still remain to be solved. Apparently the works by Israfilov [20, 21], Bilalov and Guliyeva [14], Gasymov and Guliyeva [18] have been pioneers in this field.

In this paper a perturbed system of exponents with a piecewise linear phase depending on two real parameters is considered. The sufficient conditions for these parameters are found, under which the considered system of exponents is complete, minimal, or it forms a basis for a Morrey-type space. It should be noted that the basis properties of the system (1.1) are completely different from those of the system e_β. Basis properties of e_β in Morrey-type spaces have been fully studied in the recent work by Bilalov [9].

2. Preliminaries

In this section we state some notations and facts which will be used to obtain our main results. Let us first define the Morrey space on the unit circle $\gamma=\{z\in C:|z|=1\}$ on the complex plane C. Next, $\omega=int\gamma$ will denote the unit ball in C. By $L_0(-\pi,\pi)$ we denote the linear space of all (Lebesgue-) measurable functions on $(-\pi,\pi)$. X^* will denote the conjugate space of a space X. T^* will denote the adjoint operator of a operator T.

By $L^{p,\alpha}(\gamma)$, $1\le p<+\infty$, $0\le\alpha\le 1$, we will denote the normed space of all measurable functions $f(\cdot)$

on γ with the finite norm

$$\|f\|_{L^{p,\alpha}(\gamma)} = \sup_B \left(|B\cap\gamma|_\gamma^{\alpha-1} \int_{B\cap\gamma} |f(\xi)|^p |d\xi| \right)^{1/p} < +\infty,$$

($|B\cap\gamma|_\gamma$ −is the linear measure of intersection $B\cap\gamma$), where sup has taken over all balls centered at γ with an arbitrary positive radius. $L^{p,\alpha}(\gamma)$ is a Banach space with respect to this norm. We also define the space $L^{p,\alpha}(-\pi,\pi)$, $1\le p<+\infty$, $0\le\alpha\le1$, which consists of measurable functions $f(\cdot)$ on $(-\pi,\pi)$ with the finite norm

$$\|f\|_{L^{p,\alpha}(-\pi,\pi)} = \sup_{I\subset[-\pi,\pi]} \left(|I|^{\alpha-1} \int_I |f(t)|^p |dt| \right)^{1/p} < +\infty,$$

where sup has taken over all intervals $I\subset[-\pi,\pi]$. It is not difficult to see that the correspondence $f(t) =: F\left(e^{it}\right)$, $t\in(-\pi,\pi)$, $F(\cdot)\in L^{p,\alpha}(\gamma)$, establishes an isometric isomorphism between the spaces $L^{p,\alpha}(\gamma)$ and $L^{p,\alpha}(-\pi,\pi)$. Therefore, in what follows we will equate these spaces and denote $L^{p,\alpha}$ with the norm $\|\cdot\|_{p,\alpha}$.

It is not difficult to see that for $0\le\alpha_1\le\alpha_2\le1$ the following continuous embedding holds $L^{p,\alpha_1}\subset L^{p,\alpha_2}$. Moreover, it is clear that $L^{p,1}=L_1(-\pi,\pi)$ and $L^{p,0}=L_\infty(-\pi,\pi)$. We also have $L^{p,\alpha}\subset L_1(-\pi,\pi)$, $\forall\alpha\in[0,1]$, $\forall p\ge1$.

Weighted version of the space $L^{p,\alpha}$ is defined in a natural way. Namely, if $\rho:[-\pi,\pi]\to R_+=(0,+\infty)$ (or $\rho:\gamma\to R_+$) is some weight function, then the weighted version of the space $L^{p,\alpha}$ is a normed space of measurable functions with the norm

$$\|f\|_{p,\alpha:\rho} = \|f\rho\|_{p,\alpha}, \ \forall f\in L^{p,\alpha}_\rho.$$

In the sequel we will use the notation $f(x)\sim g(x)$, $x\in M$, which means

$$\exists\delta\in(0,1):\ \ \delta\le\left|\frac{f(x)}{g(x)}\right|\le\delta^{-1},\ \ \forall x\in M.$$

The similar meaning is carried by $f(x)\sim g(x)$, $x\to a$.

The lemma below was proved in [9]:

Lemma 2.1 *[9] The space L_∞ (and so $C[-\pi,\pi]$ too) is not dense in $L^{p,\alpha}$ for $1\le p<+\infty$ and $\forall\alpha\in(0,1)$.*

It follows that the sequence of bounded functions cannot be complete in $L^{p,\alpha}$. In what follows, we will assume, if needed, that the function $f\in L^{p,\alpha}$ is periodically (with period 2π) extended to the whole real axis R. Following Lemma 2.1, we will consider the subspace $M^{p,\alpha}$ of functions $f(\cdot)$ the shifts of which are continuous in $L^{p,\alpha}$, i.e.

$$\|f(\cdot+\delta)-f(\cdot)\|_{p,\alpha}\to0,\ \ \delta\to0.$$

The following lemma holds.

Lemma 2.2 *[9] The space $M^{p,\alpha}$, $1\le p<+\infty$, $0<\alpha\le1$, is a Banach space and $C_0^\infty[-\pi,\pi]$ is dense in it, where $C_0^\infty[-\pi,\pi]$ is the subspace of functions , which has finite support and infinite differentiable on $[-\pi,\pi]$.*

It is not difficult to see that the system $E_{\beta,\mu}$ belongs to $M^{p,\alpha}$: $E_{\beta,\mu} \subset M^{p,\alpha}$. Therefore it is clear that the closure of the linear span of $E_{\beta,\mu}$ also belongs to $M^{p,\alpha}$, i.e. $\overline{spanE_{\beta,\mu}} \subset M^{p,\alpha}$. So it is quite natural to study basis properties of the system $E_{\beta,\mu}$ in the space $M^{p,\alpha}$.

We will use in this work the following result obtained in [10].

Lemma 2.3 *Let* $f(\cdot) \in L_\infty(-\pi,\pi)$ *and* $g(\cdot) \in M^{p,\alpha}$, $1 \le p < +\infty$, $0 < \alpha \le 1$. *Then* $f(\cdot)\, g(\cdot) \in M^{p,\alpha}$.

Finally, we state the following easy-to-prove lemma which will be frequently used throughout this work.

Lemma 2.4 *Let* $\{\tau_k\}_{k=\overline{1,m}} \subset \gamma-$ *be different points. Then the finite product*

$$\omega(\tau) = \prod_{k=1}^{m} |\tau - \tau_k|^{\alpha_k}, \quad \tau \in \gamma,$$

belongs to the space $L^{p,\alpha}$, $1 \le p < +\infty$, $0 < \alpha < 1$, *if and only if the inequalities* $\alpha_k \ge -\frac{\alpha}{p}$, $\forall k = \overline{1,m}$, *hold.*

We will also use the following

Lemma 2.5 *Let* $\gamma > -\frac{\alpha}{p}$, $1 \le p < +\infty$, $0 < \alpha \le 1$. *Then* $\left\| |t|^{\gamma} \chi_E(t) \right\|_{p,\alpha} \to 0$ *as* $|E| \to 0$, *where* $E \subset [-\pi,\pi]-$ *is an arbitrary interval and* $|E|-$ *is the length of this interval.*

Remark 2.6 *It is obvious that Lemma 2.5 stays true for the function* $|t - t_0|^{\gamma}$, *too, where* $t_0 \in [-\pi,\pi]$ *is an arbitrary point. In other words, if* $\gamma > -\frac{\alpha}{p}$, $1 \le p < +\infty$, $0 < \alpha \le 1$, *then*

$$\lim_{|E| \to 0} \left\| |t - t_0|^{\gamma} \chi_E(t) \right\|_{p,\alpha} = 0.$$

Moreover, it is not difficult to see that this does not hold true for $\gamma = -\frac{\alpha}{p}$.

Using the results of Lemma 2.5 and Remark 2.6, it is easy to prove the validity of the following

Lemma 2.7 *Let* $\gamma_k > -\frac{\alpha}{p}$, $k = \overline{0,r}$; *and* $\{t_k\}_0^r \in [-\pi,\pi]-$ *be different points. Then the following relation is true*

$$\lim_{|E| \to 0} \|\omega(t)\, \chi_E(t)\|_{p,\alpha} = 0,$$

where

$$\omega(t) = \prod_{k=0}^{r} |t - t_k|^{\gamma_k}.$$

3. Morrey–Hardy spaces

Let us state some facts about the theory of Hardy spaces. Define the Morrey–Hardy class $H_+^{p,\alpha}$, $1 \le p < +\infty$, $0 \le \alpha \le 1$, of functions $f(\cdot)$ analytic inside ω endowed with the norm

$$\|f\|_{H_+^{p,\alpha}} = \sup_{0<r<1} \|f_r(\cdot)\|_{p,\alpha},$$

where $f_r(t) = f\left(re^{it}\right)$. It is not difficult to see that the inclusion $H_+^{p,\alpha} \subset H_1^+$, $1 \le p < +\infty$, holds, where $H_1^+ -$ is a usual Hardy class. Therefore, every function $f(\cdot) \in H_+^{p,\alpha}$ has nontangential boundary values $f^+(\cdot)$ on γ.

Take $\forall f(\cdot) \in L^{p,\alpha}$, $1 \le p < +\infty$, $0 < \alpha \le 1$, and consider the following Cauchy-type integral

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)\,d\xi}{\xi - z}, \quad z \notin \gamma.$$

Recall the following well-known Sokhotski–Plemelj formula

$$F^{\pm}(\tau) = \pm\frac{1}{2}f(\tau) + (Sf)(\tau), \quad \tau \in \gamma,$$

where $F^+(\tau)$ $(F^-(\cdot))-$ are nontangential boundary values of $F(\cdot)$ inside (outside) the unit ball ω on γ, and $S-$ is a singular integral

$$(Sf)(\tau) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)\,d\xi}{\xi - \tau}, \quad \tau \in \gamma.$$

Later in this work we will often use the following result of [35].

Theorem 3.1 *[35] Let the weight function $\rho(\cdot)$ be defined as follows*

$$\rho(t) = \prod_{k=1}^{m} |t - t_k|^{\alpha_k}, \ \{t_k\}_1^m \subset \gamma, \ t_i \ne t_j, \quad i \ne j.$$

Then the singular operator $S-$ is bounded in the weighted space $L_\rho^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha < 1$, if and only if the following inequalities hold

$$-\frac{\alpha}{p} < \alpha_k < -\frac{\alpha}{p} + 1, \quad k = \overline{1,m}.$$

The following theorem is true.

Theorem 3.2 *[9] Let $f(\cdot) \in H_+^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \le 1$. Then $f^+(\cdot) \in L^{p,\alpha}$ and the Cauchy formula*

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f^+(\tau)\,d\tau}{\tau - z}, \quad z \in \omega, \tag{3.1}$$

holds, where $f^+(\cdot)-$are nontangential boundary values of $f(\cdot)$ on γ. Conversely, if $f^+(\cdot) \in L^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \le 1$, then the function $f(\cdot)$, defined by the Cauchy-type integral (3.1), belongs to the class $H_+^{p,\alpha}$.

Using this theorem, it is easy to prove the analog of the Smirnov theorem.

Theorem 3.3 *Let $f \in H_+^{p,\alpha}$, $1 \le p < +\infty$, $0 < \alpha \le 1$ and $f^+ \in L^{q,\beta}$; $q > p$, $0 < \beta \le 1$, where $f^+-$are nontangential boundary values of $f(\cdot)$ on γ. Then $f \in H_+^{q,\beta}$.*

Consider the space $H_+^{p,\alpha}$. Denote by $L_+^{p,\alpha}$ the subspace of $L^{p,\alpha}$ generated by the restrictions of the functions from $H_+^{p,\alpha}$ to γ, i.e. $L_+^{p,\alpha} = H_+^{p,\alpha}/_\gamma$. From the uniqueness theorem for analytic functions and Theorem 3.2 it follows that the spaces $H_+^{p,\alpha}$ and $L_+^{p,\alpha}$ are isomorphic and the restriction operator $J{:}H_+^{p,\alpha} \leftrightarrow L_+^{p,\alpha}$; $(Jf)(\tau) = f^+(\tau)$, $\tau \in \gamma$, $\forall f \in H_+^{p,\alpha}$, performs the corresponding isomorphism. Let $M_+^{p,\alpha} = M^{p,\alpha} \cap L_+^{p,\alpha}$ with the norm $\|\cdot\|_{p,\alpha}$. It is clear that $M_+^{p,\alpha}$ is a subspace of $M^{p,\alpha}$ (because $M^{p,\alpha}$ and $L_+^{p,\alpha}$ both are the closed subspaces of $L^{p,\alpha}$). Let $MH_+^{p,\alpha} = J^{-1}\left(M_+^{p,\alpha}\right)$. Obviously, $MH_+^{p,\alpha}$ is a subspace of $H_+^{p,\alpha}$. It follows from the above considerations that for $\forall f \in H_+^{p,\alpha}$ the norm $\|f\|_{H_+^{p,\alpha}}$ can be defined also by the relation $\|f\|_{H_+^{p,\alpha}} = \|f^+\|_{p,\alpha}$, where f^+- are nontangential boundary values of f on γ.

Absolutely similar to the classical case, we define the Morrey–Hardy class outside the unit circle ω. Let $\omega^- = C\backslash\bar{\omega}$ $(\bar{\omega} = \omega \cup \gamma)$. We will say that the function f analytic in ω^- has a finite order m at infinity, if its Laurent decomposition at infinitely remote point has the following form

$$f(z) = \sum_{k=-\infty}^{m} a_k z^k, \quad a_m \neq 0. \tag{3.2}$$

Thus, for $m > 0$ the function f has a pole of order m at $z = \infty$; for $m = 0$, it is bounded in the vicinity of $z = \infty$; and in case $m < 0$ it has a zero of order $(-m)$ at $z = \infty$. Let $f(z) = f_0(z) + f_1(z)$, where $f_0(z)$ is the principal part (i.e. $f_0(z) = \sum_{k=0}^{m} a_k z^k$), and $f_1(z)$ is the regular part of decomposition (3.2). Consequently, $f_0(z) \equiv 0$, for $m < 0$, and f_0 is a polynomial of degree m, i.e. $\deg f_0 = m$, if $m \geq 0$. We will say that the function f belongs to the class ${}_mH_-^{p,\alpha}$, if $\deg f_0 \leq m$ and $F \in H_+^{p,\alpha}$, where $F(z) = \overline{f_1\left(\frac{1}{\bar{z}}\right)}$, $z \in \omega$.

Absolutely similar to the case of $MH_+^{p,\alpha}$, we define the class ${}_mMH_-^{p,\alpha}$. In other words, ${}_mMH_-^{p,\alpha}$ is a subspace of functions from ${}_mH_-^{p,\alpha}$, whose shifts are continuous on γ with respect to the norm $\|\cdot\|_{p,\alpha}$.

Consider the weighted versions of above spaces. First define the weighted space $M_\rho^{p,\alpha}$ with some weight function $\rho : [-\pi, \pi] \to R_+ = (0, +\infty)$. Consider the weighted space $L_\rho^{p,\alpha}$ and let $f \in L_\rho^{p,\alpha}-$ be some function. If needed, we will assume that the function $f(\cdot)$ is extended outside $[-\pi, \pi]$ by evenness, i.e.

$$f(x) = \begin{cases} f(-2\pi - x), & x \in [-3\pi, -\pi), \\ f(2\pi - x), & x \in (\pi, 3\pi]. \end{cases}$$

(the extended function will also be denoted by $f(\cdot)$). Let

$$M_\rho = \left\{ f \in L_\rho^{p,\alpha} : \|f(\cdot + \delta) - f(\cdot)\|_{p,\alpha;\rho} \to 0, \quad \delta \to 0 \right\}.$$

It is absolutely clear that M_ρ is a linear subspace of $L_\rho^{p,\alpha}$. Denote the closure of M_ρ in $L_\rho^{p,\alpha}$ by $M_\rho^{p,\alpha}$, i.e. $M_\rho^{p,\alpha}$ is a subspace of $L_\rho^{p,\alpha}$. It is not difficult to see that if $\rho \in L^{p,\alpha}$, then $C[-\pi, \pi] \subset M_\rho^{p,\alpha}$. In fact, let $f \in C[-\pi, \pi]$ be an arbitrary function and δ be a number sufficiently small in absolute value. Obviously, the function $f(\cdot)$ extended to $[-3\pi, 3\pi]$ is also continuous. We have

$$\|f(\cdot + \delta) - f(\cdot)\|_{p,\alpha;\rho} = \sup_{I \subset (-\pi,\pi)} \left(\frac{1}{|I|^{1-\alpha}} \int_I |(f(t+\delta) - f(t))\,\rho(t)|^p \, dt \right)^{1/p} \leq$$

$$\leq \sup_{t\in(a,b)} |f(t+\delta) - f(t)| \, \|\rho\|_{p,\alpha} \to 0, \quad \delta \to 0.$$

The last relation follows from the uniform continuity of $f(\cdot)$ in $[-3\pi, 3\pi]$.

Based on the restriction operator J, absolutely similar to previous cases we define the corresponding Morrey–Hardy classes of functions analytic in ω and ω^-, respectively. Namely, a weighted Hardy–Morrey space $H^{p,\alpha}_{\rho;+}$ is defined by means of the norm

$$\sup_{0<r<1} \|f_r(t)\|_{p,\alpha;\rho} < +\infty, \ \ \forall f \in H^{p,\alpha}_{\rho;+},$$

where $f_r(t) = f\left(re^{it}\right)$. Denote $L^{p,\alpha}_{\rho;+} = H^{p,\alpha}_{\rho;+}/\gamma$. The spaces $L^{p,\alpha}_{\rho;+}$ and $H^{p,\alpha}_{\rho;+}$ are isomorphic. Let $M^{p,\alpha}_{\rho;+} = M^{p,\alpha}_{\rho} \cap L^{p,\alpha}_{\rho;+}$ and $MH^{p,\alpha}_{\rho;+} = J^{-1}\left(M^{p,\alpha}_{\rho;+}\right)$. Then, $MH^{p,\alpha}_{\rho;+}$ is a subspace of $H^{p,\alpha}_{\rho;+}$. Absolutely similar to the previous case, we define the weighted Hardy–Morrey classes ${}_mH^{p,\alpha}_{\rho;-}$ and ${}_mMH^{p,\alpha}_{\rho;-}-$ of functions analytic outside the unit circle γ.

The following theorem is true.

Theorem 3.4 *Let the singular operator S be bounded in $L^{p,\alpha}_{\rho}$, $1 < p < +\infty$, $0 < \alpha \leq 1$. Then $M^{p,\alpha}_{\rho}$ is its invariant subspace.*

4. The space $(L^{p,\alpha})'$

When treating basis properties of systems, one often has to use a conjugate space. As we do not yet have a description for a space conjugate to $L^{p,\alpha}$, in the form of functional space (see, e.g., [1]), it suffices to consider some subspace of $(L^{p,\alpha})^*$, denoted $(L^{p,\alpha})'$ and defined by

$$(L^{p,\alpha})' = \left\{ g \in L_0(-\pi,\pi) : \sup_{f\in S_{p,\alpha}} \|fg\|_{L_1(-\pi,\pi)} < +\infty \right\},$$

with the norm

$$\|g\|_{(p,\alpha)'} = \sup_{f\in S_{p,\alpha}} \|fg\|_{L_1}, \tag{4.1}$$

where $S_{p;\alpha} = \left\{ f \in L^{p,\alpha} : \ \|f\|_{p,\alpha} = 1 \right\} -$ is a unit sphere in $L^{p,\alpha}$.

The following theorem is true.

Theorem 4.1 $(L^{p,\alpha})'$, $1 \leq p < +\infty$, $0 \leq \alpha \leq 1$, *is a Banach space with respect to the norm (4.1).*

In what follows, we will often use the following lemma.

Lemma 4.2 *Let $t_0 \in [-\pi, \pi] -$ be an arbitrary point. Then the function $g(t) = |t - t_0|^{\beta}$ belongs to the space $(L^{p,\alpha})'$, $1 \leq p < +\infty$, $0 < \alpha \leq 1$, if and only if $\beta \in \left(\frac{\alpha}{p} - 1, \infty\right)$.*

Remark 4.3 *Let $J \subset [-\pi;\pi]$ be an arbitrary interval and $t_0 \in J$. It is not difficult to see that the proof of Lemma 4.2 is also applicable if J is taken instead of $[-\pi;\pi]$, i.e. $|t - t_0|^{\beta} \in (L^{p,\alpha}(J))'$ holds if and only if $\beta \in \left(-1 + \frac{\alpha}{p}, \ +\infty\right)$.*

Using Lemma 4.2, it is easy to prove the following lemma.

Lemma 4.4 *The finite product*

$$\nu(t)=\prod_{k=1}^{m}|t-t_k|^{\beta_k}, \{t_k\}_{k=\overline{1,m}}\subset[-\pi,\pi], \quad t_i\neq t_j, \quad i\neq j,$$

belongs to the space $\left(L^{p,\alpha}\right)'$, $1\leq p<+\infty$, $0<\alpha\leq 1$, *if and only if* $\beta_k\in\left(-1+\frac{\alpha}{p},+\infty\right)$, $\forall k=\overline{1,m}$.

This lemma has the following immediate corollary.

Corollary 4.5 *Let* $-\pi=s_0<s_1<...<s_r<\pi-$ *be arbitrary points. Then the finite product*

$$\mu(t)=\prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{\alpha_k}, t\in(-\pi,\pi),$$

belongs to the space $\left(L^{p,\alpha}\right)'$, $1\leq p<+\infty$, $0\leq\alpha\leq 1$, *if and only if* $\alpha_k\in\left(-1+\frac{\alpha}{p},+\infty\right)$, $\forall k=\overline{0,r}$.

5. General solution of homogeneous Riemann problem in Hardy–Morrey classes

To establish the basicity of the exponential system (1.1) for Morrey-type spaces $M^{p,\alpha}$, we will use the method of Riemann boundary value problems developed by Bilalov (see, e.g., [2, 3, 7, 8, 10–14]). Consider the following homogeneous Riemann problem

$$\left.\begin{array}{l}F^+(\tau)-G(\tau)F^-(\tau)=0, \qquad \tau\in\gamma,\\ F^+(\cdot)\in H_+^{p,\alpha}; F^-(\cdot)\in_m H_-^{p,\alpha},\end{array}\right\}\tag{5.1}$$

$1<p<+\infty$, $0<\alpha\leq 1$, where $G\left(e^{it}\right)=\left|G\left(e^{it}\right)\right|e^{i\theta(t)}$, $t\in[-\pi,\pi]-$ is the coefficient of the problem. By the solution of the problem (5.1) we mean a pair of functions $(F^+;F^-)\in H_+^{p,\alpha}\times_m H_-^{p,\alpha}$ such that the nontangential boundary values $F^+(\tau)$ inside ω and $F^-(\tau)$ outside ω satisfy the relation (5.1) a.e. on γ. Introduce the following piecewise analytic functions on the complex plane cut by γ:

$$X_1(z)\equiv\exp\left\{\frac{1}{4\pi}\int_{-\pi}^{\pi}\ln\left|G\left(e^{it}\right)\right|\frac{e^{it}+z}{e^{it}-z}dt\right\},$$

$$X_2(z)\equiv\exp\left\{\frac{1}{-4i\pi}\int_{-\pi}^{\pi}\theta(t)\frac{e^{it}+z}{e^{it}-z}dt\right\}, \ z\notin\gamma.$$

Let

$$Z_k(z)=\left\{\begin{array}{ll}X_k(z), & |z|<1,\\ (X_k(z))^{-1}, & |z|>1,\end{array}\right.$$

and $Z(z)=Z_1(z)Z_2(z)$, $z\notin\gamma$.

Function $Z(\cdot)$ *will be called a canonical solution of homogeneous problem.*

Regarding the coefficient $G(\cdot)$ of the problem (5.1), we will assume that the following conditions hold:

i) $G^{\pm 1}(\cdot) \in L_\infty(-\pi, \pi)$;

ii) $\theta(t) = \arg G(e^{it})$ – is a piecewise Hölder function on $[-\pi, \pi]$, and let $h_k = \theta(s_k + 0) - \theta(s_k - 0)$, $k = \overline{1, r}$– be the jumps of this function at the points of discontinuity $\{s_k\}_1^r : -\pi < s_1 < .... < s_r < \pi$.

Bilalov [9] proved the following:

Theorem 5.1 *[9] Let the coefficient $G(\cdot)$ of the problem (5.1) satisfy the conditions i), ii) and the jumps $\{h_k\}_0^r$ of the function $\theta(t) = \arg G(e^{it})$ on $[-\pi, \pi]$, where $h_0 = \theta(\pi) - \theta(-\pi)$, satisfy the inequalities*

$$-1 + \frac{\alpha}{p} < \frac{h_k}{2\pi} \leq \frac{\alpha}{p}, \quad k = \overline{0, r}.$$

Then:

α) for $m \geq 0$ the problem (5.1) has a general solution of the form

$$F(z) \equiv Z(z) P_k(z), \tag{5.2}$$

where $Z(z)$ – is a canonical solution of this problem, and $P_k(z)$ – is an arbitrary polynomial of degree $k \leq m$;

β) for $m < 0$ the problem (5.1) has only a trivial solution.

Using this theorem, it is easy to prove the following one.

Theorem 5.2 *[9] Let the coefficient $G(\cdot)$ satisfy the conditions i), ii) and the jumps $\{h_k\}_0^r$ of the function $\theta(\cdot)$ satisfy the inequalities*

$$-1 + \frac{\alpha}{p} < \frac{h_k}{2\pi} < \frac{\alpha}{p}, \qquad k = \overline{0, r}.$$

Then:

α) for $m \geq 0$ the problem (5.1) has a general solution of the form (5.2) in the Morrey–Hardy classes $MH_+^{p,\alpha} \times_m MH_-^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha \leq 1$;

β) for $m < 0$ the problem (5.1) has only a trivial solution in the Morrey–Hardy classes $MH_+^{p,\alpha} \times_m MH_-^{p,\alpha}$, $0 < \alpha < 1$, $1 < p < +\infty$.

6. Nonhomogeneous Riemann problem in Morrey–Hardy classes

Consider the nonhomogeneous Riemann problem

$$F^+(\tau) - G(\tau) F^-(\tau) = f(\arg \tau), \qquad \tau \in \gamma, \tag{6.1}$$

in the Morrey–Hardy classes $H_+^{p,\alpha} \times_m H_-^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha < 1$, where $f \in L^{p,\alpha}$– is some given function. $f(\cdot)$ is called the right-hand side, and $G(\cdot)$– is called the coefficient of the problem (6.1).

Let us state the result obtained by Bilalov [9] concerning the solvability of the problem (6.1).

Theorem 6.1 *[9] Let the coefficient $G(\cdot)$– of the problem (6.1) satisfy the conditions i); ii) and $\{h_k\}_1^r$ – be the jumps of the function $\theta(t) = \arg G(e^{it})$ in $(-\pi, \pi)$ at the points of discontinuity $\{s_k\}_1^r \subset (-\pi, \pi)$: $h_0 = \theta(\pi) - \theta(-\pi)$. Let the inequalities*

$$-1 + \frac{\alpha}{p} < \frac{h_k}{2\pi} < \frac{\alpha}{p}, \quad k = \overline{0, r}$$

hold. Then, the following assertions are true regarding the solvability of nonhomogeneous Riemann problem (6.1) *in the Morrey–Hardy classes* $H_{+}^{p,\alpha} \times_m H_{-}^{p,\alpha}, \quad 1 < p < +\infty, \quad 0 < \alpha < 1$*:*

α) *for* $m \geq -1$ *the problem* (6.1) *has a general solution of the form*

$$F(z) = Z(z) P_m(z) + F_1(z),$$

where $Z(\cdot) -$ *is a canonical solution of corresponding homogeneous problem* (5.1), $P_m(\cdot) -$ *is an arbitrary polynomial of degree* $k \leq m \quad (P_{-1}(z) \equiv 0)$, $F_1(\cdot) -$ *is a particular solution of the problem* (6.1) *of the form*

$$F_1(z) = \frac{Z(z)}{2\pi} \int_{-\pi}^{\pi} \frac{f(t)}{Z^{+}(e^{it})} K_z(t)\, dt, \tag{6.2}$$

where $K_z(\cdot) = \frac{1}{e^{it} - z}$ *is a Cauchy kernel, and* $f \in L^{p,\alpha} -$ *is an arbitrary function;*

β) *for* $m < -1$ *the problem* (6.1) *is solvable if and only if the right-hand side* $f(\cdot) \in L^{p,\alpha}$ *satisfies the orthogonality conditions*

$$\int_{-\pi}^{\pi} \frac{f(t)}{Z^{+}(e^{it})} e^{ikt} dt = 0, \quad k = \overline{1, -m-1}, \tag{6.3}$$

and then the problem (6.1) *has a unique solution* $F(z) = F_1(z)$, *where* $F_1(\cdot) -$ *is defined by* (6.2).

This theorem has the following immediate corollary.

Corollary 6.2 *Let all the conditions of Theorem 6.1 hold. Then for* $\forall f \in L^{p,\alpha}, \quad 1 < p < +\infty, \quad 0 < \alpha < 1$, *the problem* (6.1) *has a unique solution in the Morrey–Hardy classes* $H_{+}^{p,\alpha} \times_{-1} H_{-}^{p,\alpha}$, *which can be represented in terms of Cauchy-type integral of the form* (6.2).

Consider the case where the right-hand side $f(\cdot)$ of the problem (6.1) belongs to the space $M^{p,\alpha}$, i.e. $f(\cdot) \in M^{p,\alpha}$. In this case, the solution is sought in the classes $MH_{+}^{p,\alpha} \times_m M_{-}^{p,\alpha}$.

Bilalov [9] proved the following:

Theorem 6.3 *[9] Let all the conditions of Theorem 6.1 hold and the right-hand side* $f(\cdot)$ *of the nonhomogeneous Riemann problem* (6.1) *belong to* $M^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha < 1$. *Then the following assertions are true concerning the solvability of this problem in classes* $MH_{+}^{p,\alpha} \times_m MH_{-}^{p,\alpha}$*:*

α) *for* $m \geq -1$ *the problem* (6.1) *has a general solution of the form*

$$F(z) = Z(z) P_m(z) + F_1(z),$$

where $Z(\cdot) -$ *is a canonical solution of the problem* (5.1), $P_m(\cdot) -$ *is an arbitrary polynomial of degree* $k \leq m$ $(P_{-1}(z) \equiv 0)$, $F_1(\cdot) -$ *is a particular solution of the problem* (6.1) *defined by* (6.2)*;*

β) *for* $m < -1$ *the problem* (6.1) *is solvable if and only if* $f(\cdot)$ *satisfies the orthogonality conditions* (6.3), *and then the problem* (6.1) *has a unique solution* $F(z) = F_1(z)$, *defined by* (6.2).

This theorem has the following immediate

Corollary 6.4 *Let all the conditions of Theorem 6.3 hold. Then, for* $\forall f \in M^{p,\alpha}$, $1 < p < +\infty$, $0 < \alpha < 1$, *the problem* (6.1) *has a unique solution in Morrey–Hardy classes* $MH_{+}^{p,\alpha} \times_{-1} MH_{-}^{p,\alpha}$, *defined by the Cauchy-type integral* (6.2).

7. Main results

To determine the basicity of the exponential system $E_{\beta;\mu}$ in Morrey spaces $M^{p,\alpha}$, we will use the method of boundary value problems. This method requires the determination of basicity of parts of exponential system for Morrey–Hardy spaces $MH_{+}^{p,\alpha}$ and ${}_{-1}MH_{-}^{p,\alpha}$. Based on the results of Section 3, we equate these spaces with the spaces $M_{+}^{p,\alpha}$ and ${}_{-1}M_{-}^{p,\alpha} ={}_{-1} MH_{-}^{p,\alpha}/_{\gamma}-$ (restriction to γ). It is not difficult to prove the following

Theorem 7.1 *[9] The system* $\left\{e^{int}\right\}_{n\in Z_+}\left(\left\{e^{-int}\right\}_{n\in N}\right)$ *forms a basis for the space* $M_{+}^{p,\alpha}\left(for\ {}_{-1}M_{-}^{p,\alpha}\right)$, $0<\alpha<1,\ \ 1<p<+\infty$.

Now we pass to the basicity of the perturbed exponential system $E_{\beta;\mu}$ for Morrey space $M^{p,\alpha}$, $0<\alpha<1$, $1<p<+\infty$. We will follow the techniques used in [2, 3, 12]. Consider the following nonhomogeneous Riemann boundary value problem

$$F^{+}\left(e^{it}\right)-e^{i2\lambda(t)}F^{-}\left(e^{it}\right)=e^{i\lambda(t)}f\left(t\right),\ t\in\left(-\pi,\pi\right), \tag{7.1}$$

where $f\left(\cdot\right)\in M^{p,\alpha}$ is some function and $\lambda\left(t\right)=-\left(\beta\,t+\mu\ sign\,t\right)$. The solution of the problem (7.1) is sought in Morrey–Hardy classes $MH_{+}^{p,\alpha}\times_{-1}MH_{-}^{p,\alpha}$. Let us make use of Theorem 6.3. Let $\beta;\ \mu\in R$ be real parameters. We have

$$\theta\left(t\right)=2\lambda\left(t\right)=-2\left(\beta t+\mu\,sign\,t\right),\ t\in\left[-\pi,\pi\right].$$

The function $\theta\left(\cdot\right)$ has a unique discontinuity point $t=0$ in the interval $\left(-\pi,\pi\right)$. The corresponding jump at this point is

$$h_1=\theta\left(+0\right)-\theta\left(-0\right)=-4\mu\,.$$

We have

$$h_0=\theta\left(\pi\right)-\theta\left(-\pi\right)=-4\left(\beta+\mu\right)\,.$$

We will follow Corollary 6.4. Suppose that the inequalities

$$-1+\frac{\alpha}{p}<-\frac{2\mu}{\pi}<\frac{\alpha}{p};\ -1+\frac{\alpha}{p}<-2\beta-\frac{2\mu}{\pi}<\frac{\alpha}{p}$$

hold. Then, as follows from Corollary 6.2, problem (7.1) has a unique solution in the Morrey–Hardy classes $MH_{+}^{p,\alpha}\times_{-1}MH_{-}^{p,\alpha}$ for an arbitrary right-hand side $f\left(\cdot\right)\in M^{p,\alpha}$ and this solution is representable by a Cauchy type integral

$$F\left(z\right)=\frac{Z\left(z\right)}{2\pi}\int_{-\pi}^{\pi}\frac{f\left(t\right)}{Z^{+}\left(e^{it}\right)}K_z\left(t\right)dt,$$

where $Z\left(\cdot\right)-$ is a canonical solution of the corresponding homogeneous problem, $K_z\left(\cdot\right)=\frac{1}{e^{it}-z}$ is a Cauchy kernel.

In what follows we will need some properties of a canonical solution. Let us represent the function $\theta\left(\cdot\right)$ in the following form

$$\theta\left(t\right)=\theta_0\left(t\right)+\theta_1\left(t\right),$$

where $\theta_0\left(\cdot\right)-$ is its continuous (Hölder) part, and $\theta_1\left(\cdot\right)$ is a jump function defined by

$$\theta_1\left(-\pi\right)=0\,,\ \theta_1\left(s\right)=\sum_{k:-\pi<s_k<s}h_k\ ,\ \forall s\in\left(-\pi,\pi\right].$$

Let

$$h_0^{(0)} = \theta_0(\pi) - \theta_0(-\pi) ,$$

and

$$u_0(t) = \left|\sin\frac{t+\pi}{2}\right|^{-\frac{h_0^{(0)}}{2\pi}} \exp\left(-\frac{1}{4\pi}\int_{-\pi}^{\pi}\theta_0(\tau)\, ctg\frac{t-\tau}{2}dt\right).$$

By the results of [9], the boundary values of the canonical solution $Z(\cdot)$ can be represented as

$$\left|Z^-\left(e^{it}\right)\right| = \left|G\left(e^{it}\right)\right|^{-\frac{1}{2}} |u_0(t)| \prod_{k=0}^{r}\left|\sin\frac{t-s_k}{2}\right|^{-\frac{h_k}{2\pi}}.$$

It is absolutely clear that the function $\theta_0(\cdot)$ is a Hölder function on $[-\pi,\pi]$. Then again, by the results of [35], we have

$$\sup_{[-\pi,\pi]} vrai\ |u_0(t)|^{\pm 1} < +\infty.$$

As $\theta(\cdot)$ is a Hölder function on $[-\pi,\pi]$, then the corresponding jump function vanishes, i.e. $\theta_1(t) \equiv 0, t \in [-\pi,\pi]$, and it is clear that $\theta(t) \equiv \theta_0(t), t \in [-\pi,\pi]$.

Taking into account the equality

$$\left|Z^+\left(e^{it}\right)\right| = \left|G\left(e^{it}\right)\right| \left|Z^-\left(e^{it}\right)\right|,$$

from the previous relations we obtain

$$\left|Z^+\left(e^{it}\right)\right| \sim const, t \in (-\pi,\pi).$$

By the results of Corollary 6.4 for $\forall f \in M^{p,\alpha}$, the problem (7.1) has a unique solution $(F^+; F^-)$ in the classes $MH_+^{p,\alpha} \times_{-1} MH_-^{p,\alpha}$. Denote by $T_+^+\left(T_-^-\right)$ the operator which maps the function $f(\cdot)$ to the function $F^+(\cdot)$ $(F^-(\cdot)), T_+^+\left(T_-^-\right)$ i.e. $T_+ f = F^+ (T_- f = F^-)$. It is absolutely clear that $T^{\pm}$ is a linear operator. In the sequel, we will equate the function $F^+(\cdot) \in MH_+^{p,\alpha}\left(F^-(\cdot) \in {}_{-1}MH_-^{p,\alpha}\right)$ to its boundary values $F^+\left(e^{it}\right) \in MH_+^{p,\alpha}\left(F^-\left(e^{it}\right) \in {}_{-1}MH_-^{p,\alpha} \equiv {}_{-1}MH_-^{p,\alpha}/\gamma\right)$.

Applying Sokhotski–Plemelj formulas, we obtain

$$F^{\pm}(\tau) = Z^{\pm}(\tau)\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{Z^+\left(e^{it}\right)}\frac{e^{it}dt}{e^{it}-z}\right]_{\gamma}^{\pm} =$$

$$= Z^{\pm}(\tau)\left(\pm\frac{1}{2}\left[Z^+(\tau)\right]^{-1} f(\arg\tau) - \left[Z^+(\tau)\right]^{-1}(Kf)(\tau)\right),$$

where $[\cdot]_{\gamma}^{\pm} -$ denotes the boundary values on γ from inside ω (with "+") and outside ω (with "-"), respectively, and $K-$ is a singular Cauchy integral of the form

$$(Kf)(\tau) = \frac{Z^+(\tau)}{2\pi}\int_{-\pi}^{\pi}\frac{f(t)}{Z^+\left(e^{it}\right)}K_{\tau}(t)\,dt, \quad \tau \in \gamma.$$

It is absolutely clear that

$$f(t)\left[Z^{+}\left(e^{it}\right)\right]^{-1}\in L^{p,\alpha}.$$

Then, as the singular operator acts boundedly in $L^{p,\alpha}$, from the previous relation we obtain

$$\exists M>0:\|T_{+}f\|_{\rho,\alpha}=\left\|F^{+}\right\|_{\rho,\alpha}\leq M\|f\|_{\rho,\alpha}.$$

Consider the relation

$$e^{-i\lambda(t)}F^{+}\left(e^{it}\right)+e^{i\lambda(t)}F^{-}\left(e^{it}\right)=f(t),t\in(-\pi,\pi).$$

Expand the functions $F^{+}\left(e^{it}\right),F^{-}\left(e^{it}\right)$ in the space $MH^{p,\alpha}$ with respect to the systems $\left\{e^{int}\right\}_{n\in Z_{+}}$ and $\left\{e^{-int}\right\}_{n\in N}$, respectively. We have

$$e^{-i\lambda(t)}\sum_{n=0}^{\infty}V_{n}^{+}\left(F^{+}\right)e^{int}+e^{i\lambda(t)}\sum_{n=0}^{\infty}V_{n}^{-}\left(F^{-}\right)e^{-int}=f(t),$$

where

$$V_{n}^{+}(g)=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(t)e^{int}dt,n\in Z_{+},$$

$$V_{k}^{-}(g)=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(t)e^{ikt}dt,k\in N.$$

For convenience, let

$$V_{n}^{+}(g)=\left(g;e^{-int}\right),n\in Z_{+},V_{k}^{-}(g)=\left(g;e^{ikt}\right),k\in N,$$

where $(g;f)=\int_{-\pi}^{\pi}g(t)\overline{f(t)}dt$. Thus, the following relations hold

$$V_{n}^{+}\left(F^{+}\right)=\left(F^{+};e^{-int}\right)=\left(T^{+}f;e^{-int}\right)=\left(f;T_{+}^{*}e^{-int}\right),n\in Z^{+},$$

$$V_{k}^{-}\left(F^{-}\right)=\left(F^{-};e^{ikt}\right)=\left(T^{-}f;e^{ikt}\right)=\left(f;T_{-}^{*}e^{ikt}\right),k\in N,$$

$$V_{n}^{+}\in\left(MH_{+}^{p,\alpha}\right)^{*},\forall n\in Z_{+},V_{n}^{-}\in\left({}_{-1}MH_{-}^{p,\alpha}\right)^{*},\forall k\in N.$$

We have

$$T_{+}:M^{p,\alpha}\rightarrow M_{+}^{p,\alpha};T_{-}:M^{p,\alpha}\rightarrow{}_{-1}M_{-}^{p,\alpha}.$$

Consequently

$$T_{+}^{*}:\left(M_{+}^{p,\alpha}\right)^{*}\rightarrow\left(M^{p,\alpha}\right)^{*},$$

$$T_{-}^{*}:\left({}_{-1}M_{-}^{p,\alpha}\right)^{*}\rightarrow\left(M^{p,\alpha}\right)^{*}.$$

Then it is clear that the following inclusions hold

$$T_{+}^{*}V_{n}^{+}\in\left(M^{p,\alpha}\right)^{*},\forall n\in Z_{+},\ T_{-}^{*}V_{k}^{-}\in\left(M^{p,\alpha}\right)^{*},\forall k\in N.$$

Denote $V_{n}^{\pm}=T_{\pm}^{*}V_{n}^{\pm}$. Then we have

$$V_{n}^{\pm}\left(F^{+}\right)=V_{n}^{\pm}\left(T_{\pm}f\right)=T_{\pm}^{*}V_{n}^{\pm}(f)=V_{n}^{\pm}(f).$$

Considering these expressions for $f(\cdot)$, we have the expansion

$$f(t) = e^{i\lambda(t)} \sum_{n=0}^{\infty} V_n^+(f) e^{int} + e^{-i\lambda(t)} \sum_{n=0}^{\infty} V_n^-(f) e^{-int}.$$

It is absolutely clear that

$$F^+\left(e^{it}\right) = \sum_{n=0}^{\infty} V_n^+(f) e^{int};$$

$$F^-\left(e^{it}\right) = \sum_{n=1}^{\infty} V_n^-(f) e^{-int}.$$

Now take $f(t) = e^{-i\lambda(t)} e^{ikt}$ as a function $f(\cdot)$, where $k \in Z_+-$ is some fixed number. In this case, the solution of the problem (7.1) is

$$\left.\begin{array}{l} F^+(z) = \sum_{n=0}^{\infty} V_n^+\left[e^{-i\lambda(t)} e^{ikt}\right] z^n, \\ F^-(z) = \sum_{n=1}^{\infty} V_n^-\left[e^{-i\lambda(t)} e^{ikt}\right] z^{-n}. \end{array}\right\} \tag{7.2}$$

On the other hand, the functions below are also the solution

$$\left.\begin{array}{l} F^+(z) = z^k, |z| < 1, \\ F^-(z) = 0, |z| > 1. \end{array}\right\} \tag{7.3}$$

Comparing the relations (7.4) and (7.3), from the uniqueness of the solution we obtain

$$V_n^+\left[e^{-i\lambda(t)} e^{ikt}\right] = \left\{\begin{array}{l} 1, n = k, \\ 0, n \neq k, \end{array}\right.$$

$$V_n^+\left[e^{-i\lambda(t)} e^{ikt}\right] = 0, \forall n \in N, \forall k \in Z.$$

Similarly, if we take $f(t) = e^{i\lambda(t)} e^{-ikt}$, as a function $f(\cdot)$, where $k \in N$ is some fixed number, then we obtain the following relations

$$V_n^+\left[e^{i\,\lambda(t)} e^{-ikt}\right] = 0, \forall n \in Z^+, \forall k \in N,$$

$$V_n^-\left[e^{i\lambda(t)} e^{-ikt}\right] = \delta_{n,k}, \forall n, k \in N.$$

From these relations it immediately follows that the system $\left\{V_n^+, V_{n+1}^-\right\}_{n \in Z_+}$ is biorthogonal to the system (1.1); therefore, the system (1.1) is minimal in $M^{p,\alpha}$. Thus, the following theorem is proved.

Theorem 7.2 *Let the real parameters $\beta; \mu \in R$ satisfy the following inequalities*

$$-1 + \frac{\alpha}{p} < -\frac{2\mu}{\pi} < \frac{\alpha}{p};\; -1 + \frac{\alpha}{p} < -2\beta - \frac{2\mu}{\pi} < \frac{\alpha}{p}.$$

Then the system of exponents $E_{\beta;\mu}$ forms a basis for $M^{p,\alpha}, 0 < \alpha < 1,\; 1 < p < +\infty$.

Further we will consider the most general case. Taking into account the periodicity of the exponent, the coefficient $G(\cdot)$ of the problem (7.1) is defined by the following expression

$$G(t) = e^{i\tilde{\theta}(t)},\ t \in [-\pi, \pi],$$

where

$$\tilde{\theta}(t) = \begin{cases} -2\beta t + 2\mu + 2m_1\pi,\ t \in [-\pi, 0), \\ -2\beta t - 2\mu + 2m_2\pi, t \in (0, \pi], \end{cases}$$

$m_1; m_2 \in Z$ are some integers. The function $\tilde{\theta}(\cdot)$ has a discontinuity point $t = 0$ and its jump at this point is equal to

$$\tilde{h}_1 = \tilde{\theta}(+0) - \tilde{\theta}(-0) = -2\mu + 2m_2\pi - (2\mu + 2m_1\pi) = -4\mu + 2(m_2 - m_1)\pi.$$

We also have

$$\begin{array}{l} \tilde{h}_0 = \tilde{\theta}(\pi) - \tilde{\theta}(-\pi) = -2\beta\pi - 2\mu - 2m_1\pi + \\ (-2\beta\pi - 2\mu + 2m_2\pi) = -4\beta\pi - 4\mu - 2(m_1 - m_2)\pi. \end{array}$$

Following Corollary 6.4, we choose the integers $m_1; m_2$ from the following conditions

$$-1 + \frac{\alpha}{p} < \frac{\tilde{h}_k}{2\pi} < \frac{\alpha}{p},\ k = 0, 1.$$

We have

$$\left.\begin{array}{l} -1 + \frac{\alpha}{p} < -\frac{2\mu}{\pi} + m_2 - m_1 < \frac{\alpha}{p}, \\ 1 + \frac{\alpha}{p} < -2\beta - \frac{2\mu}{\pi} - m_1 + m_2 < \frac{\alpha}{p}. \end{array}\right\} \tag{7.4}$$

Applying the previous scheme to the system of exponents

$$\tilde{E}_{\beta;\mu} \equiv \left\{ e^{i(nt + \tilde{\lambda}_n(t))} \right\}_{n \in Z},$$

where

$$\tilde{\lambda}_n(t) = -\frac{1}{2}\tilde{\theta}(t) sign\, n, n \in Z,$$

we obtain that if there exist integers $m_1; m_2$ such that inequalities (7.4) hold, then the system of exponents $\tilde{E}_{\beta;\mu}$ forms a basis for $M^{p,\alpha}$, $0 < \alpha < 1,\ 1 < p < +\infty$. Considering that the systems $E_{\beta;\mu}$ and $\tilde{E}_{\beta;\mu}$ coincide and putting $m = m_2 - m_1$, we get the validity of the following

Theorem 7.3 *Let there exist an integer m such that the inequalities*

$$\left.\begin{array}{l} -1 + \frac{\alpha}{p} < -\frac{2\mu}{\pi} + m < \frac{\alpha}{p}, \\ -1 + \frac{\alpha}{p} < -2\beta - \frac{2\mu}{\pi} - m < \frac{\alpha}{p}, \end{array}\right\}$$

hold. Then the system of exponents $\tilde{E}_{\beta;\mu}$ forms a basis for a space $M^{p,\alpha}$, $0 < \alpha < 1$, $1 < p < +\infty$.

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