

## On surfaces endowed with a canonical principal direction in Euclidean 4-spaces

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**Abstract:** In this paper, we introduce canonical principal direction (CPD) submanifolds with higher codimension in Euclidean spaces. Then we obtain a classification of surfaces endowed with CPD relative to a fixed direction in Euclidean 4-space.

**Key words:** Euclidean spaces, canonical principal direction, angle function

### 1. Introduction

Let  $N$  be a Riemannian manifold and  $X$  a vector field tangent to  $N$ . An immersed hypersurface  $M$  of  $N$  is said to have a canonical principal direction (CPD) relative to  $X$  if the projection of  $X$  onto the tangent space of  $M$  gives one of the principal directions of  $M$  [11]. A common example of such hypersurfaces can be constructed as follows. Consider a rotational hypersurface  $M$  in a Euclidean space  $\mathbb{E}^n$  with the rotation axis  $l$ . Then  $M$  has CPD relative to the fixed direction parallel to  $l$ .

When the ambient space is Euclidean, the notion of constant angle hypersurfaces is related to hypersurfaces with CPD. A hypersurface in the Euclidean space is said to be a constant angle hypersurface if there exists a constant direction  $k$  that makes a constant angle with the tangent space of the hypersurface at every point. The applications of constant angle surfaces in the theory of liquid crystals and layered fluids were first considered in [2]. Furthermore, Munteanu and Nistor gave another approach concerning surfaces in Euclidean spaces for which the unit normal makes a constant angle with a fixed direction in [18]. Moreover, many classification results for constant angle hypersurfaces in 3-dimensional pseudo-Euclidean spaces have been obtained so far [12, 13, 15, 16]. Constant angle hypersurfaces in high-dimensional spaces have also attracted the attention of some mathematicians. For example, a classification of such surfaces in  $\mathbb{E}^4$  was given in [1]. Furthermore, a local construction of constant angle hypersurfaces was given in [8]. We want to note that some geometers call constant angle hypersurfaces helix hypersurfaces [1, 8, 9].

It is well known that a surface  $M$  in Euclidean 3-space has CPD relative to  $k$  if it is a constant angle surface. For this reason, hypersurfaces in Euclidean spaces with CPD relative to a fixed direction  $k$  have attracted the interest of some geometers in recent years. For example, surfaces with CPD in the Euclidean 3-space  $\mathbb{E}^3$  were studied in [19]. This study was moved into the Minkowski 3-space  $\mathbb{E}_1^3$  in [14, 20].

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CPD surfaces and constant angle surfaces in product spaces also attract the attention of some geometers. For example, the study of constant angle surfaces was extended to product spaces in [5–7, 10]. On the other hand, some classification results on surfaces with CPD relative to  $\partial_t$  in  $\mathbb{S}^2 \times \mathbb{R}$  and  $\mathbb{H}^2 \times \mathbb{R}$  were obtained in [4, 5, 7]. See also [10], where  $\partial_t$  denotes the parallel unit vector field tangent to the second factor. Also, Tojeiro studied CPD hypersurfaces of  $\mathbb{S}^n \times \mathbb{R}$  and  $\mathbb{H}^n \times \mathbb{R}$  in [21]. Later, Mendonça and Tojeiro gave a generalization of the notion of CPD hypersurfaces into higher codimensional submanifolds. For this purpose, they gave the definition class  $\mathcal{A}$  in [17]. An immersion  $f : M^n \rightarrow Q_c^n \times \mathbb{R}$  is said to belong to class  $\mathcal{A}$  if the tangential part of  $\partial_t$  is one of the principal directions of all shape operators of  $f$ . In a similar way, we would like to give the following definition of CPD submanifolds in Euclidean spaces:

**Definition 1.1** *Let  $M^n$  be a submanifold in  $\mathbb{E}^m$  and  $k$  be a fixed direction in  $\mathbb{E}^m$ .  $M$  is said to be a submanifold endowed with a canonical principal direction (shortly, a CPD submanifold) if the tangential component  $k^T$  of  $k$  is one of the principal directions of all shape operators of  $M$ .*

The aim of this paper is to obtain a complete classification of CPD surfaces in the Euclidean 4-space  $\mathbb{E}^4$ . In Section 2, we introduce the notations that we will use and give a brief summary of basic definitions in the theory of submanifolds in Euclidean spaces. In Section 3, we obtain the complete classification of CPD surfaces in Euclidean 4-space.

## 2. Preliminaries

Let  $\mathbb{E}^m$  denote the Euclidean  $m$ -space with the canonical Euclidean metric tensor given by

$$\tilde{g} = \langle , \rangle = \sum_{i=1}^m dx_i^2,$$

where  $(x_1, x_2, \dots, x_m)$  is a rectangular coordinate system in  $\mathbb{E}^m$ .

Consider an  $n$ -dimensional Riemannian submanifold of the space  $\mathbb{E}^m$ . We denote Levi-Civita connections of  $\mathbb{E}^m$  and  $M$  by  $\tilde{\nabla}$  and  $\nabla$ , respectively. The Gauss and Weingarten formulas are respectively given by

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{1}$$

$$\tilde{\nabla}_X \xi = -S_\xi(X) + D_X \xi, \tag{2}$$

whenever  $X, Y$  are tangent and  $\xi$  is a normal vector field on  $M$ , where  $h$ ,  $D$ , and  $S$  are the second fundamental form, the normal connection, and the shape operator of  $M$ , respectively. It is well known that the shape operator and the second fundamental form are related by

$$\langle h(X, Y), \xi \rangle = \langle S_\xi X, Y \rangle.$$

The Gauss and Codazzi equations are respectively given by

$$\langle R(X, Y)Z, W \rangle = \langle h(Y, Z), h(X, W) \rangle - \langle h(X, Z), h(Y, W) \rangle, \tag{3}$$

$$\langle R^D(X, Y)\xi, \eta \rangle = \langle [S_\xi, S_\eta]X, Y \rangle, \tag{4}$$

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z), \tag{5}$$

whenever  $X, Y, Z, W$  are tangent to  $M$ , where  $R, R^D$  are the curvature tensors associated with connections  $\nabla$  and  $D$ , respectively. We note that  $\bar{\nabla}h$  is defined by

$$(\nabla_X h)(Y, Z) = D_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z).$$

A submanifold  $M$  is said to have a flat normal bundle if  $R^D = 0$  identically. On the other hand, a normal vector field  $\xi$  is said to be parallel on the normal bundle of  $M$ , or shortly parallel, if  $D\xi = 0$ . It is well known that having a flat normal bundle is equivalent to the existence of a parallel base field of the normal bundle (see, for example, [3]). Also, because of the Ricci equation (4), a submanifold of a Euclidean space satisfies  $R^D = 0$  if and only if all shape operators can be diagonalized simultaneously. Therefore, the following corollary immediately follows from Definition 1.1:

**Corollary 2.1** *A CPD surface  $M$  in a Euclidean space  $\mathbb{E}^m$  has a flat normal bundle. Consequently, there exists a parallel orthonormal base field  $\{e_3, e_4, \dots, e_m\}$  of the normal bundle of  $M$ .*

Now consider a local orthonormal frame field  $\{e_1, e_2, \dots, e_n; e_{n+1}, \dots, e_m\}$  on  $M$ . The mean curvature vector field  $H$  of  $M$  is defined by

$$H = \frac{1}{n} \sum_{\beta=n+1}^m \text{tr} S_\beta e_\beta, \tag{6}$$

where we put  $S_\beta = S_{e_\beta}$ .  $M$  is said to be minimal if  $H = 0$ .

Let  $M$  be a surface, i.e.  $n = 2$ . Then the Gaussian curvature  $K$  of  $M$  is defined by

$$K = R(e_1, e_2, e_2, e_1).$$

$M$  is said to be flat if its Gaussian curvature vanishes identically.

### 3. CPD surfaces in $\mathbb{E}^4$

In this section, we obtain the classification of CPD surfaces in  $\mathbb{E}^4$ .

Let  $M$  be a surface in  $\mathbb{E}^4$  with CPD relative to  $k$ . Up to isometries of  $\mathbb{E}^4$ , we may assume that  $k = (1, 0, 0, 0)$ . Then one can define a tangent vector field  $e_1$  and a normal vector field  $e_3$  with the equation

$$k = \cos \theta e_1 + \sin \theta e_3 \tag{7}$$

for a smooth function  $\theta$ . Let  $e_2$  and  $e_4$  be a unit tangent vector field and a unit normal vector field, satisfying  $\langle e_1, e_2 \rangle = 0$  and  $\langle e_3, e_4 \rangle = 0$ , respectively. By a simple computation considering (7) we obtain the following lemma, where we use the notation  $h_{ij}^\beta = \langle h(e_i, e_j), e_\beta \rangle = \langle S_\beta e_i, e_j \rangle$  and  $S_\beta = S_{e_\beta}$  with  $i, j = \{1, 2\}, \beta = \{3, 4\}$ :

**Lemma 3.1** *The Levi-Civita connection  $\nabla$  and normal connection  $D$  of  $M$  are given by*

$$\nabla_{e_1} e_1 = \nabla_{e_1} e_2 = 0, \tag{8a}$$

$$\nabla_{e_2} e_1 = \tan \theta h_{22}^3 e_2, \quad \nabla_{e_2} e_2 = -\tan \theta h_{22}^3 e_1, \tag{8b}$$

$$D_{e_1} e_3 = -\cot \theta h_{11}^4 e_4, \quad D_{e_2} e_3 = 0, \tag{8c}$$

and the matrix representations of shape operator  $S$  of  $M$  with respect to  $\{e_1, e_2\}$  are

$$S_3 = \begin{pmatrix} -e_1(\theta) & 0 \\ 0 & h_{22}^3 \end{pmatrix}, \quad S_4 = \begin{pmatrix} h_{11}^4 & 0 \\ 0 & h_{22}^4 \end{pmatrix} \tag{9}$$

for some smooth functions  $h_{11}^4, \theta, h_{22}^3$ , and  $h_{22}^4$  satisfying

$$e_1(h_{22}^3) = -\cot \theta h_{11}^4 h_{22}^4 - \tan \theta h_{22}^3 (e_1(\theta) + h_{22}^3), \tag{10a}$$

$$e_1(h_{22}^4) = \cot \theta h_{22}^3 h_{11}^4 + \tan \theta h_{22}^3 (h_{11}^4 - h_{22}^4), \tag{10b}$$

$$e_2(\theta) = 0, \tag{10c}$$

$$e_2(h_{11}^4) = 0. \tag{10d}$$

**Proof** By the assumption of being the CPD of  $M$ ,  $e_1, e_2$  are principal directions of all shape operators. Therefore,  $h_{12}^\beta = 0, \beta = 3, 4$ . Thus, we have the second equation of (9). On the other hand, by differentiating (7) along  $e_i$  we get

$$\begin{aligned} 0 = & -e_i(\theta) \sin \theta e_1 + \cos \theta \nabla_{e_i} e_1 + \delta_{1i} \cos \theta (h_{11}^3 e_3 + h_{11}^4 e_4) \\ & + e_i(\theta) \cos \theta e_3 - \sin \theta h_{ii}^3 e_i + \sin \theta D_{e_i} e_3 \end{aligned} \tag{11}$$

for  $i = 1, 2$ . Then we observe that (11) for  $i = 1$  gives  $\nabla_{e_1} e_1 = 0, h_{11}^3 = -e_1(\theta)$  and the first equation of (8c). Therefore, we have the first equation of (9) and (8a).

On the other hand, (11) for  $i = 2$  gives (10c), the second equation of (8c), and  $\nabla_{e_2} e_1 = \tan \theta h_{22}^3 e_2$ , which implies (8b).

By considering the Codazzi equation (5), we obtain (10a) and (10b), and (10d) follows from the Ricci equation.  $\square$

Now let  $p \in M$ . We construct a local coordinate system by the following lemma:

**Lemma 3.2** *There exists a local coordinate system  $(s, t)$  defined in a neighborhood  $\mathcal{N}_p$  of  $p$  such that the induced metric of  $M$  is*

$$g = ds^2 + m^2 dt^2 \tag{12}$$

for a smooth function  $m$  satisfying

$$e_1(m) - \tan \theta h_{22}^3 m = 0. \tag{13}$$

Furthermore, the vector fields  $e_1, e_2$  defined above become  $e_1 = \partial_s, e_2 = \frac{1}{m} \partial_t$  in  $\mathcal{N}_p$ .

**Proof** We have  $[e_1, e_2] = -\tan \theta h_{22}^3 e_2$  because of (8). Thus, if  $m$  is a nonvanishing smooth function on  $M$  satisfying (13), then we have  $[e_1, m e_2] = 0$ . Therefore, there exists a local coordinate system  $(s, t)$  such that  $e_1 = \partial_s$  and  $e_2 = \frac{1}{m} \partial_t$ . Thus, the induced metric of  $M$  is as given in (12).  $\square$

Now we are ready to obtain the classification theorem.

**Theorem 3.3** *Let  $M$  be a surface in  $\mathbb{E}^4$  with a CPD relative to a fixed direction  $k \in \mathbb{R}^4$ . Assume that the unit normal vector field along  $k$  is parallel on the normal bundle. Then  $M$  is locally congruent to one of the following surfaces:*

(i) A CPD surface lying on an hyperplane  $\mathbb{E}^3$  of  $\mathbb{E}^4$ .

(ii) A surface given by

$$x(s, t) = \alpha_1(s)(1, 0, 0, 0) + \alpha_2(s)\phi(t) + \int_{t_0}^t \psi(\tau)\phi'(\tau)d\tau, \tag{14}$$

where  $(\alpha_1, \alpha_2)$  is an arc-length parametrized curve in  $\mathbb{E}^2$ ,  $\phi(t) = (0, \phi_2(t), \phi_3(t), \phi_4(t))$  is an arc-length parametrized curve lying on  $\mathbb{S}^2(1)$ , and  $\psi$  is a smooth function.

(iii) A flat translation surface given by

$$x(s, t) = \nu(s) + \omega(t), \tag{15a}$$

$$\nu(s) = (\nu_1(s), c_1\nu_2(s), c_2\nu_2(s), c_3\nu_2(s)), \tag{15b}$$

where  $c_1, c_2, c_3$  are some constants such that  $c_1^2 + c_2^2 + c_3^2 = 1$  and  $\omega$  is an arc-length parametrized curve satisfying

$$\langle k, \omega \rangle = \langle \nu', \omega' \rangle = 0. \tag{15c}$$

(iv) A flat ruled surface given by

$$x(s, t) = s\Phi_1(t) + \Phi_2(t) \tag{16a}$$

with the director curve

$$\Phi_1(t) = (\cos \theta_0, \sin \theta_0\Phi_{12}(t), \sin \theta_0\Phi_{13}(t), \sin \theta_0\Phi_{14}(t)) \tag{16b}$$

and directrix

$$\begin{aligned} \Phi_2(t) = & \left( 0, \int_{t_0}^t \Psi(\tau)\Phi_{12}'(\tau)d\tau, \int_{t_0}^t \Psi(\tau)\Phi_{13}'(\tau)d\tau, \right. \\ & \left. \int_{t_0}^t \Psi(\tau)\Phi_{14}'(\tau)d\tau \right), \end{aligned} \tag{16c}$$

where  $\theta_0$  is a nonzero constant,  $\Phi(t) = (0, \Phi_{12}(t), \Phi_{13}(t), \Phi_{14}(t))$  is an arc-length parametrized curve lying on  $\mathbb{S}^2(1)$ , and  $\Psi$  is a smooth function.

Conversely, the surfaces described above are CPD relative to  $k = (1, 0, 0, 0)$ .

**Remark 3.4** Because of Corollary 2.1, we know the existence of parallel normal vector fields on a CPD surface in  $\mathbb{E}^4$ .

**Remark 3.5** For the case (i) of Theorem 3.3, see [19], where the classification of CPD surfaces in  $\mathbb{E}^3$  was obtained.

**Proof** First, assume that  $M$  is a surface endowed with a CPD relative to  $k = (1, 0, 0, 0)$  and consider the case  $De_3 = 0$ . Then (8c) implies  $h_{11}^4 = 0$ . Note that if  $M$  lies on a hyperplane of  $\mathbb{E}^4$ , then we have case (i) of the theorem. Therefore, we assume that  $M$  is not contained by a hyperplane of  $\mathbb{E}^4$ . Note that if  $Se_3 = 0$ , then we have

$$\tilde{\nabla}_X e_3 = -S_{e_3}X + D_X e_3 = 0,$$

whenever  $X$  is tangent to  $M$ . Therefore,  $e_3$  is a constant normal vector field. However, this case contradicts our assumption. Thus, we consider the case where  $S_{e_3} \neq 0$ . Similarly, we assume  $h_{22}^4 \neq 0$ .

By these assumptions, we see that (9), (10a), and (10b) turn into

$$S_3 = \begin{pmatrix} -e_1(\theta) & 0 \\ 0 & h_{22}^3 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & 0 \\ 0 & h_{22}^4 \end{pmatrix} \tag{17}$$

and

$$e_1(h_{22}^3) = -\tan \theta h_{22}^3(e_1(\theta) + h_{22}^3), \tag{18}$$

$$e_1(h_{22}^4) = -\tan \theta h_{22}^3 h_{22}^4. \tag{19}$$

Let  $x : M \rightarrow \mathbb{E}^4$  be the position vector of  $M$ . Consider the local orthonormal frame field  $\{e_1, e_2; e_3, e_4\}$  described at the beginning of this section and let  $(s, t)$  be a local coordinate system given in Lemma 3.2. Note that (18), (19), and (13) become, respectively,

$$(h_{22}^3)_s = -\tan \theta h_{22}^3(\theta' + h_{22}^3), \tag{20a}$$

$$(h_{22}^4)_s = -\tan \theta h_{22}^3 h_{22}^4, \tag{20b}$$

$$m_s - \tan \theta h_{22}^3 m = 0. \tag{20c}$$

Moreover, we have

$$e_1 = x_s. \tag{21}$$

Because of (17),  $e_1(\theta) = 0$  is equivalent to  $h_{11}^3 = 0$ . We are going to consider this particular case separately.

**Case A.** Assume that  $e_1(\theta) \neq 0$  at  $p$ . If necessary, shrink  $\mathcal{N}_p$  so that  $e_1(\theta)|_{\mathcal{N}_p}$  does not vanish.

By combining (20c) and (20b) with (17) we obtain the shape operators of  $M$  as

$$S_3 = \begin{pmatrix} -\theta' & 0 \\ 0 & \cot \theta \frac{m_s}{m} \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{m} \end{pmatrix}, \tag{22}$$

where  $'$  denotes ordinary differentiation with respect to the appropriated variable.

By combining (20c) and (20a) we obtain

$$m_{ss} - \theta' \cot \theta m_s = 0,$$

whose general solution is

$$m(s, t) = \psi_1(t) \int_{s_0}^s \sin \theta(\tau) d\tau + \psi_2(t)$$

for some smooth functions  $\psi_1, \psi_2$ . Therefore, by redefining  $t$  properly, we may assume either

$$m(s, t) = \int_{s_0}^s \sin \theta(\tau) d\tau + \psi(t) \tag{23a}$$

for a smooth function  $\psi$  or

$$m(s, t) = m(t). \tag{23b}$$

**Case A1.** Let  $m$  satisfy (23a). In this case, by considering equation (8) with (21), we get that the Levi-Civita connection of  $M$  satisfies

$$\nabla_{\partial_s} \partial_s = 0, \quad \nabla_{\partial_s} \partial_t = \nabla_{\partial_t} \partial_s = \frac{m_s}{m} \partial_t, \quad \nabla_{\partial_t} \partial_t = -mm_s \partial_s + \frac{m_t}{m} \partial_t. \tag{24}$$

By combining the first equation of (24) with (22) and using Gauss formula (1), we have

$$x_{ss} = -\theta' e_3. \tag{25}$$

On the other hand, we have  $\langle x_s, k \rangle = \cos \theta$  and  $\langle x_t, k \rangle = 0$  from the decomposition (7). By considering these equations, we see that  $x$  has the form of

$$x(s, t) = \left( \int_{s_0}^s \cos \theta(\tau) d\tau, x_2(s, t), x_3(s, t), x_4(s, t) \right) + \gamma(t) \tag{26}$$

for an  $\mathbb{E}^4$ -valued smooth function  $\gamma = (0, \gamma_2, \gamma_3, \gamma_4)$ . On the other hand, by considering (21) and (25) in (7), we obtain

$$(1, 0, 0, 0) = \cos \theta x_s - \frac{\sin \theta}{\theta'} x_{ss}. \tag{27}$$

Next, we combine (27) and (26) to get

$$\theta' \cos \theta \frac{\partial x_j(s, t)}{\partial s} - \sin \theta \frac{\partial^2 x_j(s, t)}{\partial s^2} = 0, \quad j = 2, 3, 4.$$

By solving this equation and using (26), we obtain

$$x(s, t) = \alpha_1(s)(1, 0, 0, 0) + \alpha_2(s)\phi(t) + \gamma(t) \tag{28}$$

for a smooth curve  $\phi(t) = (0, \phi_2(t), \phi_3(t), \phi_4(t))$ , where we put  $\alpha_1(s) = \int_{s_0}^s \cos \theta(\tau) d\tau$  and  $\alpha_2(s) = \int_{s_0}^s \sin \theta(\tau) d\tau$ .

Therefore, the curve  $(\alpha_1, \alpha_2)$  is an arc-length parametrized curve in  $\mathbb{E}^2$  with the curvature  $\theta'$ . Also, (23a) becomes

$$m(s, t) = \alpha_2(s) + \psi(t). \tag{29}$$

Note that (28) also implies  $\langle \phi, \phi \rangle = 1$ , i.e.  $\phi$  lies on  $\mathbb{S}^3(1)$ , because  $\langle x_s, x_s \rangle = 1$ .

On the other hand, since  $h(\partial_s, \partial_t) = 0$ , the second equation of (24) implies

$$x_{st} = \frac{m_s}{m} x_t.$$

By combining this equation with (28) and (23a), we obtain  $\gamma' = \psi\phi'$ . Therefore, (28) turns into (14). Furthermore, (29) and  $\langle x_t, x_t \rangle = m^2$  imply that  $\langle \phi', \phi' \rangle = 1$ . Hence, we have case (ii) of the theorem.

**Case A2.** Let  $m$  satisfy (23b). Here, we can take  $m(t) = 1$  by redefining  $t$  properly. In this case, the induced metric given in (12) of  $M$  becomes  $g = ds^2 + dt^2$  and the Levi-Civita connection of  $M$  satisfies

$$\nabla_{\partial_s} \partial_s = 0, \quad \nabla_{\partial_s} \partial_t = 0, \quad \nabla_{\partial_t} \partial_t = 0. \tag{30}$$

On the other hand, because  $m = 1$ , (20b) and (20c) give  $(h_{22}^4)_s = 0$  and  $h_{22}^3 = 0$ . Thus, (17) becomes

$$S_3 = \begin{pmatrix} -\theta' & 0 \\ 0 & 0 \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & 0 \\ 0 & a(t) \end{pmatrix} \tag{31}$$

for a function  $a$ . Therefore,  $x$  and the normal vectors  $e_3, e_4$  satisfy

$$\begin{aligned} x_{ss} &= -\theta' e_3, & x_{st} &= 0, & x_{tt} &= a(t)e_4. \\ (e_3)_s &= -\theta' x_s, & (e_3)_t &= 0, \\ (e_4)_s &= 0, & (e_4)_t &= -a(t)x_t. \end{aligned} \tag{32a}$$

(32a) implies (15a) for some smooth curves  $\nu, \omega$ . By a straightforward computation, we obtain (15b) for some constants  $c_1, c_2, c_3$ , where we put  $\nu_1(s) = \int_{s_0}^s \cos \theta(\tau) d\tau$  and  $\nu_2(s) = \int_{s_0}^s \sin \theta(\tau) d\tau$ . Thus,  $(\nu_1, \nu_2)$  is an arc-length parametrized curve in  $\mathbb{E}^2$ . Furthermore, by considering  $\langle k, x_t \rangle = 0$  and  $g = ds^2 + dt^2$ , we obtain (15c). Hence, we have case (iii) of the theorem.

**Case B.**  $e_1(\theta) = 0$  on  $\mathcal{N}_p$ . Then we have  $\theta = \theta_0$  for a nonzero constant  $\theta_0$ . In this case, by combining (20c) with (17) and considering  $h_{11}^3 = 0$ , we obtain the shape operators of  $M$  as

$$S_3 = \begin{pmatrix} 0 & 0 \\ 0 & \cot \theta_0 \frac{m_s}{m} \end{pmatrix}, \quad S_4 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{m} \end{pmatrix}. \tag{33}$$

Thus, the Gauss equation yields that  $M$  is flat.

By combining (20c) and (20a) we get

$$m(s, t) = \psi_1(t) \left( \tan \theta_0 s + \psi_2(t) \right)$$

for some smooth functions  $\psi_1, \psi_2$ . Therefore, by redefining  $t$  properly, we may assume either

$$m(s, t) = s \sin \theta_0 + \Psi(t) \tag{34a}$$

for a smooth function  $\Psi$  or

$$m(s, t) = m(t). \tag{34b}$$

Because of (33), we see that the case (34b) gives  $S_3 = 0$ , which gives case (i) of the theorem. Therefore, we assume that  $m$  satisfies (34a). Then, by considering equation (8) with (21), we get that the Levi-Civita connection of  $M$  satisfies (24). By combining the first equations in (24) with (33) and using Gauss formula (1), we obtain

$$x_{ss} = 0,$$

which yields that  $M$  is a ruled surface given by (16a) for some  $\Phi_1, \Phi_2$ .

On the other hand, from the decomposition (7), we have  $\langle x_s, k \rangle = \cos \theta_0$ . By considering this equation and (16a), we obtain (16b) for some functions  $\Phi_{12}(t), \Phi_{13}(t), \Phi_{14}(t)$ . Since  $\langle x_s, x_s \rangle = 1$ , the curve  $(\Phi_{12}(t), \Phi_{13}(t), \Phi_{14}(t))$  lies on  $\mathbb{S}^2$ .

Next, we consider the equation

$$x_{st} = \frac{m_s}{m} x_t, \tag{35}$$



which follows from the second equation in (24) and  $h(\partial_s, \partial_t) = 0$ . By combining this equation and (16a), we obtain  $\Phi'_2 = \Psi\Phi'_1$ , which implies (16c) because  $\langle x_t, k \rangle = 0$ . Finally, by using  $\langle x_t, x_t \rangle = m^2$ , we observe that the curve  $(\Phi_{12}(t), \Phi_{13}(t), \Phi_{14}(t))$  is parametrized by its arc-length. Thus, we have case (iv) of the theorem. Hence, the proof for the necessary condition is obtained.

The proof of the converse follows from a direct computation. □

We would like to state the following immediate results of Theorem 3.3.

**Corollary 3.6** *Let  $M$  be a surface in  $\mathbb{E}^4$  with a CPD relative to a fixed direction  $k \in \mathbb{R}^4$ . Assume that the unit normal vector field along  $k$  is parallel on the normal bundle. If  $M$  is flat, then it is congruent to one of the surfaces given in case (i), case (iii), or case (iv) of Theorem 3.3.*

**Proof** By considering the proof of Theorem 3.3, one can see that the surface given by case (ii) of Theorem 3.3 is not flat. □

**Corollary 3.7** *Let  $M$  be a surface in  $\mathbb{E}^4$  with a CPD relative to a fixed direction  $k \in \mathbb{R}^4$ . Assume that the unit normal vector field along  $k$  is parallel on the normal bundle. If  $M$  is minimal, then it lies on a hyperplane of  $\mathbb{E}^4$ .*

**Proof** Let  $M$  be a minimal surface endowed with a CPD relative to a fixed direction  $k$  and assume that the vector field  $e_3$  defined by (7) is parallel. Then (6) and (17) imply that

$$H = \frac{-e_1(\theta) + h_{22}^3}{2}e_3 + \frac{h_{22}^4}{2}e_4 = 0,$$

which gives  $S_4 = 0$ . Since the codimension of  $M$  is 2, we also have  $De_4 = 0$ . Therefore, we have

$$\tilde{\nabla}e_4 = 0,$$

which implies that  $e_4$  is a constant, normal vector. Put  $e_4 = C_0 \in \mathbb{S}^3(1)$ . Hence,  $M$  lies on a hyperplane of  $\mathbb{E}^4$  whose normal is  $C_0$ . □

### 3.1. Explicit examples

In this subsection we would like to present some explicit examples for the nontrivial cases of Theorem 3.3.

**Example 3.8** *Consider the constant curvature curve*

$$\phi(t) = \left( 0, \frac{\cos(\sqrt{2}t)}{\sqrt{2}}, \frac{\sin(\sqrt{2}t)}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

of  $\mathbb{S}^2(1)$  and let  $\psi(t) = t$ . Then the surface given by (14) turns into

$$\begin{aligned} x(s, t) = & \left( \alpha_1(s), \frac{\cos(\sqrt{2}t)(\alpha_2(s) + t)}{\sqrt{2}} - \frac{1}{2}\sin(\sqrt{2}t), \right. \\ & \left. \frac{1}{2}(\sqrt{2}\sin(\sqrt{2}t)(\alpha_2(s) + t) + \cos(\sqrt{2}t) - 1), \frac{\alpha_2(s)}{\sqrt{2}} \right) \end{aligned} \tag{36}$$

for some smooth functions  $\alpha_1, \alpha_2$  such that  $\alpha_1'^2 + \alpha_2'^2 = 1$ . The induced metric of this surface is

$$g = ds^2 + (\alpha_2(s) + t)^2 dt^2$$

and we have  $\langle k, x_s \rangle = 0$ , which yields that  $e_1 = \partial_s$  is the unit vector along  $k^T$ , where  $k = (1, 0, 0, 0)$ . A direct computation yields that  $h(\partial_s, \partial_t) = 0$ , which yields that  $\partial_s$  is a principal direction of all shape operators of  $M$  in  $\mathbb{E}^4$ . Therefore, the surface given by (36) is endowed with a CPD relative to  $k$ .

Next, we get the following examples of flat surfaces in  $\mathbb{E}^4$  endowed with a CPD relative to  $k = (1, 0, 0, 0)$ .

**Example 3.9** If we put  $\omega(t) = \left(0, \frac{t}{\sqrt{2}}, -\frac{t}{\sqrt{2}}, 0\right)$  and  $\nu(s) = \left(f(s), \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}, \frac{s}{\sqrt{3}}\right)$  in (15a), then we get the surface

$$x(s, t) = \left(f(s), \frac{s}{\sqrt{3}} + \frac{t}{\sqrt{2}}, \frac{s}{\sqrt{3}} - \frac{t}{\sqrt{2}}, \frac{s}{\sqrt{3}}\right). \tag{37}$$

A direct computation yields that the surface given by (37) is flat and  $g(\partial_s, \partial_t) = \tilde{g}(k, \partial_t) = 0$  and  $h(\partial_s, \partial_t) = 0$ . Therefore, this surface is endowed with a CPD relative to  $k$ .

**Example 3.10** We consider the constant curvature curve

$$\Phi(t) = \left(0, \frac{\cos(\sqrt{2}t)}{\sqrt{2}}, \frac{\sin(\sqrt{2}t)}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

of  $\mathbb{S}^2(1)$  and put  $\theta_0 = \frac{\pi}{4}$ ,  $\Psi(t) = t$ . In this case, the surface given by (16a) turns into

$$x(s, t) = \left(\frac{s}{\sqrt{2}}, \frac{1}{2} \left( (s + \sqrt{2}t) \cos(\sqrt{2}t) - \sin(\sqrt{2}t) \right), \right. \\ \left. \frac{1}{2} \left( (s + \sqrt{2}t) \sin(\sqrt{2}t) + \cos(\sqrt{2}t) - 1 \right), \frac{s}{2} \right). \tag{38}$$

Similar to Example 3.9, we see that this surface is flat and endowed with a CPD relative to  $k$ .

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