


## On uniformly $pr$ -ideals in commutative rings

Rabia Nagehan UREGEN\*

<sup>1</sup>Department of Mathematics and Science Education, Faculty of Education, Erzincan Binali Yıldırım University, Erzincan, Turkey

Received: 22.02.2019

Accepted/Published Online: 14.05.2019

Final Version: 31.07.2019

**Abstract:** Let  $R$  be a commutative ring with nonzero identity and  $I$  a proper ideal of  $R$ . Then  $I$  is called a uniformly  $pr$ -ideal if there exists  $N \in \mathbb{N}$  such that  $ab \in I$  with  $ann(a) = 0$  then  $b^N \in I$ . We say that the smallest  $N \in \mathbb{N}$  is called order of  $I$  and denoted by  $ord_R(I) = N$ . In this paper, we give some examples and characterizations of this new class of ideals.

**Key words:**  $r$ -ideal,  $pr$ -ideal, uniformly  $pr$ -ideal

### 1. Introduction

In this article all rings will be assumed to be commutative with a unity and all modules are nonzero unital. Let  $R$  always be such a ring and  $M$  be such an  $R$ -module. The aim of this article is to introduce uniformly  $pr$ -ideals of commutative rings and to give relations with some classical ideals such as uniformly primary ideal, strongly primary ideal,  $r$ -ideal. In [12], Mohammadian introduces  $r$ -ideals of commutative rings which is the generalization of pure ideals. Recall that a proper ideal  $I$  of  $R$  is called an  $r$ -ideal if whenever  $ab \in I$  and  $ann(a) = 0$  then  $b \in I$  ( $b^n \in I$ , for some  $n \in \mathbb{N}$ ). In terms of  $r$ -ideals the author characterizes quasiregular rings, rings satisfying property A (See [12], Theorem 4.2) and ([12], Proposition 3.5).

In commutative algebra, prime ideal and its generalizations have an important role. There have been lots of studies on this issue (See [2, 15]). Recall that a proper ideal  $Q$  is called a primary ideal if  $ab \in Q$  implies either  $a \in Q$  or  $b^n \in Q$ . In [4], Cox also studies a special class of primary ideals fixing the power of an element  $b \in R$  in the above definition. A proper ideal  $Q$  is called uniformly primary ideal if there exists  $N \in \mathbb{N}$  and whenever  $ab \in Q$  then either  $a \in Q$  or  $b^N \in Q$  in that case  $N$  is called order of  $Q$  and denoted by  $ord(Q) = N$ . Also  $Q$  is called a strongly primary ideal if  $Q$  is a primary ideal and  $\sqrt{Q}^N \subseteq Q$  for some  $N \in \mathbb{N}$  where  $\sqrt{Q}$  is the radical of  $Q$ . In this case  $N$  is called the exponent of  $Q$  and denoted by  $\exp(Q) = N$ . Note that the classes of primary ideals contain uniformly primary ideals and also the classes of uniformly primary ideals contain strongly primary ideals. With these motivations in this paper uniformly  $pr$ -ideals and strongly  $pr$ -ideals are investigated. For the completeness of the article, we begin with some definitions and notations which will be followed through the study. The set of maximal ideals, prime ideals and minimal prime ideals are denoted by  $MaxR$ ,  $SpecR$ , and  $MinR$ , respectively. Also the set of all zero divisors of  $R$  is denoted by  $zd(R)$ . A commutative ring  $R$  is called von Neumann regular if for each  $a \in R$ , there exists  $x \in R$  such that

\*Correspondence: [rabia.uregen@erzincan.edu.tr](mailto:rabia.uregen@erzincan.edu.tr)

2010 AMS Mathematics Subject Classification: 13A15, 13C99

$a = a^2x$ . Recently there have been many studies on von Neumann regular rings. See for example, [8, 11, 16].

Recall that  $R$  is said to satisfy (\*) condition if for all family of ideals  $\{I_i\}_{i \in \Delta}$ ,  $\sqrt{\bigcap_{i \in \Delta} I_i} = \bigcap_{i \in \Delta} \sqrt{I_i}$ .  $R$  satisfies

(\*) property if and only if  $R$  is a  $\pi$ -regular ring, i.e, Krull dimension of  $R$  is zero. Note that a ring  $R$  is von Neumann Ring if and only if  $R$  is reduced  $\pi$ -regular ring. A proper ideal  $I$  of  $R$  is called a uniformly  $pr$ -ideal if there exists  $N \in \mathbb{N}$  such that whenever  $ab \in I$  with  $ann(a) = 0$ , then  $b^N \in I$ . An ideal  $I$  is called a strongly  $pr$ -ideal if  $I$  is a  $pr$ -ideal and  $\sqrt{I}^N \subseteq I$  for some  $N \in \mathbb{N}$ . Among other results in this paper it is shown that the classes of  $pr$ -ideals contain uniformly  $pr$ -ideals and also uniformly  $pr$ -ideals contain strongly  $pr$ -ideals (See Corollary 2.6). Further it is proved that  $pr$ -ideals, uniformly  $pr$ -ideals, and strongly  $pr$ -ideals are equal in any Noetherian ring. Moreover, in Corollary 2.9 it is shown that any power of minimal prime ideals are strongly  $pr$ -ideals. When a primary ideal becomes a  $pr$ -ideal and a uniformly primary ideal becomes a uniformly  $pr$ -ideal are demonstrated in the study (See Propositions 2.13 and 2.14). Also the behaviour of uniformly  $pr$ -ideals in factor rings, in direct product of rings, and in idealization of a module are investigated (See Proposition 2.15, Theorem 2.16, and Proposition 2.20). Finally, uniformly  $pr$ -ideals in polynomial rings and formal power series rings are examined (See Theorem 2.24, Proposition 2.25, and Theorem 2.27).

## 2. Characterization of uniformly $pr$ -ideals

**Definition 2.1** An ideal  $I$  of  $R$  is called a uniformly  $pr$ -ideal if there exists  $N \in \mathbb{N}$  such that whenever  $ab \in I$  with  $ann(a) = 0$ , then  $b^N \in I$ . The smallest  $N \in \mathbb{N}$  is called order of  $I$  and denoted by  $ord_R(I) = N$ .

It is easily obtained by the definition that every uniformly  $pr$ -ideal is a  $pr$ -ideal.

**Example 2.2** Let  $R$  be a finite ring and  $I$  be a proper ideal of  $R$ . Assume that  $ab \in I$  with  $ann(a) = 0$  for some  $a, b \in R$ . As  $R$  is a finite ring, then the set of all units in  $R$  and all regular elements in  $R$  are equal so that  $a$  has an inverse in  $R$ . Thus, we conclude that  $a^{-1}(ab) = b \in I$ . Hence,  $I$  is a uniformly  $pr$ -ideal with  $ord_R(I) = 1$ .

**Example 2.3** Consider the ring of integers  $\mathbb{Z}_n$  of modulo  $n$ , where  $n > 1$  is an integer. Then by Example 2.2, every proper ideal of  $\mathbb{Z}_n$  is a uniformly  $pr$ -ideal of order 1.

**Definition 2.4** An ideal  $I$  is called a strongly  $pr$ -ideal if  $I$  is a  $pr$ -ideal and  $\sqrt{I}^N \subseteq I$  for some  $N \in \mathbb{N}$ . The smallest  $N \in \mathbb{N}$  which has the aforementioned property is called the exponent of  $I$  and denoted by  $e_R(I)$ .

**Proposition 2.5**  $J$  is a  $pr$ -ideal if and only if  $\sqrt{J}$  is an  $r$ -ideal.

**Proof** It follows from [12, Proposition 2.6]. □

**Corollary 2.6** Every strongly  $pr$ -ideal is also a uniformly  $pr$ -ideal.

**Proof** Let  $I$  be a strongly  $pr$ -ideal. Thus,  $I$  is a  $pr$ -ideal. Since  $I$  is a strongly  $pr$ -ideal, there exists  $N \in \mathbb{N}$  such that  $\sqrt{I}^N \subseteq I$ . Let  $ab \in I$  with  $ann(a) = 0$  for some  $a, b \in R$ . Since  $I$  is a  $pr$ -ideal, then  $b \in \sqrt{I}$ . As  $\sqrt{I}^N \subseteq I$ , we have  $b^N \in \sqrt{I}^N \subseteq I$ . Hence,  $I$  is a uniformly  $pr$ -ideal. Furthermore,  $ord_R(I) \leq N$  and so  $ord_R(I) \leq e_R(I)$ . □

By Proposition 2.5, we have the following explicit result.

**Corollary 2.7** *Let  $I$  be an ideal with  $\sqrt{I}$  that is finitely generated. Then  $I$  is a  $pr$ -ideal if and only if  $I$  is a uniformly  $pr$ -ideal if and only if  $I$  is a strongly  $pr$ -ideal if and only if  $\sqrt{I}$  is an  $r$ -ideal.*

**Corollary 2.8** *Suppose that  $R$  is a Noetherian ring. Let  $I$  be a proper ideal of  $R$ . Then the followings are equivalent.*

- (i)  $I$  is a  $pr$ -ideal.
- (ii)  $\sqrt{I}$  is an  $r$ -ideal.
- (iii)  $I$  is a uniformly  $pr$ -ideal
- (iv)  $I$  is a strongly  $pr$ -ideal.

**Corollary 2.9** *For every minimal prime ideal  $P$  of  $R$ ,  $P^n$  is a strongly  $pr$ -ideal for every  $n \in \mathbb{N}$ .*

**Proof** Let  $P \in \text{Min}(R)$ , where  $\text{Min}(R)$  denotes the set of minimal prime ideals. By [7, Corollary 1.2],  $P \subseteq \text{zd}(R)$ . Thus, by [12, Remark 2.3 (f)],  $P$  is an  $r$ -ideal and so by Proposition 2.5,  $P^n$  is a strongly  $pr$ -ideal of  $R$ . □

**Corollary 2.10** *Suppose that  $I$  is a uniformly  $pr$ -ideal with  $S \subseteq R$  is a nonempty subset and  $S \not\subseteq I$ . If  $S$  contains a regular element, then  $(I : S)$  is a uniformly  $pr$ -ideal.*

**Proof** Let  $ab \in (I : S)$  with  $\text{ann}(a) = 0$ . Then  $abs \in I$  for every  $s \in S$ . Assume that  $s_0 \in S$  is a regular element. Then  $as_0$  is a regular element. Since  $abs_0 \in I$  and  $I$  is a uniformly  $pr$ -ideal, then  $b^N \in I$  for some  $N \in \mathbb{N}$ . Thus,  $b^N \in I \subseteq (I : S)$ . □

**Lemma 2.11** (i) *Assume that  $I_i$ 's are uniformly  $pr$ -ideals for each  $i \in \Delta$  with  $\text{ord}(I_i) = N_i$ . Let  $\sup\{N_i : i \in \Delta\} < \infty$ . Then  $I = \bigcap_{i \in \Delta} I_i$  is a uniformly  $pr$ -ideal with  $\text{ord}(I) \leq \sup\{N_i : i \in \Delta\}$ .*

(ii) *Assume that  $I_i$ 's are strongly  $pr$ -ideals for each  $i = 1, 2, \dots, n$  with  $\text{exp}(I_i) = N_i$ . Then  $I = \bigcap_{i=1}^n I_i$  is a strongly  $pr$ -ideal with  $\text{exp}(I) \leq \max\{N_1, N_2, \dots, N_n\}$ .*

**Proof** (i) Let  $N = \sup\{N_i : i \in \Delta\}$ . Then for every  $i \in \Delta$ ,  $N_i \leq N$ . Let  $ab \in I$  with  $\text{ann}(a) = 0$  for some  $a, b \in R$ . This implies that  $ab \in I_i$  for each  $i \in \Delta$ . Since  $I_i$ 's are uniformly  $pr$ -ideals, we conclude that  $b^{N_i} \in I_i$  and so  $b^N \in I_i$ . Then we have  $b^N \in \bigcap_{i \in \Delta} I_i = I$ . The rest can be easily seen.

(ii) By (i), it is clear that  $I = \bigcap_{i=1}^n I_i$  is a uniformly  $pr$ -ideal. Let  $\max\{N_1, N_2, \dots, N_n\} = N$ . For every  $i = 1, 2, \dots, n$ ,  $N_i \leq N$ . Then note that  $\sqrt{I} = \sqrt{\bigcap_{i=1}^n I_i} = \bigcap_{i=1}^n \sqrt{I_i}$ . Since  $I_i$ 's are strongly  $pr$ -ideals,  $\sqrt{I_i}^{N_i} \subseteq I_i$  and so  $\sqrt{I_i}^N \subseteq I_i$ . As  $\sqrt{I} \subseteq \sqrt{I_i}$ , we get  $\sqrt{I}^N \subseteq \sqrt{I_i}^N \subseteq I_i$  and so  $\sqrt{I}^N \subseteq \bigcap_{i=1}^n I_i = I$ . Hence  $I$  is a strongly  $pr$ -ideal. □

Recall that  $R$  is said to satisfy the  $(*)$  condition if for all family of ideals  $\{I_i\}_{i \in \Delta}$ ,

$$\sqrt{\bigcap_{i \in \Delta} I_i} = \bigcap_{i \in \Delta} \sqrt{I_i}.$$

$R$  satisfies the  $(*)$  property if and only if  $R$  is a  $\pi$ -regular ring, i.e, Krull dimension of  $R$  is zero.

The following corollary generalizes (ii) in the previous proposition for  $\pi$ -regular rings.

**Corollary 2.12** *Let  $R$  be a  $\pi$ -regular ring. Assume that  $I_i$ 's are strongly  $pr$ -ideals for each  $i \in \Delta$  with  $\exp(I_i) = N_i$ . Let  $\sup\{N_i : i \in \Delta\} < \infty$ . Then  $I = \bigcap_{i \in \Delta} I_i$  is a strongly  $pr$ -ideal with  $\exp(I) \leq \sup\{N_i : i \in \Delta\}$ .*

**Proof** By the previous proposition (i),  $I = \bigcap_{i=1}^n I_i$  is a uniformly  $pr$ -ideal. From the  $(*)$  property, radical commutes with intersection. The rest can be proved similar to the previous proposition (ii).  $\square$

**Proposition 2.13** *Let  $Q$  be a primary ideal of  $R$ . Then  $Q$  is a  $pr$ -ideal if and only if  $Q \subseteq zd(R)$ .*

**Proof** Suppose that  $Q$  is a primary ideal with  $Q \subseteq zd(R)$ . Let  $ab \in Q$  and  $ann(a) = 0$  for some  $a, b \in R$ . It is easy to see that  $a \notin Q$  and so  $b^n \in Q$  for some  $n \in \mathbb{N}$ . Thus,  $Q$  is a  $pr$ -ideal. Conversely, assume that  $Q$  is  $pr$ -ideal. Then  $\sqrt{Q}$  is an  $r$ -ideal and so by [12, Remark 2.3 (d)], we have  $Q \subseteq \sqrt{Q} \subseteq zd(R)$ .  $\square$

Recall that a proper ideal  $Q$  of  $R$  is said to be a uniformly primary ideal if there exists  $N \in \mathbb{N}$  whenever  $ab \in Q$  with  $a \notin Q$  then  $b^N \in Q$ . A proper ideal  $Q$  is a strongly primary ideal if  $Q$  is a primary ideal and  $\sqrt{Q}^N \subseteq Q$  for some  $N \in \mathbb{N}$  [4].

**Proposition 2.14** (i) *Let  $Q$  be a uniformly primary ideal. Then  $Q$  is a uniformly  $pr$ -ideal if and only if  $Q \subseteq zd(R)$ .*

(ii) *Let  $Q$  be a strongly primary ideal. Then  $Q$  is a strongly  $pr$ -ideal if and only if  $Q \subseteq zd(R)$ .*

**Proof** (i) Let  $Q$  be a uniformly primary ideal. Assume that  $Q$  is a uniformly  $pr$ -ideal. Then  $Q$  is a  $pr$ -ideal. Also note that uniformly primary ideals are primary ideals so  $Q \subseteq zd(R)$  by the previous proposition. Conversely, assume that  $Q \subseteq zd(R)$  and  $ab \in Q$  with  $ann(a) = 0$ . Then  $a \notin zd(R)$  and so  $a \notin Q$ . Assume that  $ord(Q) = N$ . Since  $Q$  is a uniformly primary ideal, we have  $b^N \in Q$  and so  $Q$  is a uniformly  $pr$ -ideal.

(ii) It is similar to (i).  $\square$

**Proposition 2.15** *Let  $I$  be an  $r$ -ideal and  $I \subseteq J$  for some ideal  $J$  of  $R$ . If  $J/I$  is a uniformly  $pr$ -ideal then  $J$  is a uniformly  $pr$ -ideal.*

**Proof** Suppose that  $ab \in J$  with  $ann(a) = 0$  for some  $a, b \in R$ . If  $ab \in I$ , then  $b \in I$  since  $I$  is an  $r$ -ideal and so  $b \in J$ . Now assume that  $ab \notin I$ . We will show that  $ann(a + I) = 0_{R/I}$ . Let  $(x + I)(a + I) = 0_{R/I}$  for some  $x \in R$ . Then  $xa \in I$ . Since  $ann(a) = 0$  and  $I$  is an  $r$ -ideal, we have  $x \in I$  and so  $x + I = 0_{R/I}$ . Thus we have  $ann(a + I) = 0_{R/I}$ . As  $J/I$  is a uniformly  $pr$ -ideal and  $(a + I)(b + I) = ab + I \in J/I$ , there exists  $N \in \mathbb{N}$  such that  $(b + I)^N = b^N + I \in J/I$ . This implies  $b^N \in J$  and so  $J$  is a uniformly  $pr$ -ideal.  $\square$

Let  $R_1, R_2$  be two commutative rings and  $R = R_1 \times R_2$  be a direct product of these rings. It is well known that all ideals of  $R$  have the form  $I = I_1 \times I_2$ , where  $I_1, I_2$  are ideals of  $R_1$  and  $R_2$ , respectively.

**Theorem 2.16** Let  $R_1, R_2$  be two commutative rings and  $R = R_1 \times R_2$ . Suppose that  $I = I_1 \times I_2$ , where  $I_1, I_2$  are ideals of  $R_1$  and  $R_2$ , respectively. Then the followings are equivalent:

- (i)  $I$  is a uniformly  $pr$ -ideal.
- (ii)  $I_1 = R_1$  and  $I_2$  is a uniformly  $pr$ -ideal or  $I_2 = R_2$  and  $I_1$  is a uniformly  $pr$ -ideal or  $I_1, I_2$  are uniformly  $pr$ -ideals.

**Proof** (i)  $\Rightarrow$  (ii) : Suppose that  $I$  is a uniformly  $pr$ -ideal and  $I_2 = R_2$ . Let  $ab \in I_1$  with  $ann(a) = 0$ . Then note that  $ann(a, 1) = (0, 0)$  and also  $(a, 1)(b, 0) = (ab, 0) \in I$ . Since  $I$  is a uniformly  $pr$ -ideal, we conclude that  $(b, 0)^N = (b^N, 0) \in I_1 \times I_2$  and so  $b^N \in I_1$ . Hence,  $I_1$  is a uniformly  $pr$ -ideal. Similarly, one can easily show that  $I_2$  is a uniformly  $pr$ -ideal when  $I_1 = R_1$ . Assume that  $I_1, I_2$  are proper ideals, similarly  $I_1, I_2$  are uniformly  $pr$ -ideals. Furthermore,  $ord(I_1 \times I_2) \leq \max\{ord(I_1), ord(I_2)\}$ .

(ii)  $\Rightarrow$  (i) : Let  $I_1, I_2$  be uniformly  $pr$ -ideals with  $ord(I_i) = N_i$  for every  $i = 1, 2$ . Put  $N = \max\{N_1, N_2\}$ . Let  $(a_1, a_2)(b_1, b_2) = (a_1b_1, a_2b_2) \in I_1 \times I_2$  with  $ann(a_1, a_2) = (0, 0)$ . This implies that  $ann(a_1) = ann(a_2) = 0$ . Note that  $a_i b_i \in I_i$  with  $ann(a_i) = 0$ . Since  $I_i$  is uniformly  $pr$ -ideals with  $ord(I_i) = N_i$ , we conclude that  $b_i^{N_i} \in I_i$ . Then  $b_i^N \in I_i$  and so  $(b_1, b_2)^N = (b_1^N, b_2^N) \in I_1 \times I_2$ . Thus  $I$  is a uniformly  $pr$ -ideal. Also note that  $ord(I_1 \times I_2) \leq N$ . In other cases, one can similarly prove that  $I$  is a uniformly  $pr$ -ideal. □

**Theorem 2.17** Let  $R_1, R_2, \dots, R_n$  be commutative rings and  $R = R_1 \times R_2 \times \dots \times R_n$ . Suppose that  $I = I_1 \times I_2 \times \dots \times I_n$ , where  $I_i$ 's are ideals of  $R_i$ , respectively. Then the followings are equivalent:

- (i)  $I$  is a uniformly  $pr$ -ideal.
- (ii) There exists  $k_1, k_2, \dots, k_t \in \{1, 2, \dots, n\}$  such that  $I_k = R_k$  for each  $k \in \{k_1, k_2, \dots, k_t\}$  and  $I_k$  is a uniformly  $pr$ -ideal for each  $k \in \{1, 2, \dots, n\} - \{k_1, k_2, \dots, k_t\}$

**Proof** We use the mathematical induction on  $n$ . If  $n = 1$ , the claim is true. If  $n = 2$ , the claim follows from the previous theorem. Assume that the claim is true for all  $k < n$ . Suppose that  $I = I_1 \times I_2 \times \dots \times I_n$ . Now put  $L = I_1 \times I_2 \times \dots \times I_{n-1}$ . Then by the previous theorem  $I = L \times I_n$  is a uniformly  $pr$ -ideal if and only if  $L = R_1 \times R_2 \times \dots \times R_{n-1}$  and  $I_n$  is a uniformly  $pr$ -ideal or  $I_n = R_n$  and  $L$  is a uniformly  $pr$ -ideal or  $L, I_n$  are uniformly  $pr$ -ideals. By induction hypothesis the claim is true. □

**Theorem 2.18** Let  $R_1, R_2$  be two commutative rings and  $R = R_1 \times R_2$ . Suppose that  $I = I_1 \times I_2$ , where  $I_1, I_2$  are ideals of  $R_1$  and  $R_2$ , respectively. Then the followings are equivalent:

- (i)  $I$  is a strongly  $pr$ -ideal.
- (ii)  $I_1 = R_1$  and  $I_2$  is a strongly  $pr$ -ideal or  $I_2 = R_2$  and  $I_1$  is a strongly  $pr$ -ideal or  $I_1, I_2$  are strongly  $pr$ -ideals.

**Proof** Note that  $\sqrt{I_1 \times I_2} = \sqrt{I_1} \times \sqrt{I_2}$ . The rest is similar to Theorem 2.16. □

As a consequence of the previous theorem we have the following theorem.

**Theorem 2.19** Let  $R_1, R_2, \dots, R_n$  be commutative rings and  $R = R_1 \times R_2 \times \dots \times R_n$ . Suppose that  $I = I_1 \times I_2 \times \dots \times I_n$ , where  $I_i$ 's are ideals of  $R_i$ , respectively. Then the followings are equivalent:

- (i)  $I$  is a strongly  $pr$ -ideal.

(ii) There exists  $k_1, k_2, \dots, k_t \in \{1, 2, \dots, n\}$  such that  $I_k = R_k$  for each  $k \in \{k_1, k_2, \dots, k_t\}$  and  $I_k$  is a strongly  $pr$ -ideal for each  $k \in \{1, 2, \dots, n\} - \{k_1, k_2, \dots, k_t\}$ .

Let  $M$  be an  $R$ -module. Then the idealization of  $M$ ,  $R(+)M = \{(r, m) : r \in R, m \in M\}$  is a commutative ring with componentwise addition and multiplication  $(a, m)(b, n) = (ab, an + bm)$  for every  $a, b \in R$  and  $m, n \in M$  [1, 13]. If  $I$  is an ideal of  $R$  and  $N$  is a submodule of  $M$  then  $I(+)N$  is an ideal of  $R(+)M$  if and only if  $IM \subseteq N$  [1]. In that case,  $I(+)N$  is called a homogeneous ideal of  $R(+)M$ . Note that  $zd(R(+)M) = \{(a, m) : a \in zd(R) \cup z(M)\}$  where  $z(M) = \{a \in R : am = 0 \text{ for some } 0 \neq m \in M\}$  [1]. Now we characterize the uniformly  $pr$ -ideal of  $R$  in terms of the uniformly  $pr$ -ideal of  $R(+)M$ .

**Proposition 2.20** *Let  $R$  be a ring and  $M$  be an  $R$ -module. Then*

- (i) *If  $I$  is a uniformly  $pr$ -ideal, then  $I(+)M$  is a uniformly  $pr$ -ideal of  $R(+)M$ .*
- (ii) *If  $z(M) = zd(R)$  and  $I(+)M$  is a uniformly  $pr$ -ideal of  $R(+)M$ , then  $I$  is a uniformly  $pr$ -ideal.*

**Proof** (i) Let  $(a, m)(b, n) = (ab, an + bm) \in I(+)M$  with  $ann(a, m) = (0, 0)$ . Then we have  $ann(a) = 0$ . Note that  $ab \in I$ . Since  $I$  is a uniformly  $pr$ -ideal, we conclude that  $b^N \in I$ . This implies that  $(b, n)^N = (b^N, Nb^{N-1}n) \in I(+)M$ . Hence,  $I(+)M$  is a uniformly  $pr$ -ideal of  $R(+)M$ .

(ii) Let  $ab \in I$  with  $ann(a) = 0$ . Then  $(a, 0)(b, 0) = (ab, 0) \in I(+)M$ . Since  $Z(M) = zd(R)$ , note that  $ann(a, 0) = (0, 0)$ . As  $I(+)M$  is a uniformly  $pr$ -ideal, we conclude that  $(b, 0)^N = (b^N, 0) \in I(+)M$ . This implies that  $b^N \in I$  and so  $I$  is a uniformly  $pr$ -ideal. □

**Corollary 2.21** *Let  $R$  be a ring and  $M$  be an  $R$ -module such that  $Z(M) = zd(R)$ . Then  $I(+)M$  is a uniformly  $pr$ -ideal of  $R(+)M$  if and only if  $I$  is a uniformly  $pr$ -ideal of  $R$ .*

**Proposition 2.22** *Let  $R$  be a ring and  $M$  be an  $R$ -module. Then*

- (i) *If  $I$  is a strongly  $pr$ -ideal, then  $I(+)M$  is a strongly  $pr$ -ideal of  $R(+)M$ .*
- (ii) *If  $Z(M) = zd(R)$  and  $I(+)M$  is a strongly  $pr$ -ideal of  $R(+)M$ , then  $I$  is a strongly  $pr$ -ideal.*

**Proof** (i) Let  $I$  be a strongly  $pr$ -ideal. Then  $I$  is a uniformly  $pr$ -ideal. By previous proposition (i),  $I(+)M$  is a uniformly  $pr$ -ideal and so  $I(+)M$  is a  $pr$ -ideal. Since  $I$  is a strongly  $pr$ -ideal, we have  $\sqrt{I}^N \subseteq I$  for some  $N \in \mathbb{N}$ . By [1, Theorem 3.2],  $\sqrt{I(+)M} = \sqrt{I}(+)M$  and so

$$\begin{aligned} \sqrt{I(+)M}^N &= (\sqrt{I}(+)M)^N \subseteq \sqrt{I}^N(+)M \\ &\subseteq I(+)M. \end{aligned}$$

(ii) Assume that  $I(+)M$  is a strongly  $pr$ -ideal. Then  $I(+)M$  is a uniformly  $pr$ -ideal so by the previous proposition (i)  $I$  is a uniformly  $pr$ -ideal. Then  $I$  is a  $pr$ -ideal. Since  $I(+)M$  is a strongly  $pr$ -ideal, there exists  $N \in \mathbb{N}$  such that

$$\begin{aligned} \sqrt{I(+)M}^N &= (\sqrt{I}(+)M)^N \subseteq \sqrt{I}^N(+)M \\ &\subseteq I(+)M. \end{aligned}$$

This implies that  $\sqrt{I}^N \subseteq I$  and so  $I$  is a strongly  $pr$ -ideal. □

**Corollary 2.23** *Let  $R$  be a ring and  $M$  be an  $R$ -module such that  $Z(M) = zd(R)$ . Then  $I(+)M$  is a strongly  $pr$ -ideal of  $R(+)M$  iff  $I$  is a strongly  $pr$ -ideal of  $R$ .*

A ring  $R$  is said to satisfy the annihilator condition (for short, a.c.) if every finitely generated ideal  $I$  consisting zero divisors has a nonzero annihilator.  $R[X]$  denotes the polynomial ring over  $R$  in indeterminates  $x$ . If  $f(X) = a_0 + a_1X + \dots + a_nX^n$ , where  $a_n \neq 0$ , then  $\deg(f) = n$ . Also the content of  $f$  is defined as  $c(f) = (a_0, a_1, \dots, a_n)$ . In [6, Theorem 28.1],  $c(f)^{n+1}c(g) = c(f)^nc(fg)$ , where  $\deg f = n$ . Let  $I$  be an ideal of  $R$ . Then  $I[X] = \{f(X) = a_0 + a_1X + \dots + a_nX^n : a_i \in I \text{ for } i = 0, 1, \dots, n\}$  is an ideal of  $R[X]$ . A ring  $R$  is called Armendariz ring if for any  $f(x) = a_0 + a_1x + \dots + a_nx^n$ ,  $g(x) = b_0 + b_1x + \dots + b_kx^k$  with  $f(x)g(x) = 0$ , then  $a_ib_j = 0$  for all  $i, j$ . [14] Note that all reduced rings are an example of Armendariz ring.

**Theorem 2.24** *Let  $R$  be an Armendariz ring. Then  $I$  is a uniformly  $pr$ -ideal if and only if  $I[X]$  is a uniformly  $pr$ -ideal of  $R[X]$ .*

**Proof** Suppose that  $I[X]$  is a uniformly  $pr$ -ideal of  $R[X]$ . Let  $ab \in I$  with  $ann(a) = 0$ . Now put  $f(X) = a, g(X) = b$ . Then  $f(X)g(X) = ab \in I[X]$  with  $ann_{R[X]}(f(X)) = 0$ . Since  $I[X]$  is a uniformly  $pr$ -ideal of  $R[X]$ , there exists  $N \in \mathbb{N}$  such that  $g(X)^N = b^N \in I[X]$  and so  $b^N \in I$ . Thus,  $I$  is a uniformly  $pr$ -ideal of  $R$ . Conversely, let  $I$  be a uniformly  $pr$ -ideal of  $R$  with  $ord_R(I) = N$ . Assume that  $f(X)g(X) \in I[X]$  with  $ann_{R[X]}(f(X)) = 0$ . Since  $R$  is an Armendariz ring and  $ann_{R[X]}(f(X)) = ann(c(f))[X] = 0$ , we have  $c(f) \not\subseteq zd(R)$  and so there exists  $r \in c(f)$  such that  $ann(r) = 0$ . Also note that  $c(fg) \subseteq I$ . Then by [6, Theorem 28.1],  $c(f)^{n+1}c(g) = c(f)^nc(fg) \subseteq I$ , where  $\deg f = n$ . Since  $r^{n+1} \in c(f)^{n+1}$  and  $ann(r^{n+1}) = 0$ , we get  $r^{n+1}c(g) \subseteq I$ . Now assume that  $c(g) = (b_0, b_1, \dots, b_k)$  for some  $b_i \in R$ . Then for any  $b_i \in c(g)$ , we have  $b_i^N \in I$ . It is clear that  $c(g)^{(k+1)N} \subseteq I$ . and so  $c(g)^{(k+1)N} \subseteq I$  and this yields that  $g(X)^{(k+1)N} \in I[X]$ . Hence  $I[X]$  is a uniformly  $pr$ -ideal of  $R[X]$ . □

**Proposition 2.25** *Let  $R$  be a ring satisfying property (A). Then  $I$  is a strongly  $pr$ -ideal if and only if  $I[X]$  is a strongly  $pr$ -ideal of  $R[X]$ .*

**Proof** Let  $I$  be a proper ideal of  $R$ . Then  $I[X]$  is a proper ideal and so  $R[X]/I[X]$  is isomorphic to ring  $(R/I)[X]$ . This isomorphism gives us  $\sqrt{0_{R[X]/I[X]}} = \sqrt{I[X]/I[X]} \cong \sqrt{0_{(R/I)[X]}} = (\sqrt{I}/I)[X]$  and so  $\sqrt{I[X]}/I[X] = \sqrt{I}[X]/I[X]$  and this yields  $\sqrt{I[X]} = \sqrt{I}[X]$ . Suppose that  $I[X]$  is a strongly  $pr$ -ideal of  $R[X]$ . Then  $I[X]$  is a uniformly  $pr$ -ideal and so by the previous theorem  $I$  is a uniformly  $pr$ -ideal. Then  $I$  is a  $pr$ -ideal. Now assume that  $e_{R[X]}(I[X]) = N$ . Then

$$\sqrt{I[X]}^N = (\sqrt{I}[X])^N = \sqrt{I}^N[X] \subseteq I[X]$$

and this gives  $\sqrt{I}^N \subseteq I$ . Hence,  $I$  is a strongly  $pr$ -ideal. Conversely, assume that  $I$  is a strongly  $pr$ -ideal with  $e_R(I) = N$ . Then  $I$  is a uniformly  $pr$ -ideal and so by the previous theorem  $I[X]$  is a uniformly  $pr$ -ideal. Then  $I[X]$  is a  $pr$ -ideal. Also note that  $\sqrt{I}^N \subseteq I$  and so

$$\sqrt{I[X]}^N = (\sqrt{I}[X])^N = \sqrt{I}^N[X] \subseteq I[X].$$

Hence,  $I[X]$  is a strongly  $pr$ -ideal of  $R[X]$ . □

Let  $R[[x]]$  denote the formal power series ring with coefficient in  $R$ . For any  $f(x) = \sum_{k=0}^{\infty} a_k x^k \in R[[x]]$ , we denote the ideal generated by the set  $\{a_n : n \in \mathbb{N}\}$  with  $C(f)$ . In [[5], Theorem 2.6], Dedekind Mertens Theorem for formal power series ring is given as the following.

**Theorem 2.26** *Let  $R$  be a Noetherian ring and  $0 \neq g \in R[[x]]$ . Let  $k$  be the maximum of the numbers  $\mu(c(g)_m)$ , taken all over the maximal ideals  $m$  of  $R$ . (In particular,  $\mu(c(g)_m) \geq k$ ) Then for all  $f \in R[[x]]$ , we have  $c(f)^k c(g) = c(f)^{k-1} c(fg)$ .*

Recall that the commutative ring is called a  $ps$ -Armendariz ring if for any  $f(x) = \sum_{k=0}^{\infty} a_k x^k$  and  $g(x) = \sum_{k=0}^{\infty} b_k x^k$  with  $f(x)g(x) = 0$ , then  $a_k b_n = 0$  for all  $k, n$ . Note that all  $ps$ -Armendariz rings are an Armendariz ring [9].

**Theorem 2.27** *Let  $R$  be a Noetherian  $ps$ -Armendariz ring. Then  $I$  is a uniformly  $pr$ -ideal if and only if  $I[[X]]$  is a uniformly  $pr$ -ideal of  $R[[X]]$ .*

**Proof** The "only if" part is clear. Assume that  $I$  is a uniformly  $pr$ -ideal of  $R$ . Take  $f(x) = \sum_{k=0}^{\infty} a_k x^k$  and  $g(x) = \sum_{k=0}^{\infty} b_k x^k \in R[[X]]$  such that  $f(x)g(x) \in I[[X]]$  with  $ann(f(x)) = 0$ . Since  $R$  is a  $ps$ -Armendariz ring, then  $ann(f(x)) = (ann(c(f)))[[x]]$  where  $c(f) = \langle a_n : n \in \mathbb{N} \rangle$ . Since  $R$  is a Noetherian ring,  $c(f) = \langle a_1^*, a_2^*, \dots, a_n^* \rangle$  and so  $ann(f(x)) = ann(\langle a_1^*, a_2^*, \dots, a_n^* \rangle)[[X]] = 0$ . As  $R$  satisfies the property (A),  $\langle a_1^*, a_2^*, \dots, a_n^* \rangle \not\subseteq z d(R)$  and so there exists  $r \in c(f)$  such that  $ann(r) = 0$ . Also note that  $c(fg) \subseteq I$ . Then by the previous theorem  $c(f)^{n+1} c(g) = c(f)^n c(fg) \subseteq I$  where  $k$  is the maximum of the numbers  $\mu(c(g)_m)$ . As  $r^{n+1} \in c(f)^{n+1}$ ,  $ann(r^{n+1}) = 0$ , we get  $r^{n+1} c(g) \subseteq I$ . Since  $R$  is a Noetherian ring,  $c(g) = \langle b_n : n \in \mathbb{N} \rangle = \langle b_1^*, b_2^*, \dots, b_k^* \rangle$ . Since  $r^{n+1} b_i^* \in I$  and  $I$  is a uniformly  $pr$ -ideal with  $ord(I) = N$ , we can conclude that  $(b_i^*)^N \in I$ . Now put  $t = (k+1)N$ . A similar argument of Theorem 2.24 shows that  $c(g^t) \subseteq I$  and hence  $g^t \in I$ . Therefore,  $I[[X]]$  is a uniformly  $pr$ -ideal of  $R[[X]]$ . □

**Proposition 2.28** *Let  $R$  be a Noetherian  $ps$ -Armendariz ring. Then  $I$  is a strongly  $pr$ -ideal if and only if  $I[[X]]$  is a strongly  $pr$ -ideal of  $R[[X]]$ .*

**Proof** Suppose that  $I$  is a strongly  $pr$ -ideal. Then  $I$  is a uniformly  $pr$ -ideal by Corollary 2.6. Then by the previous theorem  $I[[X]]$  is a uniformly  $pr$ -ideal and hence  $I[[X]]$  is a  $pr$ -ideal. By [[3], Corollary 2.2.3],  $\sqrt{I[[X]]} = \sqrt{I}[[X]]$ . Assume that  $exp(I) = N$  and it is easy to see that  $\sqrt{I[[X]]}^N = (\sqrt{I}[[X]])^N = \sqrt{I}^N [[X]] \subseteq I[[X]]$  since  $\sqrt{I}^N \subseteq I$ . Conversely, assume that  $I[[X]]$  is a strongly  $pr$ -ideal with  $ord(I[[X]]) = N$ . Then the similar argument shows that  $I$  is a strongly  $pr$ -ideal. □

**Theorem 2.29** *Let  $R$  be a Noetherian  $ps$ -Armendariz ring. Then the following are equivalent.*



- (i)  $I$  is a  $pr$ -ideal of  $R$ .
- (ii)  $\sqrt{I}$  is a  $r$ -ideal of  $R$ .
- (iii)  $I$  is a uniformly  $pr$ -ideal of  $R$ .
- (iv)  $I$  is a strongly  $pr$ -ideal of  $R$ .
- (v)  $I[X]$  is a uniformly  $pr$ -ideal of  $R[X]$ .
- (vi)  $I[X]$  is a strongly  $pr$ -ideal of  $R[X]$ .
- (vii)  $I[[X]]$  is a uniformly  $pr$ -ideal of  $R[[X]]$ .
- (viii)  $I[[X]]$  is a strongly  $pr$ -ideal of  $R[[X]]$ .
- (ix)  $I[[X]]$  is a  $pr$ -ideal of  $R[[X]]$ .
- (x)  $I[X]$  is a  $pr$ -ideal of  $R[X]$ .

**Proof** (i)  $\iff$  (iv) are equal from Corollary 2.9.

(i)  $\iff$  (v) – (viii) follows from Theorem 2.24, Proposition 2.25, previous proposition, and previous theorem.

(viii)  $\Rightarrow$  (ix)  $\Rightarrow$  (x)  $\Rightarrow$  (i) is clear. □

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