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Riemannian manifolds admitting a new type of semisymmetric nonmetric connection

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Abstract: We define a new type of semisymmetric nonmetric connection on a Riemannian manifold and establish its existence. It is proved that such connection on a Riemannian manifold is projectively invariant under certain conditions. We also find many basic results of the Riemannian manifolds and study the properties of group manifolds and submanifolds of the Riemannian manifolds with respect to the semisymmetric nonmetric connection. To validate our findings, we construct a nontrivial example of a 3-dimensional Riemannian manifold equipped with a semisymmetric nonmetric connection.

Key words: Riemannian manifold, semisymmetric nonmetric connection, different curvature tensors, Ricci solitons

1. Introduction

Let M^n be an n-dimensional Riemannian manifold and let ∇ denote the Levi-Civita connection corresponding to the Riemannian metric g on M^n. A linear connection $\tilde{\nabla}$ defined on M^n is said to be symmetric if the torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ defined by

$$\tilde{T}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - [X,Y]$$

is zero for all X and Y on M^n; otherwise, it is nonsymmetric. In 1924, Friedmann and Schouten [6] considered a differentiable manifold and introduced the idea of a semisymmetric linear connection on it. A linear connection $\tilde{\nabla}$ on M^n is said to be semisymmetric if

$$\tilde{T}(X,Y) = \pi(Y)X - \pi(X)Y \tag{1.1}$$

holds for all vector fields X, Y on M^n, where π is a 1-form associated with the vector field P and satisfies

$$\pi(X) = g(X,P). \tag{1.2}$$

In 1932, Hayden gave the idea of a metric connection $\tilde{\nabla}$ on a Riemannian manifold and later named such connection a Hayden connection. After a long gap, Pak [10] considered the Hayden connection $\tilde{\nabla}$ equipped with the torsion tensor $\tilde{T}$ defined as (1.1) and proved that it is a semisymmetric metric connection. A linear connection $\tilde{\nabla}$ is said to be metric on M^n if $\tilde{\nabla} g = 0$; otherwise, it is nonmetric. A systematic study of the

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semisymmetric metric connection $\tilde{\nabla}$ on a Riemannian manifold was initiated by Yano [14] in 1970. Smaranda [2] studied the properties of semisymmetric recurrent metric connections and proved many interesting geometrical results. In 1992, Agashe and Chafle [1] introduced a new class of the semisymmetric connection, called the semisymmetric nonmetric connection, on a Riemannian manifold and studied some of its geometric properties. Sengupta et al. [11] defined a new type of semisymmetric nonmetric connection on a Riemannian manifold in 2000. In this connection, Chaubey et al. in [3] and [4] defined the nonsymmetric nonmetric connection on almost contact metric manifolds. Motivated by the above studies, we define and study a new type of semisymmetric nonmetric connection on a Riemannian manifold.

We organize our present work as follows: after an introduction in Section 1, we define a new class of semisymmetric nonmetric connection on a Riemannian manifold and prove its existence in Section 2. We also present some basic results that will help in further study. In Section 3, we establish the relation between curvature tensors of the Levi-Civita and semisymmetric nonmetric connections and prove some basic properties of the curvature tensor of $\tilde{\nabla}$. The necessary and sufficient conditions for projectively invariant curvature tensors are proved and we also bridge the gaps between the curvature, conformal curvature, concircular curvature, and conharmonic curvature tensors. Section 4 deals with the study of a group manifold with respect to a semisymmetric nonmetric connection. The properties of submanifolds of a Riemannian manifold with respect to the semisymmetric nonmetric connection are studied in Section 5. In the last section, we construct a nontrivial example of a 3-dimensional Riemannian manifold endowed with a semisymmetric nonmetric connection and prove some results.

2. Semisymmetric nonmetric connection

Let (M^n, g) be a Riemannian manifold of dimension n endowed with a Levi-Civita connection ∇ corresponding to the Riemannian metric g. A linear connection $\tilde{\nabla}$ on (M^n, g) defined by

$$\tilde{\nabla}_X Y = \nabla_X Y + \frac{1}{2}\{\pi(Y)X - \pi(X)Y\} \tag{2.1}$$

for arbitrary vector fields X and Y on M^n is said to be a semisymmetric nonmetric connection if the torsion tensor $\tilde{T}$ of M^n with respect to $\tilde{\nabla}$ satisfies equations (1.1) and (1.2), and the metric g holds the relation

$$(\tilde{\nabla}_X g)(Y, Z) = \frac{1}{2}\{2\pi(X)g(Y, Z) - \pi(Y)g(X, Z) - \pi(Z)g(X, Y)\} \tag{2.2}$$

for all vector fields X, Y, Z on M^n. Now we prove the existence of such a connection on an n-dimensional Riemannian manifold in the following theorem.

Theorem 2.1 *Let (M^n, g) be an n-dimensional Riemannian manifold endowed with the Levi-Civita connection ∇. Then there exists a unique linear connection $\tilde{\nabla}$ on M^n, called a semisymmetric nonmetric connection, given by (2.1), and it satisfies equations (1.1) and (2.2).*

Proof We suppose that (M^n, g) is a Riemannian manifold of dimension n equipped with a linear connection $\tilde{\nabla}$. Then $\tilde{\nabla}$ and the Levi-Civita connection ∇ are connected by the relation

$$\tilde{\nabla}_X Y = \nabla_X Y + U(X, Y) \tag{2.3}$$

for arbitrary vector fields X and Y on M^n, where U is a tensor field of type $(1,2)$. By definition of the torsion tensor $\tilde{T}$ of $\tilde{\nabla}$ and equation (2.3), we can conclude that

$$\tilde{T}(X,Y) = U(X,Y) - U(Y,X), \tag{2.4}$$

which gives

$$g(\tilde{T}(X,Y),Z) = g(U(X,Y),Z) - g(U(Y,X),Z). \tag{2.5}$$

From (1.1) and (2.5), we have

$$g(U(X,Y),Z) - g(U(Y,X),Z) = \pi(Y)g(X,Z) - \pi(X)g(Y,Z). \tag{2.6}$$

In view of equation (2.1), we conclude that

$$\begin{aligned}(\tilde{\nabla}_X g)(Y,Z) &= -g(\tilde{\nabla}_X Y - \nabla_X Y, Z) - g(Y, \tilde{\nabla}_X Z - \nabla_X Z)\\ &= -U'(X,Y,Z),\end{aligned} \tag{2.7}$$

where $U'(X,Y,Z) = g(U(X,Y),Z) + g(U(X,Z),Y)$. We have

$$\begin{aligned}&g(\tilde{T}(X,Y),Z) + g(\tilde{T}(Z,X),Y) + g(\tilde{T}(Z,Y),X)\\ &= 2g(U(X,Y),Z) - U'(X,Y,Z) + U'(Z,X,Y) - U'(Y,X,Z),\end{aligned}$$

where equations (2.4), (2.5), and (2.7) are used. In consequence of equations (2.2) and (2.7), the above equation assumes the form

$$\begin{aligned}2g(U(X,Y),Z) = {}& g(\tilde{T}(X,Y),Z) + g(\tilde{T}'(X,Y),Z) + g(\tilde{T}'(Y,X),Z)\\ & -\pi(X)g(Y,Z) - \pi(Y)g(X,Z) + 2\pi(Z)g(X,Y),\end{aligned} \tag{2.8}$$

where

$$g(\tilde{T}'(X,Y),Z) = g(\tilde{T}(Z,X),Y) = \pi(X)g(Z,Y) - \pi(Z)g(X,Y) \tag{2.9}$$

for all vector fields X, Y, and Z on M^n. By using equation (2.9), equation (2.8) takes the form

$$2U(X,Y) = \pi(Y)X - \pi(X)Y \tag{2.10}$$

and thus equations (2.3) and (2.10) give (2.1). Conversely, we can easily show that if the affine connection $\tilde{\nabla}$ satisfies (2.1) then it will also satisfy equations (1.1) and (2.2). Hence, the statement of Theorem 2.1 is proved. □

The covariant derivative of equation (1.2) with respect to the semisymmetric nonmetric connection $\tilde{\nabla}$ gives

$$(\tilde{\nabla}_X \pi)(Y) = (\nabla_X \pi)(Y) + \pi(P)g(X,Y) - \pi(Y)\pi(X) \tag{2.11}$$

for arbitrary vector fields X and Y on M^n. From equation (2.11), we get

$$(\tilde{\nabla}_X \pi)(Y) - (\tilde{\nabla}_Y \pi)(X) = (\nabla_X \eta)(Y) - (\nabla_X \eta)(Y). \tag{2.12}$$

Thus, we are in position to state the following proposition:

Proposition 2.2 *Let (M^n, g) be a Riemannian manifold of dimension n endowed with a semisymmetric nonmetric connection $\tilde{\nabla}$, and then the necessary and sufficient condition for the 1-form π to be closed with respect to $\tilde{\nabla}$ is that it is also closed corresponding to the Levi-Civita connection ∇.*

Theorem 2.3 *On an n-dimensional Riemannian manifold (M^n, g) endowed with a semisymmetric nonmetric connection $\tilde{\nabla}$, we have*

$$'\tilde{T}(X,Y,Z) +' \tilde{T}(Y,X,Z) = 0,$$

$$'\tilde{T}(X,Y,Z) +' \tilde{T}(Y,Z,X) +' \tilde{T}(Z,X,Y) = 0.$$

Proof We define $'\tilde{T}(X,Y,Z) = g(\tilde{T}(X,Y),Z)$ on (M^n, g). Therefore, equation (1.1) gives

$$'\tilde{T}(X,Y,Z) = \pi(Y)g(X,Z) - \pi(X)g(Y,Z). \tag{2.13}$$

With the help of equation (2.13), we can easily prove the statement of Theorem 2.3. □

Theorem 2.4 *If (M^n, g) is an n-dimensional Riemannian manifold equipped with a semisymmetric nonmetric connection $\tilde{\nabla}$, then $\tilde{T}$ is cyclic parallel if and only if the 1-form π is closed.*

Proof Taking the covariant derivative of (1.1) with respect to the semisymmetric nonmetric connection $\tilde{\nabla}$, we find that

$$(\tilde{\nabla}_X \tilde{T})(Y,Z) = (\tilde{\nabla}_X \pi)(Z)Y - (\tilde{\nabla}_X \pi)(Y)Z. \tag{2.14}$$

The cyclic sum of (2.14) for vector fields X, Y, and Z gives

$$\begin{aligned}(\tilde{\nabla}_X \tilde{T})(Y,Z) + (\tilde{\nabla}_Y \tilde{T})(Z,X) + (\tilde{\nabla}_Z \tilde{T})(X,Y) &= \{(\tilde{\nabla}_X \pi)(Z) - (\tilde{\nabla}_Z \pi)(X)\}Y \\ &+\{(\tilde{\nabla}_Z \pi)(Y) - (\tilde{\nabla}_Y \pi)(Z)\}X + \{(\tilde{\nabla}_Y \pi)(X) - (\tilde{\nabla}_X \pi)(Y)\}Z.\end{aligned} \tag{2.15}$$

From equation (2.15) and Proposition 2.2, we can easily show that $(\tilde{\nabla}_X \tilde{T})(Y,Z)+(\tilde{\nabla}_Y \tilde{T})(Z,X)+(\tilde{\nabla}_Z \tilde{T})(X,Y) = 0$ if and only if the 1-form π is closed. Hence, Theorem 2.4 is proved. □

Proposition 2.5 *If an n-dimensional Riemannian manifold (M^n, g) admits a semisymmetric nonmetric connection $\tilde{\nabla}$, then for any arbitrary vector fields X and Y, and the vector field P defined as (1.2), the following relation holds:*

$$(\tilde{\mathfrak{L}}_P g)(X,Y) = (\mathfrak{L}_P g)(X,Y) + 2\{\pi(P)g(X,Y) - \pi(X)\pi(Y)\}, \tag{2.16}$$

where $\tilde{\mathfrak{L}}_P$ and $\mathfrak{L}_P$ denote the Lie derivatives along the vector field P corresponding to $\tilde{\nabla}$ and ∇, respectively.

Proof It is well known that

$$(\mathfrak{L}_P g)(X,Y) = g(\nabla_X P, Y) + g(X, \nabla_Y P) \tag{2.17}$$

holds for arbitrary vector fields X and Y on M^n. By using equations (2.1) and (2.17) and the definition of the Lie derivative, we find

$$\begin{aligned}(\tilde{\mathfrak{L}}_P g)(X,Y) &= Pg(X,Y) - g(\tilde{\nabla}_P X - \tilde{\nabla}_X P, Y) - g(X, \tilde{\nabla}_P Y - \tilde{\nabla}_Y P) \\ &= (\mathfrak{L}_P g)(X,Y) + 2\{\pi(P)g(X,Y) - \pi(X)\pi(Y)\}.\end{aligned}$$

Hence, the statement of Proposition 2.5 is proved. □

If the vector field P is Killing on (M^n, g), then $\mathfrak{L}_P g = 0$ and therefore in view of Proposition 2.5 we state:

Corollary 2.6 *If the vector field P defined as in (1.2) is Killing on an n-dimensional Riemannian manifold (M^n, g) equipped with a semisymmetric nonmetric connection $\tilde{\nabla}$, then*

$$(\tilde{\mathfrak{L}}_P g)(X, Y) = 2\{\pi(P)g(X, Y) - \pi(X)\pi(Y)\}. \tag{2.18}$$

3. Curvature tensor with respect to the semisymmetric nonmetric connection

Let (M^n, g) be an n-dimensional Riemannian manifold admitting a semisymmetric nonmetric connection $\tilde{\nabla}$. If the curvature tensor $\tilde{R}$ corresponding to $\tilde{\nabla}$ is defined by

$$\tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z$$

for arbitrary vector fields X, Y, and Z on (M^n, g), then the Riemannian curvature tensor R of the Levi-Civita connection ∇ is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all vector fields X, Y, and Z on (M^n, g) and connected by the relation

$$\begin{aligned}
\tilde{R}(X, Y)Z &= \tilde{\nabla}_X\{\nabla_Y Z + \frac{1}{2}\left(\pi(Z)Y - \pi(Y)Z\right)\} - \tilde{\nabla}_Y\{\nabla_X Z \\
&+ \frac{1}{2}\left(\pi(Z)X - \pi(X)Z\right)\} - \{\nabla_{[X,Y]}Z + \frac{1}{2}\left(\pi(Z)[X,Y] - \pi([X,Y])Z\right)\} \\
&= \nabla_X\{\nabla_Y Z + \frac{1}{2}(\pi(Z)Y - \pi(Y)Z)\} - \nabla_Y\{\nabla_X Z + \frac{1}{2}(\pi(Z)X \\
&- \pi(X)Z)\} - \nabla_{[X,Y]}Z + \frac{1}{2}[\pi(\nabla_Y Z)X - \pi(X)\nabla_Y Z - \pi(\nabla_X Z)Y \\
&+ \pi(Y)\nabla_X Z] - \frac{1}{4}[\pi(X)\pi(Z)Y - \pi(Y)\pi(Z)X - \pi(Z)[X,Y] + \pi([X,Y])Z] \\
&= R(X, Y)Z + \frac{1}{2}\{\theta(X, Z)Y - \theta(Y, Z)X - \left(\theta(X, Y) - \theta(Y, X)\right)Z\}
\end{aligned} \tag{3.1}$$

for arbitrary vector fields X, Y, and Z on M^n, where θ is a tensor field of type $(0, 2)$ and is defined by

$$\theta(X, Y) = g(AX, Y) = (\nabla_X \pi)(Y) - \frac{1}{2}\pi(X)\pi(Y) \tag{3.2}$$

and

$$AX = \nabla_X P - \frac{1}{2}\pi(X)P \tag{3.3}$$

for arbitrary vector fields X and Y on M^n. From equation (3.2), it is obvious that the tensor field θ is symmetric if and only if the 1-form π is closed. Taking the inner product of (3.1) with W and then setting $X = W = e_i$, $1 \le i \le n$, where $\{e_i,\ i = 1, 2, 3, ... n\}$ is an orthonormal basis of the tangent space at each point of the Riemannian manifold M^n, we have

$$\begin{aligned}\tilde{S}(Y,Z) &= S(Y,Z) + \frac{1}{2}\sum_{i=1}^{n}\{g(Ae_i,Z)g(Y,e_i) - \theta(Y,Z)g(e_i,e_i)\\ &\quad -g(Ae_i,Y)g(Z,e_i) + g(AY,e_i)g(Z,e_i)\}\\ &= S(Y,Z) - \frac{n-1}{2}\theta(Y,Z) + \frac{1}{2}\sum_{i=1}^{n}\{g(Ae_i,e_i)g(Z,e_i)g(Y,e_i)\\ &\quad -g(Ae_i,e_i)g(Z,e_i)g(Y,e_i)\},\end{aligned}$$

which is equivalent to

$$\tilde{S}(Y,Z) = S(Y,Z) - \frac{n-1}{2}\theta(Y,Z) \Longleftrightarrow \tilde{Q}Y = QY - \frac{n-1}{2}AY \tag{3.4}$$

for all vector fields Y and Z on M^n. Here $\tilde{Q}$ and Q are the Ricci operators corresponding to the Ricci tensors $\tilde{S}$ and S of the connections $\tilde{\nabla}$ and ∇, respectively; that is, $\tilde{S}(Y,Z) = g(\tilde{Q}Y,Z)$ and $S(Y,Z) = g(QY,Z)$. Again contracting (3.4) along the vector field Y, we get

$$\tilde{r} = r - (n-1)a, \tag{3.5}$$

where $\tilde{r}$ and r denote the scalar curvatures corresponding to the semisymmetric nonmetric connection $\tilde{\nabla}$ and the Levi-Civita connection ∇, respectively, and

$$a \overset{\text{def}}{=} \frac{1}{2}\, trA.$$

Here trA represents the trace of A. From equation (3.5), we can observe the following proposition:

Proposition 3.1 *Let (M^n, g) denote an n-dimensional Riemannian manifold endowed with a semisymmetric nonmetric connection $\tilde{\nabla}$. Then the necessary and sufficient condition for the scalar curvatures $\tilde{r}$ and r to coincide is that a be zero; that is, $trA = 0$.*

Interchanging Y and Z in (3.4), we have

$$\tilde{S}(Z,Y) = S(Z,Y) - \frac{n-1}{2}\theta(Z,Y). \tag{3.6}$$

Subtracting (3.6) from (3.4) and then using equation (3.2) and the symmetric property of the Ricci tensor in it, we conclude that

$$\tilde{S}(Y,Z) - \tilde{S}(Z,Y) = \frac{n-1}{2}\{\theta(Z,Y) - \theta(Y,Z)\} = -\frac{n-1}{2}d\pi(Y,Z), \tag{3.7}$$

where d denotes the exterior derivative. In view of (3.7) and Proposition 2.2, we are in a position to state the following proposition:

Proposition 3.2 *If an $n(>1)$-dimensional Riemannian manifold (M^n, g) admits a semisymmetric nonmetric connection $\tilde{\nabla}$, then the Ricci tensor $\tilde{S}$ corresponding to the connection $\tilde{\nabla}$ is symmetric if and only if the 1-form π is closed.*

Theorem 3.3 *Let (M^n, g) be an n-dimensional Riemannian manifold equipped with a semisymmetric nonmetric connection $\tilde{\nabla}$, then the following relations hold for all vector fields X, Y, Z, and U on M^n:*

$$(i)\ \tilde{R}(X,Y)Z + \tilde{R}(Y,X)Z = 0,$$

$$(ii)\ \tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0 \Longleftrightarrow \mathit{1-form}\,\pi\ \mathit{is\ closed},$$

$$\begin{aligned}(iii)\ &(\tilde{\nabla}_X\tilde{R})(Y,Z)U + (\tilde{\nabla}_Y\tilde{R})(Z,X)U + (\tilde{\nabla}_Z\tilde{R})(X,Y)U\\ &\quad= 2\{\pi(X)\tilde{R}(Y,Z)U + \pi(Y)\tilde{R}(Z,X)U + \pi(Z)\tilde{R}(X,Y)U\},\end{aligned}$$

$$\begin{aligned}(iv)\ {}'\tilde{R}(X,Y,Z,U) +{}' \tilde{R}(X,Y,U,Z) &= \frac{1}{2}\{\theta(X,Z)g(Y,U) - \theta(Y,U)g(X,Z)\\ &+\theta(X,U)g(Y,Z) - \theta(Y,Z)g(X,U)\} - d\pi(X,Y)g(U,Z),\end{aligned}$$

$$\begin{aligned}(v)\ {}'\tilde{R}(X,Y,Z,U) -{}' \tilde{R}(Z,U,X,Y) &= \frac{1}{2}\{d\pi(X,Z)g(Y,U) - d\pi(X,Y)g(Z,U)\\ &-d\pi(U,Z)g(X,Y) + \theta(U,X)g(Y,Z) - \theta(Y,Z)g(X,U)\}.\end{aligned}$$

Proof Interchanging X and Y in (3.1) and then adding with (3.1), we obtain (i). Again from (3.1), we find

$$\begin{aligned}\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y &= \{\theta(Z,Y) - \theta(Y,Z)\}X\\ &+\{\theta(X,Z) - \theta(Z,X)\}Y + \{\theta(Y,X) - \theta(X,Y)\}Z.\end{aligned}$$

This expression shows that the Riemannian manifold (M^n, g) equipped with a semisymmetric nonmetric connection $\tilde{\nabla}$ satisfies Bianchi's first identity if and only if the 1-form π is closed. Thus, result (ii). Bianchi's second identity for a semisymmetric nonmetric connection $\tilde{\nabla}$ is given by the expression

$$\begin{aligned}&(\tilde{\nabla}_X\tilde{R})(Y,Z)U + (\tilde{\nabla}_Y\tilde{R})(Z,X)U + (\tilde{\nabla}_Z\tilde{R})(X,Y)U\\ &\quad= -\tilde{R}(\tilde{T}(X,Y),Z)U - \tilde{R}(\tilde{T}(Y,Z),X)U - \tilde{R}(\tilde{T}(Z,X),Y)U\end{aligned}$$

for arbitrary vector fields X, Y, Z, and U on M^n. With the help of equation (1.1), (i), and the last expression, we can easily find (iii). If we define ${}'\tilde{R}(X,Y,Z,U) = g(\tilde{R}(X,Y)Z,U)$ and ${}'R(X,Y,Z,U) = g(R(X,Y)Z,U)$, then equation (3.1) becomes

$$\begin{aligned}{}'\tilde{R}(X,Y,Z,U) = {}'R(X,Y,Z,U) &+ \frac{1}{2}\{\theta(X,Z)g(Y,U)\\ &-\theta(Y,Z)g(X,U) - (\theta(X,Y) - \theta(Y,X))\,g(Z,U)\}\end{aligned} \tag{3.8}$$

for all vector fields X, Y, Z, and U on M^n. Expressions (iv) and (v) are obvious from equations (3.2) and (3.8) and the symmetric properties of the curvature tensor. Hence, the proof is completed. □

Theorem 3.4 *Let (M^n, g) be a Riemannian manifold of dimension $n\,(>1)$ equipped with a semisymmetric nonmetric connection $\tilde{\nabla}$ defined as in (2.1). Then the connection $\tilde{\nabla}$ is projectively invariant; that is, the projective curvature tensors with respect to $\tilde{\nabla}$ and ∇ coincide if and only if the 1-form π is closed.*

Proof We suppose that the 1-form π is closed and therefore equation (3.2) shows that θ is symmetric. This fact together with equation (3.1) gives

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \frac{1}{2}\{\theta(X,Z)Y - \theta(Y,Z)X\}. \tag{3.9}$$

Contracting (3.9) along the vector field X, we have

$$\tilde{S}(Y,Z) = S(Y,Z) - \frac{(n-1)}{2}\theta(Y,Z), \tag{3.10}$$

which gives

$$\tilde{Q}Y = QY - \frac{(n-1)}{2}AY \tag{3.11}$$

and

$$\tilde{r} = r - (n-1)a. \tag{3.12}$$

The projective curvature $\tilde{\mathcal{P}}$ [5] with respect to the semisymmetric nonmetric connection $\tilde{\nabla}$ is defined as

$$\tilde{\mathcal{P}}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\} \tag{3.13}$$

for all vector fields X, Y, and Z on M^n. In view of equations (3.9) and (3.10), equation (3.13) assumes the form

$$\tilde{\mathcal{P}}(X,Y)Z = \mathcal{P}(X,Y)Z, \tag{3.14}$$

where $\mathcal{P}$ denotes the projective curvature tensor [5] with respect to ∇ and is defined by

$$\mathcal{P}(X,Y)Z = R(X,Y)Z - \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\} \tag{3.15}$$

for arbitrary vector fields X, Y, and Z on M^n. Conversely, we suppose that (M^n,g) equipped with $\tilde{\nabla}$ satisfies (3.14). Thus, use of (3.1), (3.4), (3.11), (3.13), and (3.15) in (3.14) gives

$$\{\theta(X,Y) - \theta(Y,X)\}Z = 0.$$

Contracting the last equation along the vector field X, we find

$$\theta(Y,Z) - \theta(Z,Y) = 0,$$

which shows that $\theta(Y,Z) = \theta(Z,Y)$. Hence, the proof is completed. □

Theorem 3.5 *If (M^n,g), $n>2$, is an n-dimensional Riemannian manifold endowed with a semisymmetric nonmetric connection $\tilde{\nabla}$ whose curvature tensor $\tilde{R}$ vanishes identically, then (M^n,g) is projectively flat if and only if θ is a symmetric tensor.*

Proof We consider that the curvature tensor with respect to the semisymmetric nonmetric connection $\tilde{\nabla}$ vanishes on (M^n, g); that is, $\tilde{R} = 0$, and the tensor field θ is symmetric. Then equation (3.9) becomes

$$R(X,Y)Z = \frac{1}{2}\{\theta(Y,Z)X - \theta(X,Z)Y\}, \tag{3.16}$$

which implies that

$$S(Y,Z) = \frac{n-1}{2}\theta(Y,Z), \qquad r = (n-1)a. \tag{3.17}$$

In consequence of (3.15), (3.16), and (3.17), we find that $\mathcal{P} = 0$. Conversely, if the projective curvature tensor of ∇ is zero and the curvature tensor $\tilde{R}$ is also zero, then equations (3.1) and (3.15) become

$$R(X,Y)Z = \frac{1}{n-1}\{S(Y,Z)X - S(X,Z)Y\} \tag{3.18}$$

and

$$R(X,Y)Z = \frac{1}{2}\{\theta(Y,Z)X - \theta(X,Z)Y - (\theta(Y,X) - \theta(X,Y))\, Z\}. \tag{3.19}$$

Equating equations (3.18) and (3.19) and then using (3.4), we obtain

$$\{\theta(X,Y) - \theta(Y,X)\}Z = 0.$$

Contracting the above equation along the vector field Z, we have

$$\theta(X,Y) = \theta(Y,X).$$

Thus, the statement of Theorem 3.5 is proved. □

Theorem 3.6 *Let (M^n, g), $n > 2$, be an n-dimensional Riemannian manifold endowed with a semisymmetric nonmetric connection $\tilde{\nabla}$. If the curvature tensor with respect to $\tilde{\nabla}$ vanishes, then the tensor field θ is symmetric if and only if*

$$(n-2)\{'C(X,Y,Z,U) +' \breve{C}(X,Y,Z,U)\} = -2'R(X,Y,Z,U).$$

Proof Let the curvature tensor $\tilde{R}$ with respect to the semisymmetric nonmetric connection $\tilde{\nabla}$ vanish on M^n. To prove the necessary part, we consider that the tensor field θ is symmetric; that is, $\theta(X,Y) = \theta(Y,X)$. The conformal curvature tensor C [12] with respect to ∇ is defined by

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX \\ -g(X,Z)QY] + \frac{r}{(n-1)(n-2)}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{3.20}$$

for arbitrary vector fields X, Y, Z on M^n. The inner product of (3.20) with U gives

$$\begin{aligned} 'C(X,Y,Z,U) &= {}'R(X,Y,Z,U) - \frac{1}{n-2}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &\quad +g(Y,Z)S(X,U) - g(X,Z)S(Y,U)] \\ &\quad +\frac{r}{(n-1)(n-2)}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}, \end{aligned} \tag{3.21}$$

where $'C(X,Y,Z,U) = g(C(X,Y)Z,U)$. Using (3.16) and (3.17) in (3.21), we conclude that

$$'C(X,Y,Z,U) = -\frac{n}{(n-2)}{}'R(X,Y,Z,U) + \frac{a}{n-2}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\}. \tag{3.22}$$

The concircular curvature tensor $\breve{C}$ [13] on (M^n, g) with respect to ∇ is defined by

$$'\breve{C}(X,Y,Z,U) = {}'R(X,Y,Z,U) - \frac{r}{(n-1)(n-2)}\{g(Y,Z)g(X,U) - g(X,Z)g(Y,U)\} \tag{3.23}$$

for arbitrary vector fields X, Y, Z, U on M^n, where $'\breve{C}(X,Y,Z,U) = g(\breve{C}(X,Y)Z,U)$. Using equations (3.17) and (3.23) in (3.22), we can find

$$(n-2)\{'C(X,Y,Z,U) + {}'\breve{C}(X,Y,Z,U)\} = -2\,'R(X,Y,Z,U). \tag{3.24}$$

For the sufficient part, we suppose that the Riemannian manifold (M^n, g) equipped with a semisymmetric nonmetric connection $\tilde{\nabla}$ satisfies relation (3.24). Thus, we have the following from equations (3.1), (3.21), (3.23), and (3.24):

$$\begin{aligned} &S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY \\ &= (n-1)\{\theta(Y,Z)X - \theta(X,Z)Y - (\theta(Y,X) - \theta(X,Y))\,Z\}. \end{aligned} \tag{3.25}$$

Contracting equation (3.25) along the vector field Z, we obtain $\theta(X,Y) = \theta(Y,X)$. Hence, the proof of Theorem 3.6 is completed. □

Corollary 3.7 *If an n-dimensional Riemannian manifold (M^n, g), $n > 2$, admits a semisymmetric nonmetric connection $\tilde{\nabla}$ whose curvature tensor vanishes and whose 1-form π is closed, then*

$$(n-2)'L(X,Y,Z,U) + nR(X,Y,Z,U) = 0.$$

Proof It is obvious that a Riemannian manifold (M^n, g) satisfies

$$'C(X,Y,Z,U) + {}'\breve{C}(X,Y,Z,U) = {}'L(X,Y,Z,U) + {}'R(X,Y,Z,U), \tag{3.26}$$

where $'L$ is a conharmonic curvature tensor of type $(0,4)$ [9] defined by

$$\begin{aligned} 'L(X,Y,Z,U) = {}'R(X,Y,Z,U) - \frac{1}{n-2}\{&S(Y,Z)g(X,U) \\ -S(X,Z)g(Y,U) + g(Y,Z)S(X,U) - &g(X,Z)S(Y,U)\}. \end{aligned} \tag{3.27}$$

In view of (3.24) and (3.26), we get the statement of Corollary 3.7. □

4. Group manifolds with respect to the semisymmetric nonmetric connection

An n-dimensional Riemannian manifold (M^n, g) endowed with a semisymmetric nonmetric connection $\tilde{\nabla}$ is said to be a group manifold [14] if

$$(\tilde{\nabla}_X \tilde{T})(Y,Z) = 0 \quad \textit{and} \quad \tilde{R}(X,Y)Z = 0 \tag{4.1}$$

for arbitrary vector fields X, Y, and Z on M^n. In consequence of (2.14) and the first part of (4.1), we conclude that

$$(\tilde{\nabla}_X\pi)(Z)Y - (\tilde{\nabla}_X\pi)(Y)Z = 0,$$

which shows that

$$(\tilde{\nabla}_X\pi)(Y) = 0 \Longleftrightarrow (\nabla_X\pi)(Y) = \pi(X)\pi(Y) - g(X,Y)\pi(P), \tag{4.2}$$

where equation (2.11) is used and $n > 1$. Also, from equations (3.1) and (4.1), we have

$$R(X,Y)Z = \frac{1}{4}\{\pi(Y)\pi(Z)X - \pi(X)\pi(Z)Y\} - \frac{\pi(P)}{2}\{g(Y,Z)X - g(X,Z)Y\}. \tag{4.3}$$

Contracting (4.3) along the vector field X, we find that

$$S(Y,Z) = \frac{(n-1)}{4}\left[\pi(Y)\pi(Z) - 2g(Y,Z)\pi(P)\right], \tag{4.4}$$

which is equivalent to

$$QY = \frac{(n-1)}{4}\left[\pi(Y)P - 2\pi(P)Y\right]. \tag{4.5}$$

Changing Z with P in (4.4) and using (1.2) in it, we find that

$$S(Y,P) = -\frac{n-1}{4}\pi(P)g(Y,P).$$

This equation gives us the following proposition:

Proposition 4.1 *Let an n-dimensional group manifold (M^n, g), $n > 1$, admit a semisymmetric nonmetric connection $\tilde{\nabla}$. Then $-\frac{n-1}{4}\pi(P)$ is an eigenvalue of the Ricci tensor S corresponding to the eigenvector P.*

Also contracting (4.5) along Y, we get

$$r = -\frac{(n-1)(2n-1)\pi(P)}{4}. \tag{4.6}$$

Using equations (4.3) and (4.4) in (3.15), we have $\mathcal{P} = 0$. Hence, we are in a position to state the following theorem:

Theorem 4.2 *Every group manifold (M^n, g), $n > 1$, endowed with a semisymmetric nonmetric connection $\tilde{\nabla}$ is projectively flat.*

We will also prove the following theorems.

Theorem 4.3 *An n-dimensional group manifold (M^n, g) equipped with a semisymmetric nonmetric connection $\tilde{\nabla}$ is P-conformally flat.*

Proof In consequence of equations (4.3), (4.4), (4.5), and (4.6), equation (3.20) assumes the form

$$C(X,Y)Z = \frac{\pi(P)}{4(n-2)}[g(Y,Z)X - g(X,Z)Y] - \frac{1}{4(n-2)}\{\pi(Y)X \\ -\pi(X)Y\}\pi(Z) - \frac{n-1}{4(n-2)}[\pi(Y)g(X,Z) - \pi(X)g(Y,Z)]P. \tag{4.7}$$

An n-dimensional Riemannian manifold (M^n, g) is said to be P-conformally flat [15] if its nonvanishing conformal curvature tensor C satisfies $C(X,Y)P = 0$ for all vector fields X and Y on M^n. Replacing Z by P in (4.7), we can easily conclude that $C(X,Y)P = 0$. Hence, the statement of Theorem 4.3 is verified. □

Theorem 4.4 *Every Ricci-symmetric group manifold (M^n, g) endowed with a semisymmetric nonmetric connection $\tilde{\nabla}$ satisfies $\pi(P) = 0$.*

Proof Let (M^n, g) be an n-dimensional group manifold equipped with a semisymmetric nonmetric connection $\tilde{\nabla}$. The covariant derivative of (4.4) gives

$$(\nabla_X S)(Y,Z) = \frac{n-1}{4}[(\nabla_X \pi)(Y)\pi(Z) + \pi(Y)(\nabla_X \pi)(Z) \\ -2g(Y,Z)(\nabla_X \pi)(P) - 2g(Y,Z)\pi(\nabla_X P)], \tag{4.8}$$

which becomes

$$(\nabla_X S)(Y,Z) = \frac{n-1}{4}\{2\pi(X)\pi(Y)\pi(Z) - [\pi(Y)g(X,Z) + \pi(Z)g(X,Y)]\}, \tag{4.9}$$

where equation (4.2) is used.

A Riemannian manifold (M^n, g) of dimension n is said to be Ricci symmetric if and only if $\nabla S = 0$. If possible, we suppose that the group manifold (M^n, g) is Ricci-symmetric, and then the last equation gives $\pi(P) = 0$. Hence, the statement of Theorem 4.4 is proved. □

Theorem 4.5 *Suppose (M^n, g) is a group manifold of dimension $n (> 1)$ endowed with a semisymmetric nonmetric connection $\tilde{\nabla}$. A Ricci soliton (g, P, λ) on (M^n, g) to be shrinking, steady, and expanding according as $\pi(P)$ is $<$, $=$, and > 0, respectively.*

Proof If (M^n, g) is a group manifold equipped with a semisymmetric nonmetric connection $\tilde{\nabla}$, then equation (4.2) and Proposition 2.5 give

$$(\mathfrak{L}_P g)(X,Y) = 2\{\pi(X)\pi(Y) - g(X,Y)\pi(P)\} \tag{4.10}$$

for arbitrary vector fields X and Y on M^n.

A triplet (g, V, λ) on an n-dimensional Riemannian manifold (M^n, g) is said to be a Ricci soliton [7] if it satisfies the relation

$$\mathfrak{L}_V g + 2S + 2\lambda g = 0, \tag{4.11}$$

where V is a complete vector field on M^n and λ is a real constant. A Ricci soliton (g, V, λ) on (M^n, g) is said to be shrinking, steady, and expanding if λ is negative, zero, and positive, respectively. Changing V with P in (4.11) and then using equations (4.4) and (4.10), we find that

$$(n+3)\pi(X)\pi(Y) - 2(n+1)\pi(P)g(X,Y) + 4\lambda g(X,Y) = 0 \tag{4.12}$$

for arbitrary vector fields X and Y on M^n. Setting $Y = P$ in (4.12), we get

$$\{\lambda - \frac{(n-1)}{4}\pi(P)\}\pi(X) = 0,$$

which shows that $\lambda = \frac{(n-1)}{4}\pi(P)$ because $\pi(X) \neq 0$ on M^n (in general). In view of the last expression, we can easily observe that the Ricci soliton (g, P, λ) on M^n is shrinking, steady, and expanding if $\pi(P)$ $<, =,$ and > 0, respectively. Thus, the statement of Theorem 4.5 is satisfied. □

5. Submanifold of a Riemannian manifold with respect to the semisymmetric nonmetric connection

Let M^{n-2} be an $(n-2)$-dimensional submanifold of an n-dimensional Riemannian manifold M^n. Suppose $i : M^{n-2} \to M^n$ is an inclusion map such that for each $p \in M^{n-2} \implies pi \in M^n$. The inclusion map i induces a Jacobian map $J : T(M^{n-2}) \to T(M^n)$, where $T(M^{n-2})$ and $T(M^n)$ denote the tangent spaces to M^{n-2} at i and M^n at pi, respectively. Let G be a metric tensor of M^n and g be an induced metric tensor of the submanifold M^{n-2} at pi and i, respectively. Then we have

$$G(JX, JY) \circ p = g(X,Y), \quad \forall X, Y \in T(M^{n-2}).$$

Let N_1 and N_2 be two mutually orthogonal unit normal vector fields to the submanifold M^{n-2} satisfying the following relations:

$$\begin{aligned} &a) \quad G(JX, N_1) = G(JX, N_2) = G(N_1, N_2) = 0,\\ &b) \quad G(N_1, N_1) = G(N_2, N_2) = 1. \end{aligned} \tag{5.1}$$

Let ∇^* be the induced connection on M^{n-2} corresponding to the Levi-Civita connection ∇ of M^n. Then we can write

$$\nabla_{JX} JY = J(\nabla^*_X Y) + h_1(X,Y)N_1 + h_2(X,Y)N_2 \tag{5.2}$$

for all vector fields X and Y on M^{n-2}. Here h_1 and h_2 denote the second fundamental tensors of the submanifold M^{n-2}. Let $\tilde{\nabla}^*$ be the induced connection of the submanifold M^{n-2} corresponding to the semisymmetric nonmetric connection $\tilde{\nabla}$ of M^n defined as (2.1). Then for the unit normal vectors N_1 and N_2, we have

$$\tilde{\nabla}_{JX} JY = J(\tilde{\nabla}^*_X Y) + \mu_1(X,Y)N_1 + \mu_2(X,Y)N_2 \tag{5.3}$$

for arbitrary vector fields X and Y of M^{n-2} and μ_1, μ_2 denote the tensor fields of type $(0,2)$ of the submanifold M^{n-2}. Now we are going the prove the following theorem:

Theorem 5.1 *The induced connection $\tilde{\nabla}^*$ on the submanifold M^{n-2} of the Riemannian manifold M^n endowed with a semisymmetric nonmetric connection $\tilde{\nabla}$ is also a semisymmetric nonmetric connection.*

Proof We have from (2.1)

$$\tilde{\nabla}_{JX}JY = (\nabla_{JX}JY) + \frac{1}{2}\{\pi(JY)JX - \pi(JX)JY\} \tag{5.4}$$

for arbitrary vector fields X and Y. In consequence of equations (5.2) and (5.3), equation (5.4) assumes the form

$$J(\tilde{\nabla}^*_X Y) + \mu_1(X,Y)N_1 + \mu_2(X,Y)N_2 = J(\nabla^*_X Y)$$
$$+h_1(X,Y)N_1 + h_2(X,Y)N_2 + \frac{1}{2}\{\pi(JY)JX - \pi(JX)JY\},$$

which gives

$$\tilde{\nabla}^*_X Y = \nabla^*_X Y + \frac{1}{2}\{\pi(Y)X - \pi(X)Y\}, \tag{5.5}$$

$$a)\;\; h_1(X,Y) = \mu_1(X,Y) \quad and \quad b)\;\; h_2(X,Y) = \mu_2(X,Y). \tag{5.6}$$

Thus, the induced connections $\tilde{\nabla}^*$ and ∇^* on M^{n-2} corresponding to the semisymmetric nonmetric and Levi-Civita connections of the Riemannian manifold M^n are connected by equation (5.5). The torsion tensor T^* of $\tilde{\nabla}^*$ is defined by

$$T^*(X,Y) = \tilde{\nabla}^*_X Y - \tilde{\nabla}^*_Y X - [X,Y] = \pi(Y)X - \pi(X)Y,$$

where equation (5.5) is used. Thus, the induced connection $\tilde{\nabla}^*$ of the submanifold M^{n-2} is semisymmetric. Next, we have to prove that the connection $\tilde{\nabla}^*$ is nonmetric; that is, $\tilde{\nabla}^* g \neq 0$. We have

$$X\,g(Y,Z) = (\tilde{\nabla}^*_X g)(Y,Z) + g(\tilde{\nabla}^*_X Y, Z) + g(Y, \tilde{\nabla}^*_X Z)$$
$$= g(\nabla^*_X Y, Z) + g(Y, \nabla^*_X Z),$$

which reflects that $(\tilde{\nabla}^*_X g)(Y,Z) = \frac{1}{2}\{2\pi(X)g(Y,Z) - \pi(Y)g(X,Z) - \pi(Z)g(X,Y)\} \neq 0$. This shows that the induced connection $\tilde{\nabla}^*$ of the submanifold M^{n-2} corresponding to the semisymmetric nonmetric connection $\tilde{\nabla}$ is also semisymmetric nonmetric. Hence, the statement of the theorem is proved. □

Theorem 5.2 *Let M^{n-2} be a submanifold of the Riemannian manifold M^n. Then:*
(i) The mean curvatures of M^{n-2} corresponding to the induced connections $\tilde{\nabla}^*$ and ∇^* coincide.
(ii) The submanifold M^{n-2} will be totally geodesic with respect to $\tilde{\nabla}^*$ if and only if it is totally geodesic for ∇^*.
(iii) The submanifold M^{n-2} is totally umbilical with respect to $\tilde{\nabla}^*$ if and only if it is totally umbilical for ∇^*.
(iv) The submanifold M^{n-2} is minimal corresponding to $\tilde{\nabla}^*$ if and only if it is also minimal for ∇^*.

Proof We define

$$(\nabla^* J)(X,Y) = (\nabla^*_X J)(Y) = \nabla_{JX}JY - J(\nabla^*_X Y),$$

$$(\tilde{\nabla}^* J)(X,Y) = (\tilde{\nabla}^*_X J)(Y) = \tilde{\nabla}^*_{JX}JY - J(\tilde{\nabla}^*_X Y).$$

In view of equations (5.2) and (5.3), the above equations are considered in the forms

$$(\nabla^*_X J)(Y) = h_1(X,Y)N_1 + h_2(X,Y)N_2,$$

$$(\tilde{\nabla}^*_X J)(Y) = \mu_1(X,Y)N_1 + \mu_2(X,Y)N_2.$$

Let $\{e_1, e_2, ..., e_{n-2}\}$ be a set of $(n-2)$ orthonormal local vector fields in M^{n-2}. Then the mean curvature tensor $\mathcal{H}$ of the submanifold M^{n-2} with respect to the connection ∇^* is a function defined by $\mathcal{H} = \frac{1}{n-2}\sum_{i=1}^{n-2} h(e_i, e_i)$. Let $\mathcal{H}^* = \frac{1}{n-2}\sum_{i=1}^{n-2} h(e_i, e_i)$ denote the mean curvature of M^{n-2} with respect to the semisymmetric nonmetric induced connection $\tilde{\nabla}^*$. In particular, if $\mathcal{H} = 0$ on M^{n-2}, then the submanifold is said to be a minimal submanifold for ∇^*. Also, if $\mathcal{H}^* = 0$ on M^{n-2}, then the submanifold is said to be a minimal submanifold for $\tilde{\nabla}^*$.

On the other hand, the submanifold M^{n-2} is said to be totally geodesic with respect to the Levi-Civita connection ∇^* if and only if h_1 and h_2 vanish identically on M^{n-2}. If h_1 and h_2 are proportional to the metric g, i.e. $h_1 = \mathcal{H}g$ and $h_2 = \mathcal{H}g$, then the submanifold M^{n-2} is said to be totally umbilical with respect to the Levi-Civita connection ∇^*. In a similar fashion, we can say that the submanifold M^{n-2} is said to be totally umbilical with respect to the semisymmetric nonmetric induced connection $\tilde{\nabla}^*$ if μ_1 and μ_2 are proportional to g ($\mu_1 = \mathcal{H}^* g$ and $\mu_2 = \mathcal{H}^* g$). The statements of Theorem 5.2 are obvious from the above discussions and equation (5.6). $\square$

6. Example

Let

$$M^3 = \{(x,y,z) \in \mathbb{R}^3 : x, y, z(\neq 0) \in \mathbb{R}\}$$

be a three-dimensional differentiable manifold, where (x,y,z) denotes the standard coordinate of a point in $\mathbb{R}^3$. Let us suppose that

$$e_1 = e^{\alpha z}\frac{\partial}{\partial x},\ \ e_2 = e^{\alpha z}\frac{\partial}{\partial y},\ \ e_3 = \frac{\partial}{\partial z}$$

is a set of linearly independent vector fields at each point of the manifold M^3 and therefore it forms a basis for the tangent space $T(M^3)$. Here α is a positive real constant. We define a positive definite metric g on M^3 as

$$g_{ij} = \begin{cases} 2\alpha & \text{for} \quad i = j \\ 0 & \text{for} \quad i \neq j \end{cases},$$

where $i,j = 1,2,3$ and it is given by $g = 2\alpha\left[e^{-2\alpha z}\{dx \otimes dx + dy \otimes dy\} + dz \otimes dz\right]$. Let the 1-form π be defined by $\pi(X) = g(X,P)$, where $P = e_3$. Then it is obvious that (M^3, g) is a Riemannian manifold of dimension 3. The Lie brackets can be obtain by the above discussion as

$$[e_1, e_2] = 0, \qquad [e_1, e_3] = -\alpha e_1, \qquad [e_2, e_3] = -\alpha e_2.$$

With the help of the above results and Koszul's formula,

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y]), \end{aligned}$$

we find

$$\begin{aligned}
&\nabla_{e_1} e_1 = \alpha e_3, \qquad && \nabla_{e_1} e_2 = 0, \qquad && \nabla_{e_1} e_3 = -\alpha e_1,\\
&\nabla_{e_2} e_1 = 0, && \nabla_{e_2} e_2 = \alpha e_3, && \nabla_{e_2} e_3 = -\alpha e_2,\\
&\nabla_{e_3} e_1 = 0, && \nabla_{e_3} e_2 = 0, && \nabla_{e_3} e_3 = 0,
\end{aligned}$$

where ∇ denotes the Levi-Civita connection corresponding to the metric g. The nonvanishing components of the Riemannian curvature tensor and the Ricci tensor can be calculated by the formulae $R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$ and $S(X,Y) = \sum_{i=1}^{3} g(R(e_i, X)Y, e_i)$ as

$$\begin{aligned}
&R(e_1, e_2)e_1 = \alpha^2 e_2, \quad R(e_1, e_3)e_1 = \alpha^2 e_3, \quad R(e_1, e_2)e_2 = -\alpha^2 e_1,\\
&R(e_2, e_3)e_2 = \alpha^2 e_3, \quad R(e_1, e_3)e_3 = -\alpha^2 e_1, \quad R(e_2, e_3)e_3 = -\alpha^2 e_2,\\
&S(e_1, e_1) = S(e_2, e_2) = S(e_3, e_3) = -2\alpha^2
\end{aligned}$$

and other components can be easily obtained by the symmetric properties. It is obvious that the scalar curvature of (M^3, g) is $r = -6\alpha^2$.

In consequence of the above discussions and equation (2.1), we obtain

$$\begin{aligned}
&\tilde{\nabla}_{e_1} e_1 = \alpha e_3, \qquad && \tilde{\nabla}_{e_1} e_2 = 0, \qquad && \tilde{\nabla}_{e_1} e_3 = 0,\\
&\tilde{\nabla}_{e_2} e_1 = 0, && \tilde{\nabla}_{e_2} e_2 = \alpha e_3, && \tilde{\nabla}_{e_2} e_3 = 0,\\
&\tilde{\nabla}_{e_3} e_1 = -\alpha e_1, && \tilde{\nabla}_{e_3} e_2 = -\alpha e_2, && \tilde{\nabla}_{e_3} e_3 = 0.
\end{aligned}$$

In view of the above results we can easily prove that equation (1.1) holds for all vector fields e_i, $(i = 1, 2, 3)$, e.g., $\tilde{T}(e_1, e_3) = 2\alpha e_1$ and $\pi(e_3)e_2 - \pi(e_2)e_3 = 2\alpha e_1$. This shows that the linear connection $\tilde{\nabla}$ defined as (2.1) is a semisymmetric connection on (M^3, g). Also,

$$(\tilde{\nabla}_{e_1} g)(e_1, e_3) = -2\alpha^2 \neq 0.$$

Similarly, we can verify this for other components. Hence, the semisymmetric connection $\tilde{\nabla}$ is nonmetric on (M^3, g). It is not hard to prove that the Levi-Civita connection ∇ satisfies

$$(\nabla_X \pi)(Y) = (\nabla_Y \pi)(X)$$

for all $X, Y = e_i$ $(i = 1, 2, 3)$ on (M^3, g). Thus, the manifold (M^3, g) is closed with respect to the Levi-Civita connection ∇. Again by considering $(\tilde{\nabla}_X \pi)(Y) = X\pi(Y) - g(\tilde{\nabla}_X Y, P)$, we observe that

$$(\tilde{\nabla}_X \pi)(Y) = (\tilde{\nabla}_Y \pi)(X)$$

holds for all $X, Y = e_i$ on (M^3, g). Hence, the 1-form is closed on (M^3, g) with respect to the semisymmetric nonmetric connection $\tilde{\nabla}$. This verifies Proposition 2.2.

Let X, Y, Z be vector fields of M^3. Then it can be expressed as a linear combination of e_1, e_2, and e_3; that is,

$$X = X^1 e_1 + X^2 e_2 + X^3 e_3,\ Y = Y^1 e_1 + Y^2 e_2 + Y^3 e_3,\ \textit{and}\ Z = Z^1 e_1 + Z^2 e_2 + Z^3 e_3,$$

where X^i, Y^i, and Z^i, $i=1,2,3$, are real constants. We have

$$\begin{aligned}'\tilde{T}(X,Y,Z) &= g(\tilde{T}(X,Y),Z)\\ &= 4\alpha^2\left[(X^1Y^3-X^3Y^1)Z^1+(Y^3X^2-X^3Y^2)Z^2\right],\end{aligned}$$

$$'\tilde{T}(Y,X,Z) = -4\alpha^2\left[(X^1Y^3-X^3Y^1)Z^1+(Y^3X^2-X^3Y^2)Z^2\right],$$

$$'\tilde{T}(Y,Z,X) = 4\alpha^2\left[(Y^1Z^3-Y^3Z^1)X^1+(Z^3Y^2-Y^3Z^2)X^2\right],$$

$$'\tilde{T}(Z,X,Y) = 4\alpha^2\left[(Z^1X^3-Z^3X^1)Y^1+(X^3Z^2-Z^3X^2)Y^2\right].$$

From the above relations, we conclude that $'\tilde{T}(X,Y,Z)+'\tilde{T}(Y,X,Z)=0$ and $'\tilde{T}(X,Y,Z)+'\tilde{T}(Y,Z,X)+'\tilde{T}(Z,X,Y)=0$. Therefore, Theorem 2.3 is verified.

Straightforward calculations by considering the above facts and $\tilde{R}(X,Y)Z=\tilde{\nabla}_X\tilde{\nabla}_YZ-\tilde{\nabla}_Y\tilde{\nabla}_XZ-\tilde{\nabla}_{[X,Y]}Z$ reveal that

$$\tilde{R}(e_i,e_j)e_k=0,\ \ for\ all\ \ i,j,k=1,2,3.$$

That is, the Riemannian manifold (M^n,g) equipped with a semisymmetric nonmetric connection $\tilde{\nabla}$ is flat. The curvature tensor with respect to the Levi-Civita connection ∇ is

$$\begin{aligned}R(X,Y)Z &= \alpha^2\{\left[-X^1Y^2Z^2-X^1Y^3Z^3+X^2Y^1Z^2+X^3Y^1Z^3\right]e_1\\ &\quad+\left[X^1Y^2Z^1-X^2Y^1Z^1-X^2Y^3Z^3+X^3Y^2Z^3\right]e_2\\ &\quad+\left[X^1Y^3Z^1+X^2Y^3Z^2-X^3Y^1Z^1-X^3Y^2Z^2\right]e_3\}.\end{aligned}$$

Also,

$$\begin{aligned}S(Y,Z)X-S(X,Z)Y &= -2\alpha^2\{[X^1Z^1+X^2Z^2+X^3Z^3][Y^1e_1+Y^2e_2+Y^3e_3]\\ &\quad-[Y^1Z^1+Y^2Z^2+Y^3Z^3][X^1e_1+X^2e_2+X^3e_3]\}.\end{aligned}$$

In consequence of the last two equations and (3.15), we get $\mathcal{P}=0$. That is, the manifold (M^3,g) is projectively flat. Thus, Theorem 3.5 is verified.

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