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Fixed point properties for a degenerate Lorentz–Marcinkiewicz space

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Abstract: We construct an equivalent renorming of ℓ^1, which turns out to produce a degenerate ℓ^1-analog Lorentz–Marcinkiewicz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}} = (2, 1, 1, 1, \cdots)$ is a decreasing positive sequence in $\ell^\infty \backslash c_0$, rather than in $c_0\backslash\ell^1$ (the usual Lorentz situation). Then we obtain its isometrically isomorphic predual $\ell^0_{\delta,\infty}$ and dual $\ell_{\delta,\infty}$, corresponding degenerate c_0-analog and ℓ^∞-analog Lorentz–Marcinkiewicz spaces, respectively. We prove that both spaces $\ell_{\delta,1}$ and $\ell^0_{\delta,\infty}$ enjoy the weak fixed point property (w-fpp) for nonexpansive mappings yet they fail to have the fixed point property (fpp) for nonexpansive mappings since they contain an asymptotically isometric copy of ℓ^1 and c_0, respectively. In fact, we prove for both spaces that there exist nonempty, closed, bounded, and convex subsets with invariant fixed point-free affine, nonexpansive mappings on them and so they fail to have fpp for affine nonexpansive mappings. Also, we show that any nonreflexive subspace of $l^0_{\delta,\infty}$ contains an isomorphic copy of c_0 and so fails fpp for strongly asymptotically nonexpansive maps. Finally, we get a Goebel and Kuczumow analogy by proving that there exists an infinite dimensional subspace of $\ell_{\delta,1}$ with fpp for affine nonexpansive mappings.

Key words: Nonexpansive mapping, nonreflexive Banach space, fixed point property, weak fixed point property, closed, bounded, convex set, asymptotically isometric copy of c_0, asymptotically isometric copy of ℓ^1, Lorentz–Marcinkiewicz spaces

1. Introduction and preliminaries

A Banach space is said to have the fixed point property (fpp) for nonexpansive mappings [fpp(ne)] if every nonexpansive invariant mapping defined on any nonempty closed, bounded, and convex subset has a fixed point. Wondering how nonexpansive mappings behave if smaller sets are picked, some other fixed point properties such as w-fpp and w*-fpp have became fixed point theorists' center of interest. A Banach space (its dual) is said to have w-fpp (w*-fpp) if every nonexpansive invariant mapping defined on any nonempty weakly (weak*, respectively) compact and convex subset has a fixed point. There have been many strategies to check whether or not a Banach space X (dual space X^*) possesses w-fpp (w*-fpp, respectively) and fpp, and different types of Banach spaces have been the subject of many papers in the last 50 or so years.

In 1965, Browder [4] gave the first example of the largest class of Banach spaces with fpp(ne) until then by proving that Hilbert spaces have fpp(ne). In the same year, Browder [5] and Göhde [14] each independently generalized this to all uniformly convex Banach spaces; for example, $X = L^p,\ 1 < p < \infty$, with its usual norm $\|\cdot\|_p$.

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Again in 1965, Kirk [16] generalized Browder's theorem to all reflexive Banach spaces with normal structure.

There is a strong relationship between the concept of reflexive Banach spaces and Banach spaces having fpp(ne). As has been mentioned above, it was showed that under the conditions of some geometric properties for Banach spaces such as uniform convexity or normal structure, reflexivity implies fpp(ne). Less contingent relations have become a matter of curiosity and the role of the norm of the Banach space in the complete equivalence of the two concepts has recently become the center of interest. It has been observed that most nonreflexive classical Banach spaces such as ℓ^1, Banach space of absolutely summable scalar sequences with its usual absolute summing norm, and c_0, the Banach space of real valued sequences converging to 0 with its usual absolute supremum norm, fail fpp(ne) and it is still and has been an open question for over 50 years whether or not all nonreflexive Banach spaces can be renormed to have the fixed point property.

In 2008, Lin [19] gave the first example of a nonreflexive space that can be renormed to have fpp(ne). He verified this fact by renorming ℓ^1.

Banach spaces containing nice copies (asymptotically isometric (ai) copies) of ℓ^1 or c_0 cannot have fpp(ne). Furthermore, if a Banach space is a Banach lattice or has an unconditional basis then it is nonreflexive if and only if it contains either an isomorphic copy of ℓ^1 or c_0 [12, 15, 21]. Because these two nonreflexive Banach spaces ℓ^1 and c_0 share many common properties and they are the fundamental examples of nonreflexive Banach spaces, in order to investigate whether or not nonreflexive Banach spaces can be renormed to have the fixed point property, investigating the question of the second example c_0 has become one of the most important subjects for researchers. In my recent joint paper with Mustafa [25], we gave a positive answer to this question under affinity condition. Unconditionally resolving this problem is still a research topic, and the reason for the difficulty of solving it is that c_0 does not offer adequate tools for researchers; on the contrary, the ℓ^1 space provides possibilities facilitating the research, such as having the Schur property or having weak Opial property, although it has many common properties with c_0. Thus, for the aim of solving the aforementioned important question, the c_0-analogue of Lin's result, or to construct bridges that can yield a solution method for that, due to the possibility of offering more tools, researchers have showed interest in working on ℓ^1-like and c_0-like spaces. In the literature, c_0-analogue and ℓ^1-analogue spaces have been considered, and fixed-point theory oriented properties have been questioned, even by constructing new equivalent norms on ℓ^1 again. By Gamboa de Buen and Nuñez-Medina [11], it has been showed that Lin's result is not the only example for nonreflexive Banach spaces. In their study, first of all, they constructed an equivalent norm on c_0 and obtained isometrically isomorph dual ℓ^1, and then they proved that there exists another equivalent norm on ℓ^1 with fpp(ne) by using Lin's properties [20] generalizing [19].

A Banach space, which is very similar to ℓ^1 and offers exactly the same features, but without a norm obtained only by a linear expansion of the usual one, is rarely included in the literature (even the renorming of Gamboa de Buen and Nuñez-Medina is different, too, since it has fpp(ne)). There are some similarities such as Orlicz spaces or Lebesgue space $(L^1[0,1], \|\cdot\|_1)$ but different properties are observed. For example, by Alspach's result [1], $L^1[0,1]$ does not have w-fpp, contrary to ℓ^1.

Here, we can note that Maurey [23] showed that $(c_0, \|\cdot\|_\infty)$ and reflexive subspaces of $L^1[0,1]$ do have w-fpp using ultrafilter techniques. Conversely to Maurey's result, Downling and Lennard [8] showed that every nonreflexive subspace of $L^1[0,1]$ fails the fixed point property. Before their result, in 1996, by Carothers et al. [6], the analogous result for nonreflexive subspaces of the Lorentz function space $L_{w,1}(0,\infty)$ was established, but even before that, in 1991, Carothers et al. [7] showed that the Lorentz space $L_{w,1}(\mu)$ enjoys the weak* fixed point property if (X, Σ, μ) is a σ-finite measure space; that is, if C is a weak-star compact convex subset of $L_{w,1}(\mu)$, then every nonexpansive mapping on C has a fixed point.

One of Banach spaces offering the closest features to ℓ^1's is the ℓ^1-analogue Lorentz–Marcinkiewicz space $\ell_{w,1}$, where the weight sequence $w = (w_n)_{n\in\mathbb{N}}$ is a decreasing positive sequence in $c_0 \backslash \ell^1$ that I studied in my PhD thesis [24], written under the supervision of Chris Lennard, and its fixed point theory oriented properties were studied, but due to troubles inhered by complexity of its norm depending on its weight sequence's approaching to 0, it has not been proven whether there are completely similar features even in this space.

In this paper, as a result of valuable discussions with Lennard*, by constructing an equivalent renorming of ℓ^1, we produce a degenerate ℓ^1-analog Lorentz–Marcinkiewicz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}} = (2, 1, 1, 1, \cdots)$ is a decreasing positive sequence in $\ell^\infty \backslash c_0$, rather than in $c_0 \backslash \ell^1$ (the usual Lorentz situation). Then we obtain its isometrically isomorphic predual $\ell^0_{\delta,\infty}$ and dual $\ell_{\delta,\infty}$, corresponding degenerate c_0-analog and ℓ^∞-analog Lorentz–Marcinkiewicz spaces, respectively. We prove that both Banach spaces $\ell_{\delta,1}$ and $\ell^0_{\delta,\infty}$ enjoy w-fpp(ne) yet they fail to have fpp(ne) since, respectively, they contain an ai copy of ℓ^1 and c_0. In fact, we prove that for both spaces, there exist nonempty, closed, bounded, and convex subsets with invariant fixed point-free affine, nonexpansive mappings on them and so they fail to have fpp for affine nonexpansive mappings. Finally, we prove that there exists an infinite dimensional subspace of $\ell_{\delta,1}$ with fpp(ne) for affine mappings so we obtain an analogue result of Goebel and Kuczumow's on ℓ^1 [13] under affinity condition.

Throughout the study, we denote the set of all positive integers and the set of all real numbers by $\mathbb{N}$ and $\mathbb{R}$, respectively. Throughout this paper our scalar field is $\mathbb{R}$.

Let $(X, \|\cdot\|)$ be a Banach space and $E \subseteq X$. We will denote the convex hull of E by $\text{co}\,(E)$.

As usual, we define the Banach space $(c_0, \|\cdot\|_\infty)$ by the vector space of all scalar sequences converging to 0 such that $\|x\|_\infty := \sup_{n\in\mathbb{N}} |x_n|$, for all $x = (x_n)_{n\in\mathbb{N}} \in c_0$; and $(\ell^1, \|\cdot\|_1)$ is the vector space of all absolutely summable scalar sequences such that $\|x\|_1 := \sum_{n=1}^{\infty} |x_n|$ for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$.

Definition 1.1 *Assume that $(X, \|\cdot\|)$ is a Banach space and K is a nonempty closed, bounded, convex subset.*

1. *If $U : K \longrightarrow K$ is a mapping such that for all $\lambda \in [0,1]$ and for all $x, y \in K$, $U\big((1-\lambda)\,x + \lambda\, y\big) = (1-\lambda)\,U(x) + \lambda\, U(y)$, then U is said to be an affine mapping.*

2. *If $U : K \longrightarrow K$ is a mapping such that $\|U(x) - U(y)\| \le \|x - y\|$, for all $x, y \in K$ then U is said to be a nonexpansive mapping.*

   Also, if for every nonexpansive mapping $U : K \longrightarrow K$, there exists $z \in K$ with $U(z) = z$, then K is said to have the fixed point property for nonexpansive mappings [fpp(ne)].

3. *If $U : K \longrightarrow K$ is a mapping such that $\exists \{\beta_{n,m} : n, m \in \mathbb{N}, n \ge m \ge 0\} \subseteq [1, \infty)$ such that [$\forall x, y \in K$ and $\forall\, n \ge m$, $\|U^n x - U^n y\| \le \beta_{n,m} \|U^m x - U^m y\|$], [$\beta_{n,m} \to 1$ as $n \ge m \to \infty$], and [$\beta_{n,m} \to 1$ as $n \to \infty$, $\forall m$], then U is said to be a strongly asymptotically nonexpansive [17].*

*Lennard CJ. Personal communication, 2017.

We should note that in our PhD thesis [24], written under the supervision of Chris Lennard, we studied the usual Lorentz–Marcinkiewicz spaces and their fixed point properties; hence, we can give their definitions below to understand how different the degenerate ones are.

Let $w \in (c_0 \setminus \ell^1)^+$, $w_1 = 1$ and $(w_n)_{n\in\mathbb{N}}$ be decreasing; that is, consider a scalar sequence given by $w = (w_n)_{n\in\mathbb{N}}$, $w_n > 0, \forall n \in \mathbb{N}$ such that $1 = w_1 \geq w_2 \geq w_3 \geq \cdots \geq w_n \geq w_{n+1} \geq \ldots, \forall n \in \mathbb{N}$ with $w_n \longrightarrow 0$ as $n \longrightarrow \infty$ and $\sum_{n=1}^{\infty} w_n = \infty$. This sequence is called a weight sequence. For example, $w_n = \frac{1}{n}, \forall n \in \mathbb{N}$.

Definition 1.2

$$l_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \left| \|x\|_{w,\infty} := \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^{n} {x_j}^{\star}}{\sum_{j=1}^{n} w_j} < \infty \right. \right\} .$$

Here, $x^{\star}$ represents the decreasing rearrangement of sequence x, which is the sequence of $|x| = (|x_j|)_{j\in\mathbb{N}}$, arranged in nonincreasing order, followed by infinitely many zeros when $|x|$ has only finitely many nonzero terms.

This space is nonseparable and an analogue of l_∞ space.

Definition 1.3

$$l^0_{w,\infty} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \left| \limsup_{n\longrightarrow\infty} \frac{\sum_{j=1}^{n} {x_j}^{\star}}{\sum_{j=1}^{n} w_j} = 0 \right. \right\} .$$

This is a separable subspace of $l_{w,\infty}$ and an analogue of c_0 space.

Definition 1.4

$$l_{w,1} := \left\{ x = (x_n)_{n\in\mathbb{N}} \in c_0 \left| \|x\|_{w,1} := \sum_{j=1}^{\infty} w_j \, {x_j}^{\star} < \infty \right. \right\} .$$

This is a separable subspace of $l_{w,\infty}$ and an analogue of l_1 space with the following facts: $(l^0_{w,\infty})^{\star} \cong l_{w,1}$ and $(l_{w,1})^{\star} \cong l_{w,\infty}$, where the star denotes the dual of a space while $\cong$ denotes isometrically isomorphic.

More information about Lorentz spaces can be seen in [21].

We also need the following theorems and definitions to obtain our results.

Theorem 1.5 *[18] Let X be a Banach space. If X has an unconditional basis (e_n) with unconditional constant $\lambda < \frac{\sqrt{33}-3}{2}$, then X has the w-fpp.*

Theorem 1.6 *[17] If X is a Banach space containing an isomorphic copy of c_0, then X fails the fixed point property for affine strongly asymptotically nonexpansive mappings.*

Definition 1.7 *[9] We say that a Banach space $(X, \|\cdot\|)$ contains an ai copy of ℓ^1 if there exist a sequence $(x_n)_n$ in X and a null sequence $(\varepsilon_n)_n$ in $(0,1)$ so that*

$$\sum_{n=1}^{\infty} (1-\varepsilon_n)|a_n| \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq \sum_{n=1}^{\infty} |a_n| \ ,$$

for all $(a_n)_n \in \ell^1$.

Definition 1.8 *[9] We say that a Banach space* $(X, \|\cdot\|)$ *contains an ai copy of* c_0 *if there exist a sequence* $(x_n)_n$ *in* X *and a null sequence* $(\varepsilon_n)_n$ *in* $(0,1)$ *so that*

$$\sup_n (1-\varepsilon_n)|a_n| \leq \left\| \sum_{n=1}^{\infty} a_n x_n \right\| \leq \sup_n |a_n| \ ,$$

for all $(a_n)_n \in c_0$.

Theorem 1.9 *[9] If a Banach space* $(X, \|\cdot\|)$ *contains an ai copy of* ℓ^1 *or an ai copy of* c_0*, then* X *fails fpp(ne).*

2. Producing some degenerate Lorentz–Marcinkiewicz spaces $\ell_{\delta,1}$, $\ell^0_{\delta,\infty}$, and $\ell_{\delta,\infty}$, where the weight sequence is in $\ell^\infty \backslash c_0$

For all $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$, we define

$$|||x||| := \|x\|_1 + \|x\|_\infty = \sum_{n=1}^{\infty} |x_n| + \sup_{n\in\mathbb{N}} |x_n| \ .$$

Clearly $|||\cdot|||$ is an equivalent norm on ℓ^1 with $\|x\|_1 \leq |||x||| \leq 2\|x\|_1, \ \forall x \in \ell^1$.

We shall call $|||\cdot|||$ the $1 \boxplus \infty$-norm on ℓ^1.

Note that $\forall x \in \ell^1, \ |||x||| = 2x_1^* + x_2^* + x_3^* + x_4^* + \cdots$, where z^* is the decreasing rearrangement of $|z| = (|z_n|)_{n\in\mathbb{N}}, \ \forall z \in c_0$.

Let $\delta_1 := 2, \delta_2 := 1, \delta_3 := 1, \cdots, \delta_n := 1, \ \forall n \geq 4$.

We see that $(\ell^1, |||\cdot|||)$ is a (degenerate) Lorentz space $\ell_{\delta,1}$, where the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}}$ is a decreasing positive sequence in $\ell^\infty \backslash c_0$, rather than in $c_0 \backslash \ell^1$ (the usual Lorentz situation).

This suggests that $\ell^0_{\delta,\infty} = (c_0, \|\cdot\|)$ is an isometric predual of $(\ell^1, |||\cdot|||)$ where for all $z \in c_0$,

$$\|z\| := \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^{n} z_j^*}{\sum_{j=1}^{n} \delta_j} = \sup_{n\in\mathbb{N}} \frac{1}{n+1} \sum_{j=1}^{n} z_j^*.$$

There is a way to rewrite $\|z\|$ without using decreasing rearrangements of $|z|$. This may help with calculations involving this norm.

Fix $z \in c_0$, arbitrary.

$$\|z\| = \sup_{n\in\mathbb{N}} \frac{1}{n+1} \sum_{j=1}^{n} z_j^*.$$

Note that $\forall n \in \mathbb{N}, \ \sum_{j=1}^{n} z_j^* = \sup_{\substack{K\subseteq\mathbb{N} \\ \#(K)=n}} \sum_{i\in K} |z_i|$, where $\#(K)$ is the number of elements in K for all finite subsets $K \subseteq \mathbb{N}$.

Thus,

$$\begin{aligned}\|z\| &= \sup_{n\in\mathbb{N}} \frac{1}{n+1} \sup_{\substack{K\subseteq\mathbb{N}\\ \#(K)=n}} \sum_{i\in K} |z_i| \\ &= \sup_{n\in\mathbb{N}} \sup_{\substack{K\subseteq\mathbb{N}\\ \#(K)=n}} \frac{1}{\#(K)+1} \sum_{i\in K} |z_i|.\end{aligned}$$

Hence, for all $z \in c_0$,

$$\|z\| = \sup_{\substack{\emptyset\neq K\subseteq\mathbb{N}\\ \#(K)<\infty}} \frac{1}{\#(K)+1} \sum_{i\in K} |z_i|.$$

Also, noting that formula 2.1 can be extended to ℓ^∞: $\forall w = (w_i)_{i\in\mathbb{N}} \in \ell^\infty$, we define

$$\|w\| := \sup_{\substack{\emptyset\neq K\subseteq\mathbb{N}\\ \#(K)<\infty}} \frac{1}{\#(K)+1} \sum_{i\in K} |w_i|.$$

Lemma 2.1 *Dual space of* $(\ell^1, |||\cdot|||)$ *is isometrically isomorphic to* $(\ell^\infty, \|\cdot\|)$*; i.e.* $(\ell^1, |||\cdot|||)^* \cong (\ell^\infty, \|\cdot\|)$.

Proof First of all, considering canonical basis $(e_n)_{n\in\mathbb{N}}$, define the sequence $(Q_n)_{n\in\mathbb{N}}$ in ℓ^1 by $Q_1 := \frac{1}{2}e_1$, $Q_2 := \frac{1}{3}(e_2+e_3)$, $Q_3 := \frac{1}{4}(e_4+e_5+e_6)$, $Q_4 := \frac{1}{5}(e_7+e_8+e_9+e_{10}), \cdots$ and note that $|||Q_n||| = 1,\ \forall n \in \mathbb{N}$, and that for any $x := (\xi_n)_{n\in\mathbb{N}} \in \ell^1$, there is unique representation of x with scalars $\alpha_1 = 2\xi_1,\ \alpha_2 = 3\xi_2,\ \alpha_3 = 3\xi_3,\ \alpha_4 = 4\xi_4,\ \alpha_5 = 4\xi_5,\ \alpha_6 = 4\xi_6,\ \alpha_7 = 5\xi_7,\ \alpha_8 = 7\xi_8,\ \alpha_9 = 5\xi_9,\ \alpha_7 = 5\xi_{10},\ \cdots$ such that $x = \sum_{n=1}^{\infty} \alpha_n Q_n$.

Now, consider any $f \in {\ell^1}^*$. Then, since f is linear and bounded, $f(x) = \sum_{n=1}^{\infty} \alpha_n f(Q_n) = \sum_{n=1}^{\infty} \alpha_n \gamma_n,\ \gamma_n := f(Q_n),\ \forall n \in \mathbb{N}$.

Thus, $(\gamma_n)_{n\in\mathbb{N}} \in \ell^\infty$.

Also, $\sup_{\substack{\emptyset\neq K\subseteq\mathbb{N}\\ \#(K)<\infty}} \frac{1}{\#(K)+1} \sum_{i\in K} |\gamma_i| \le \sup_{k\in\mathbb{N}} |\gamma_k| \le \|f\|_* = \sup_{\substack{x\in\ell^1\\ |||x|||=1}} |f(x)|$. ♡ where $\|\cdot\|_*$ denotes the operator norm.

Furthermore, if $(\gamma_n)_{n\in\mathbb{N}} \in \ell^\infty$ is given arbitrarily, we can construct linear functionals in ${\ell^1}^*$ with the elements of ℓ^∞ as follows:

$$g(x) = \sum_{k=1}^{\infty} \alpha_k \gamma_k.$$

Showing linearity is no problem, and for the boundedness of the linear functional g, we will use the inequality 5.2(5) in [22, page 52]. Then we have

$$\begin{aligned}|g(x)| &\leq \sum_{k=1}^{\infty}|\alpha_k\gamma_k| \\ &\leq \sup_{\substack{\emptyset\neq K\subseteq\mathbb{N}\\ \#(K)<\infty}} \frac{1}{\#(K)+1}\sum_{i\in K}|\gamma_i|\sum_{k=1}^{\infty}\alpha_k^* \\ &\leq \sup_{\substack{\emptyset\neq K\subseteq\mathbb{N}\\ \#(K)<\infty}} \frac{1}{\#(K)+1}\sum_{i\in K}|\gamma_i|\,|||x|||\,.\end{aligned}$$

Thus, indeed the functional g is bounded, and by the linearity, $g \in {\ell^1}^*$. Also, the above inequality says that for any $f \in {\ell^1}^*$ and for any $x \in \ell^1$,

$$\|f\|_* = \sup_{|||x|||=1}|f(x)| \leq \sup_{\substack{\emptyset\neq K\subseteq\mathbb{N}\\ \#(K)<\infty}} \tfrac{1}{\#(K)+1}\sum_{i\in K}|\gamma_i| = \|(\gamma_k)_{k\in\mathbb{N}}\|. \quad \heartsuit\heartsuit$$

Hence, $\heartsuit$ together with $\heartsuit\heartsuit$ tells us that the norm is preserved; i.e., $\|f\|_* = \|(\gamma_k)_{k\in\mathbb{N}}\|$. Therefore, the isomorphism between the two given normed spaces ${\ell^1}^*$ and ℓ^∞ is a fact. Taking an element out of ${\ell^1}^*$ is thus in certain sense the same as speaking about an element out of ℓ^∞. □

Remark 2.2 $B_{(\ell^\infty,\|\cdot\|_\infty)} = \{x \in \ell^\infty : \|x\|_\infty \leq 1\}$ *has the fixed point property for nonexpansive mappings since it is a hyperconvex metric space and hyperconvex metric spaces have fpp(ne) by Soardi [29].*

3. Fixed point properties for $l^0_{\delta,\infty}$

Here we take the degenerate c_0-analog Lorentz–Marcinkiewicz space $l^0_{\delta,\infty}$ and begin our section by showing that $\ell^0_{\delta,\infty}$ has w-fpp. Then, in the following two subsections, first we show that $\ell^0_{\delta,\infty}$ contains an ai copy of c_0 and so it fails to have fpp(ne), but in the next subsection, in fact, we see that it fails to have fpp for affine nonexpansive mappings. Then, in the final subsection, we show that any nonreflexive vector subspace of $l^0_{\delta,\infty}$ contains an isomorphic copy of c_0 and so fails fpp for strongly asymptotically nonexpansive maps.

3.1. $\ell^0_{\delta,\infty}$ has w-fpp

Theorem 3.1 *Let* $X := l^0_{\delta,\infty}$ *and let* $\|\cdot\| = \|\cdot\|_{\delta,\infty}$ *as we discussed in the above notes. Then Banach space* $(X, \|\cdot\|)$ *has the weak fixed point property.*

Proof To prove this theorem, we will use Theorem 1.5. First, we need to recall the definition of the unconditional constant of an unconditional basis $(u_n)_{n\in\mathbb{N}}$ and for this we can assume the sequence $(u_n)_{n\in\mathbb{N}}$ is normalized; i.e. $\|u_n\| = 1, \forall n \in \mathbb{N}$. Then $(u_n)_{n\in\mathbb{N}}$ is said to be an unconditional basis if for every convergent series $\sum_{j=1}^{\infty} s_j\, u_j$ (where $(s_j)_{j\in\mathbb{N}}$ is a sequence of scalars) and for every sequence of signs $(\varepsilon_n)_{n\in\mathbb{N}}(\varepsilon_n = \pm 1)$ there exists a constant $\lambda \geq 1$ such that

$$\left\|\sum_{j=1}^{\infty} s_j\,\varepsilon_j\,u_j\right\| \leq \lambda \left\|\sum_{j=1}^{\infty} s_j\,u_j\right\|.$$

Then, the unconditionality constant of the unconditional basis is the smallest λ satisfying the above condition; that is,

$$\lambda := \sup_{\substack{\varepsilon = (\varepsilon_n)_{n\in\mathbb{N}} \\ \varepsilon_n = \pm 1}} \frac{\| \sum_{j=1}^{\infty} s_j \, \varepsilon_j \, u_j \|}{\| \sum_{j=1}^{\infty} s_j \, u_j \|}.$$

Now, for our space, we have a normalized basis $(u_n)_{n\in\mathbb{N}} = ((n+1)e_n)_{n\in\mathbb{N}}$. Define $S_{\mathbb{N}} := \{\pi : \mathbb{N} \to \mathbb{N} \,|\pi \text{ 1-1 and onto map}\}$ set of all permutations. Then $\forall \varepsilon_j = \pm 1$, $\forall \pi \in S_{\mathbb{N}}$, and for any sequence of scalars $(s_j)_{j\in\mathbb{N}}$,

$$\left\| \sum_{n=1}^{\infty} \varepsilon_n \, s_{\pi(n)} \, u_n \right\|_{\delta,\infty} = \left\| \sum_{n=1}^{\infty} s_n \, u_n \right\|_{\delta,\infty} = \left\| \sum_{n=1}^{\infty} s_n^{\star} \, u_n \right\|_{\delta,\infty}.$$

Hence, $\lambda = 1 < \frac{\sqrt{33}-3}{2}$. □

3.2. $\ell^0_{\delta,\infty}$ contains an ai copy of c_0

Theorem 3.2 *$\forall \delta \in c_0 \setminus \ell^1$, $\exists Z \subseteq l^0_{\delta,\infty}$, such that Z is an ai copy of c_0 and so $l^0_{\delta,\infty}$ fails the fixed point property for affine, $\|\cdot\|_{\delta,\infty}$-nonexpansive mappings.*

Proof Let $\delta \in c_0 \setminus \ell^1$ be given. Fix $\varepsilon_j \downarrow_j 0, \varepsilon_j \in (0,1)$. Then choose a sequence $(r_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ so that $1 \leq r_1 < r_2 < r_3 < \dots$ and each term is large enough such that

$$\frac{\sum_{j=1}^{r_n} \delta_j}{\sum_{j=1}^{r_{n+1}} \delta_j} \leq \varepsilon_{n+1}, \forall n \in \mathbb{N} \;\; \left(\text{ since } \sum_{j=1}^{\infty} \delta_j = \infty\right),$$

i.e. increasing to ∞ fast enough so that

$$\frac{\sum_{j=1}^{r_n} \delta_j}{\sum_{j=1}^{r_{n+1}} \delta_j} \leq \varepsilon_{n+1}, \forall n \in \mathbb{N} \, .$$

Define $\eta_1 = g_1 = (\delta_1, \delta_2, \delta_3, \dots, \delta_{r_1}, 0, 0, 0, \dots)$.

Then notice $\|\eta_1\|_{\delta,\infty} = \|g_1\|_{\delta,\infty} = 1$, and next define

$$g_2 = (0, 0, 0, \dots, \underset{\substack{\uparrow \\ r_1 \;^{st} \text{ term},}}{0}, \delta_{r_1+1}, \delta_{r_1+2}, \dots, \delta_{r_2-1}, \delta_{r_2}, 0, 0, 0, \dots)$$

and then define $\eta_2 := g_1 + g_2$ and notice $\|\eta_2\|_{\delta,\infty} = \|g_1 + g_2\|_{\delta,\infty} = 1$, and next define

$$g_3 = (0, 0, 0, \dots, \underset{\substack{\uparrow \\ r_2 \;^{nd} \text{ term},}}{0}, \delta_{r_2+1}, \delta_{r_2+2}, \dots, \delta_{r_3-1}, \delta_{r_3}, 0, 0, 0, \dots)$$

and then define $\eta_3 := g_1 + g_2 + g_3$ and notice $\|\eta_3\|_{\delta,\infty} = \|g_1 + g_2 + g_3\|_{\delta,\infty} = 1$. Continuing in this way, we obtain a sequence $(g_n)_{n\in\mathbb{N}}$ and so $(\eta_n)_{n\in\mathbb{N}}$ such that $\|\eta_n\|_{\delta,\infty} = \|g_1+g_2+g_3+\cdots+g_n\|_{\delta,\infty} = 1, \forall n \in \mathbb{N}$. Then define the set

$$Z := \left\{ x = t_1 g_1 + t_2 g_2 + t_3 g_3 + \cdots = \sum_{n=1}^{\infty} t_n \, g_n \ \mid t = (t_n)_{n\in\mathbb{N}} \in c_0 \right\} .$$

Then let $t \in c_0$ and $x \in Z$ be arbitrarily given. Then,

$$\begin{aligned}
\|x\|_{\delta,\infty} &= \|t_1 g_1 + t_2 g_2 + t_3 g_3 + \ldots\|_{\delta,\infty} = \left\|\sum_{n=1}^{\infty} t_n g_n\right\|_{\delta,\infty} \\
&= \||t_1| g_1 + |t_2| g_2 + |t_3| g_3 + \ldots\|_{\delta,\infty} = \left\|\sum_{n=1}^{\infty} |t_n| g_n\right\|_{\delta,\infty} \\
&\leq \|\|t\|_\infty g_1 + \|t\|_\infty g_2 + \|t\|_\infty g_3 + \ldots\|_{\delta,\infty} = \left\|\sum_{n=1}^{\infty} \|t\|_\infty g_n\right\|_{\delta,\infty} \\
&= \|t\|_\infty \|g_1 + g_2 + g_3 + \ldots\|_{\delta,\infty} = \|t\|_\infty \left\|\sum_{n=1}^{\infty} g_n\right\|_{\delta,\infty} \\
&= \|t\|_\infty \|\delta\|_{\delta,\infty} \\
&= \|t\|_\infty.
\end{aligned}$$

Also,

$$\|x\|_{\delta,\infty} \geq \frac{|t_1| \sum_{j=1}^{r_1} \delta_j}{\sum_{j=1}^{r_1} \delta_j} = |t_1| \geq |t_1|(1 - \varepsilon_1).$$

Next,

$$\begin{aligned}
\|x\|_{\delta,\infty} &\geq \|t_2 g_2\|_{\delta,\infty} \\
&\geq \frac{|t_2| \sum_{j=r_1+1}^{r_2} \delta_j}{\sum_{j=1}^{r_2-r_1} \delta_j} = |t_2| \frac{\sum_{j=1}^{r_2} \delta_j - \sum_{j=1}^{r_1} \delta_j}{\sum_{j=1}^{r_2-r_1} \delta_j} \\
&\geq |t_2| \frac{\sum_{j=1}^{r_2} \delta_j - \sum_{j=1}^{r_1} \delta_j}{\sum_{j=1}^{r_2} \delta_j} \\
&\geq |t_2|(1 - \varepsilon_2).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\|x\|_{\delta,\infty} &\geq \|t_3 g_3\|_{\delta,\infty} \\
&\geq \frac{|t_3| \sum_{j=r_2+1}^{r_3} \delta_j}{\sum_{j=1}^{r_3-r_2} \delta_j} = |t_3| \frac{\sum_{j=1}^{r_3} \delta_j - \sum_{j=1}^{r_2} \delta_j}{\sum_{j=1}^{r_3-r_2} \delta_j} \\
&\geq |t_3| \frac{\sum_{j=1}^{r_3} \delta_j - \sum_{j=1}^{r_2} \delta_j}{\sum_{j=1}^{r_3} \delta_j} \\
&\geq |t_3|(1 - \varepsilon_3).
\end{aligned}$$

Then, inductively, we obtain $\|x\|_{\delta,\infty} \geq |t_n|(1-\varepsilon_n), \forall n \in \mathbb{N}$. Hence,

$$\sup_{\nu\in\mathbb{N}}(1-\varepsilon_\nu)|t_\nu| \leq \|x\|_{\delta,\infty} \leq \sup_{\nu\in\mathbb{N}}|t_\nu| \leq \sup_{\nu\in\mathbb{N}}(1+\varepsilon_\nu)|t_\nu|$$

$$\sup_{\nu\in\mathbb{N}}(1-\varepsilon_\nu)|t_\nu| \leq \left\|\sum_{\nu=1}^{\infty} t_\nu\delta_\nu\right\|_{\delta,\infty} \leq \sup_{\nu\in\mathbb{N}}(1+\varepsilon_\nu)|t_\nu|.$$

□

3.3. $\ell^0_{\delta,\infty}$ fails fpp for affine $\|\cdot\|_{\delta,\infty}$-nonexpansive mappings

Theorem 3.3 *Define*

$$E := \{u = (s_1\,\delta_1, s_2\,\delta_2, s_3\,\delta_3, \dots) \mid s \in c_0 \ \ , 1 = s_1 \geq s_2 \geq \dots \geq 0\}.$$

Then $E \subseteq l^0_{\delta,\infty}$ is a convex, closed, bounded set, and $\exists\, T : E \to E$ s.t. T is fixed point free, $\|\cdot\|_{\delta,\infty}$-nonexpansive, affine mapping.

Proof It is clear that E is convex. We need to show $E \subseteq l^0_{\delta,\infty}$. Let $u \in E$ be given. Then $u^\star = u$ and

$$\|u\|_{\delta,\infty} = \sup_{n\in\mathbb{N}} \frac{\sum_{j=1}^{n} u_j^\star}{\sum_{j=1}^{n} \delta_j} \ .$$

$$\Psi_n(u) = \frac{\sum_{j=1}^{n} s_j\,\delta_j}{\sum_{j=1}^{n} \delta_j} \leq \frac{\sum_{j=1}^{n} 1\,\delta_j}{\sum_{j=1}^{n} \delta_j} = 1 \ .$$

Hence, $\|u\|_{\delta,\infty} \leq 1, \forall u \in E$. Thus, $E \subseteq l_{\delta,\infty}$ and, in fact, $E \subseteq \mathcal{B}_{l_{\delta,\infty}}$. Now, to show $E \subseteq l^0_{\delta,\infty}$, we need to show for $u \in E$, $\Psi_n(u) \underset{n}{\longrightarrow} 0$, and then we would prove $u \in l^0_{\delta,\infty}$.

Now, first of all,

$$0 \leq \Psi_n(u) = \sum_{j=1}^{n} s_j\,\gamma_j^{(n)}$$

where

$$\gamma_j^{(n)} = \frac{\delta_j}{\sum_{k=1}^{n} \delta_k} > 0 \,\text{and}\, \sum_{j=1}^{n} \gamma_j^{(n)} = 1 \ .$$

Then fix $\varepsilon > 0$ and choose $N = N_\varepsilon$ s.t. $\forall j \geq N_\varepsilon$, $s_j < \frac{\varepsilon}{2}$.

Let $n > N_\varepsilon$, n arbitrary.

$$\begin{aligned}
\Psi_n(u) &= \sum_{j=1}^{N} s_j\gamma_j^{(n)} + \sum_{j=N+1}^{n} s_j\gamma_j^{(n)} \\
&\leq (1)\sum_{j=1}^{N} \gamma_j^{(n)} + \sum_{j=N+1}^{n} s_j\gamma_j^{(n)} \\
&< \sum_{j=1}^{N} \gamma_j^{(n)} + \frac{\varepsilon}{2}\sum_{j=N+1}^{n} \gamma_j^{(n)} \text{ but since } \sum_{j=N+1}^{n} \gamma_j^{(n)} < 1 \\
&< \sum_{j=1}^{N} \gamma_j^{(n)} + \frac{\varepsilon}{2} \\
&= \sum_{j=1}^{N} \left(\frac{\delta_j}{\sum_{k=1}^{n} \delta_k}\right) + \frac{\varepsilon}{2} \\
&= \frac{\sum_{j=1}^{N_\varepsilon} \delta_j}{\sum_{k=1}^{n} \delta_k} + \frac{\varepsilon}{2} =: \frac{K(\varepsilon)}{\sum_{k=1}^{n} \delta_k} + \frac{\varepsilon}{2} \ .
\end{aligned}$$

Choose $M_\varepsilon > N_\varepsilon$ s.t. $\forall n \geq M_\varepsilon$,

$$\frac{K(\varepsilon)}{\sum_{k=1}^{n} \delta_k} < \frac{\varepsilon}{2} \ .$$

Then $\forall n \geq M_\varepsilon \,(> N_\varepsilon)$,

$0 \leq \Psi_n(u) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$, and so

$$\lim_{n\longrightarrow\infty} \Psi_n(u) = 0 \ .$$

Hence, $E \subseteq l^0_{\delta,\infty}$. □

Claim 3.4 *E is closed.*

Proof Let $(u_n)_{n\in\mathbb{N}}$ be any sequence in E norm convergent to a $u_0 \in l_{\delta,\infty}$, i.e. $u_n \underset{n}{\longrightarrow} u_0$ in norm then $u_n \underset{n}{\longrightarrow} u_0$ coordinatewise.

Say $u_0 := (u_{0,1}, u_{0,2}, u_{0,3}, \ldots, u_{0,j}, \ldots)$ and $u_n := (s_1^n\delta_1, s_2^n\delta_2, s_3^n\delta_3, \ldots, s_j^n\delta_j, \ldots)$. Then $u_{0,1} = \delta_1$ and $s_j^n\delta_j \underset{n}{\longrightarrow} u_{0,j}$ pointwise. Then $s_j^n \underset{n}{\longrightarrow} \frac{u_{0,j}}{\delta_j} =: s_j$ $s_1 = \frac{u_{0,1}}{\delta_1} = 1$, and each $s_j^n \geq 0 \Rightarrow s_j \geq 0$, by pointwise convergence. Now, fix $j \in \mathbb{N}$, and we can show $s_j \geq s_{j+1}$ by taking the limit as n goes to ∞ using the inequality $s_j^n \geq s_{j+1}^n$, since $s_j^n \underset{n}{\longrightarrow} s_j$. Hence, $s_j \geq s_{j+1}, \forall j \in \mathbb{N}$. Now, we need to show $s_j \underset{j}{\longrightarrow} 0$. Since $u_0 = (u_{0,j})_{j\in\mathbb{N}} \in l^0_{\delta,\infty} \subseteq c_0$, $u_{0,j} \underset{j}{\longrightarrow} 0$. Now we know that $u_0 = (s_1\delta_1, s_2\delta_2, \ldots, s_j\delta_j, \ldots)$, $1 = s_1 \geq s_2 \geq \ldots s_j \geq \cdots \geq 0$, and so $u_0^\star = u_0$. Recall that $u_0 \in l^0_{\delta,\infty}$ by hypothesis.

$$\Leftrightarrow \lim_{n\longrightarrow\infty} \frac{\sum_{j=1}^{n} (u_0^\star)_j}{\sum_{j=1}^{n} \delta_j} = 0 \ .$$

$$\text{Set for each n} \in \mathbb{N}\,,\ \Lambda_n := \frac{\sum_{j=1}^{n} (u_0^\star)_j}{\sum_{j=1}^{n}\ \delta_j}\ ,\ \text{and then}\ \Lambda_n = \frac{\sum_{j=1}^{n}\ s_j\delta_j}{\sum_{j=1}^{n}\ \delta_j} \underset{n}{\longrightarrow} 0.$$

Suppose we get a contradiction that $s_j \underset{j}{\not\longrightarrow} 0$.

However, we know $s_j \downarrow_j$ and $s_j \geq 0, \forall j \in \mathbb{N}$. Thus, $\exists\, L > 0$ s.t. $s_j \underset{j}{\longrightarrow} L$. Then,

$$\Lambda_n = \frac{\sum_{j=1}^{n}\ s_j\delta_j}{\sum_{j=1}^{n}\ \delta_j} \geq \frac{\sum_{j=1}^{n}\ L\,\delta_j}{\sum_{j=1}^{n}\ \delta_k} = L > 0, \forall n \in \mathbb{N}\ .$$

This is a clear contradiction since we obtain that $\forall n \in \mathbb{N}, \Lambda_n \geq L > 0$, whereas $\Lambda_n \underset{n}{\longrightarrow} 0$. In conclusion, $E \subseteq l^0_{\delta,\infty}$ is a closed bounded convex subset. □

Now we will prove the following claim.

Claim 3.5 *$\exists\, T : E \longrightarrow E$ $\|\cdot\|_{\delta,\infty}$-nonexpansive and fixed point free.*

Proof Indeed, consider T as the right shift.

That is,

$$\begin{aligned} T : u = (s_j\delta_j)_{j\in\mathbb{N}} \longrightarrow Tu &= (1\delta_1, s_1\delta_2, s_2\delta_3, s_3\delta_4, \dots) \\ &= (1\delta_1, 1\delta_2, s_2\delta_3, s_3\delta_4, \dots). \end{aligned}$$

T is clearly fixed point free. Indeed, assume $\exists\, u \in E$ s.t. $Tu = u$ but then that means $1 = s_1 = s_2 = s_3 = \cdots = s_k \Leftrightarrow u = (\delta_1, \delta_2, \delta_3, \dots) \in l_{\delta,\infty} - l^0_{\delta,\infty} \Rightarrow u \notin E$, which would be a contradiction. Now, let us see that T is nonexpansive. Hence, let $u, z \in E$ and say $u = (s_1\delta_1, s_2\delta_2, s_3\delta_3, \dots)$, $z = (s_1\delta_1, s_2\delta_2, s_3\delta_3, \dots)$ for some scalars s_n and v_n.

Then $Tu = (1\delta_1, s_1\delta_2, s_2\delta_3, s_3\delta_4, \dots) = (1\delta_1, 1\delta_2, s_2\delta_3, s_3\delta_4, \dots)$ and

$Tz = (1\delta_1, v_1\delta_2, v_2\delta_3, v_3\delta_4, \dots) = (1\delta_1, 1\delta_2, v_2\delta_3, v_3\delta_4, \dots)$.

Note that if $\beta = (\beta_j)_{j\in\mathbb{N}} \in l_{\delta,\infty}$ is s.t. $0 \leq \alpha_j \leq \beta_j, \forall j \in \mathbb{N}$,

then $\alpha = (\alpha_j)_{j\in\mathbb{N}} \in l_{\delta,\infty}$ and $[\|\alpha\|_{\delta,\infty} \leq \|\beta\|_{\delta,\infty}]$ $[\diamondsuit\diamondsuit]$

Also,

$$\forall u \in c_0, \forall n \in \mathbb{N}, \sum_{j=1}^{n} u_j^\star = \max_{\substack{F \subseteq \mathbb{N} \\ \#(F) = n}} \sum_{k\in F} |u_k| \qquad [\diamondsuit^{\diamondsuit}_{\diamondsuit}]$$

and so it follows that $[\diamondsuit^{\diamondsuit}_{\diamondsuit}] \Rightarrow [\diamondsuit\diamondsuit]$. Hence,

$$\begin{aligned} \|u - z\|_{\delta,\infty} &= \|(0, (s_2 - v_2)\delta_2, (s_3 - v_3)\delta_3, (s_4 - v_4)\delta_4, \dots)\|_{\delta,\infty} \\ &= \|(|s_2 - v_2|\delta_2, |s_3 - v_3|\delta_3, |s_4 - v_4|\delta_4, \dots)\|_{\delta,\infty} \\ &= \sup_{n\in\mathbb{N}} \left[\frac{\sum_{k=1}^{n} |s_{k+1} - v_{k+1}|\delta_{k+1}}{\sum_{k=1}^{n}\ \delta_j} \right]. \end{aligned}$$

Furthermore,

$$\begin{aligned}\|Tu - Tz\|_{\delta,\infty} &= \|(0, 0, (s_2 - v_2)\delta_3, (s_3 - v_3)\delta_4, (s_4 - v_4)\delta_5, \dots)\|_{\delta,\infty} \\ &= \|(0, 0, |s_2 - v_2|\delta_3, |s_3 - v_3|\delta_4, |s_4 - v_4|\delta_5, \dots)\|_{\delta,\infty} \\ &= \|(|s_2 - v_2|\delta_3, |s_3 - v_3|\delta_4, |s_4 - v_4|\delta_5, \dots)\|_{\delta,\infty} \\ &= \sup_{n\in\mathbb{N}} \left[\frac{\sum_{k=1}^{n} |s_{k+1} - v_{k+1}|\delta_{k+2}}{\sum_{k=1}^{n} \delta_k}\right].\end{aligned}$$

Thus, $\|Tu - Tz\|_{\delta,\infty} \leq \|u - z\|_{\delta,\infty}$. □

The proof of Theorem 3.3 is complete.

3.4. Any closed nonreflexive vector subspace Z of $\ell^0_{\delta,\infty}$ contains an isomorphic copy of c_0

Theorem 3.6 *Let Z be any closed, nonreflexive vector subspace of $l^0_{\delta,\infty}$. Then Z contains an isomorphic copy of c_0 and so $(Z, \|\cdot\|_{\delta,\infty})$ fails the fixed point property for strongly asymptotically nonexpansive maps.*

Proof We know that if Z is nonreflexive then, by [21, Theorem 1.c.12], $\exists$ an isomorphic copy of ℓ^1 or c_0 inside Z. □

Claim 3.7 *Z does not contain an isomorphic copy of ℓ^1 and so it contains an isomorphic copy of c_0.*

Proof By contradiction, assume not, i.e. $\ell^1 \lesssim l^0_{\delta,\infty}$. Then, by well-known facts in the literature [3, 24, Proposition 5.1.1, 5.1.2, 5.1.3], there exists a subspace Z of the dual space of $(l^0_{\delta,\infty})^\star$, i.e. $Z \leq (l^0_{\delta,\infty})^\star$ such that $(\ell^1)^\star$ is isometrically identical to $(l^0_{\delta,\infty})^\star/Z$, i.e. $(\ell^1)^\star \cong (l^0_{\delta,\infty})^\star/Z$. Moreover, $(\ell^1)^\star \cong l^\infty$ and $(l^0_{\delta,\infty})^\star/Z \cong (l_{\delta,1}/Z)$. However, we know that $l_{\delta,1}$ is separable and so is the quotient space $l_{\delta,1}/Z$. However, l^∞ is nonseparable. Hence, we get a contradiction; that is, $(\ell^1)^\star \cong (l^0_{\delta,\infty})^\star/Z$ cannot be true and so Z cannot contain an isomorphic copy of ℓ^1. Therefore, Z has to contain an isomorphic copy of c_0. □

The proof of Theorem 3.6 is complete.

4. Fixed Point Properties for $\ell_{\delta,1}$

As we discussed in Section 2, recall that ℓ^1 with equivalent norm $|||\cdot|||$ is a (degenerate) Lorentz space $\ell_{\delta,1}$ with the weight sequence $\delta = (\delta_n)_{n\in\mathbb{N}} = (2, 1, 1, 1, \cdots) \in c_0 \setminus \ell^1$ such that for all $x = (x_n)_{n\in\mathbb{N}} \in \ell^1$, $|||x||| = \sum_{n=1}^{\infty} \delta_n x_n^*$, where $x = (x_n^*)_{n\in\mathbb{N}}$ is decreasing rearrangement of x.

In the following two subsections, first we show that $\ell_{\delta,1}$ contains an ai of ℓ^1 and so it fails to have fpp(ne), but in the next subsection, in fact, we see that it fails to have fpp for affine nonexpansive mappings. Then, in the final subsection, we show that $\ell_{\delta,1}$ has w-fpp(ne).

4.0.1. $\ell_{\delta,1}$ contains an ai copy of ℓ^1

Theorem 4.1 *$\ell_{\delta,1}$ contains an ai copy of ℓ^1 and so it fails the fixed point property for $|||\cdot|||$-nonexpansive mappings.*

Proof Consider the sequence $(Q_n)_{n\in\mathbb{N}}$ constructed as in Lemma 2.1 using canonical basis $(e_n)_{n\in\mathbb{N}}$ given by $Q_1 := \frac{1}{2}e_1,\ Q_2 := \frac{1}{3}(e_2+e_3),\ Q_3 := \frac{1}{4}(e_4+e_5+e_6),\ Q_4 := \frac{1}{5}(e_7+e_8+e_9+e_{10}),\cdots$ and note that $|||Q_n||| = 1,\ \forall n\in\mathbb{N}$.

Then for all $t=(t_n)_{n\in\mathbb{N}}\in\ell^1$,

$$\begin{aligned}\left|\left|\left|\sum_{n=1}^{\infty} t_n Q_n\right|\right|\right| &= \left|\left|\left| \begin{array}{l} t_1\frac{1}{2}e_1 + t_2\frac{1}{3}(e_2+e_3) + t_3\frac{1}{4}(e_4+e_5+e_6) \\ +t_4\frac{1}{5}(e_7+e_8+e_9+e_{10}) + \cdots \end{array} \right|\right|\right| \\ &= \begin{array}{l} \left[\frac{1}{2}|t_1| + \frac{2}{3}|t_2| + \frac{3}{4}|t_3| + \frac{4}{5}|t_4| + \cdots\right] \\ + \left[\frac{1}{2}|t_1| \bigvee \frac{1}{3}|t_2| \bigvee \frac{1}{4}|t_3| \bigvee \frac{1}{5}|t_4| + \cdots\right] \end{array} \\ &\le |t_1| + |t_2| + |t_3| + |t_4| + \cdots \end{aligned}$$

and

$$\left|\left|\left|\sum_{n=1}^{\infty} t_n Q_n\right|\right|\right| \ge \sum_{n=1}^{\infty}(1-\varepsilon_n)|t_n|$$

where $\varepsilon_n := \frac{1}{n+1},\ \forall n\in\mathbb{N}$. □

4.0.2. $\ell_{\delta,1}$ fails fpp for affine $|||\cdot|||$-nonexpansive mappings

Theorem 4.2 *$\ell_{\delta,1}$ fails fpp for affine $|||\cdot|||$-nonexpansive mappings*

Proof Consider the set

$$C := \left\{ x = (x_n)_{n\in\mathbb{N}} \in \ell^1 \ :\ \text{each } x_n \ge 0 \ \&\ \sum_{n=1}^{\infty} x_n = 1 \right\}.$$

Clearly, $C \subseteq \ell^1$ is a closed, bounded, and convex subset. Now consider the right shift mapping $U : C \longrightarrow C$ defined by $Ux := (0, x_1, x_2, x_3, \cdots),\ \forall x\in C$. Then U is a fixed point free, affine $|||\cdot|||$-nonexpansive mapping. □

4.0.3. $\ell_{\delta,1}$ has w-fpp(ne)

Lemma 4.3 *$\ell_{\delta,1}$ has the weak fixed point property.*

Proof Recall that we named the renorming of ℓ^1 with the equivalent norm $|||\cdot|||$ by a degenerate Lorentz–Marcinkiewicz space $\ell_{\delta,1}$. Let $\emptyset \ne K \subseteq \ell^1$ be weakly compact. Let $T : K \longrightarrow K$ be a $|||\cdot|||$-nonexpansive map. Since ℓ^1 has the Schur property, i.e. weak convergence is equivalent to norm convergence, K is norm compact. Also, T is norm-to-norm continuous. Hence, by Schauder's theorem [28], T has a fixed point. □

5. An infinite-dimensional subspace of $\ell_{\delta,1} = (\ell^1, |||\cdot|||)$ with fpp(ne)

In this section, we will show that there exists a large class of nonweakly* compact, closed, bounded, and convex subsets of $\ell_{\delta,1} = (\ell^1, |||\cdot|||)$ with fpp(ne) under affinity condition using the ideas of Goebel and Kuczumow [13] where they show that there exists a large class of nonweakly* compact, closed, bounded, and convex subsets of $(\ell^1, \|\cdot\|_1)$ with fpp(ne).

In 1979, Goebel and Kuczumow showed the above fact, generalizing the result for the following class of sets.

Example 5.1 *Fix $b \in (0,1)$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by setting $f_1 := b\,e_1$, $f_2 := b\,e_2$, and $f_n := e_n$, for all integers $n \geq 3$. Next, define the closed, bounded, convex subset $E = E_b$ of ℓ^1 by*

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : \text{ each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1 \right\} .$$

However, we will be working on the following class.

Example 5.2 *Fix $b \in \left(0, \frac{2}{3}\right]$. Define the sequence $(f_n)_{n\in\mathbb{N}}$ in c_0 by setting $f_1 := b\,e_1$, $f_2 := b\,e_2$, and $f_n := e_n$, for all integers $n \geq 3$. Next, define the closed, bounded, convex subset $E = E_b$ of ℓ^1 by*

$$E := \left\{ \sum_{n=1}^{\infty} t_n\, f_n : \text{ each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n = 1 \right\} .$$

In order to obtain our results, we will be using an easy fact below such that its proof is straightforward.

Lemma 5.3 *Let $(X, \|.\|)$ be a Banach space.*

1. *If X has the Banach–Saks property and $x \in X$ is the weak limit of a bounded sequence $(x_n)_n$, then there exists a subsequence $(x_{n_k})_k$ whose Cesaro norm limit is x such that if s is defined by*

   $s(y) = \limsup_m \left\| \frac{1}{m} \sum_{k=1}^{m} x_{n_k} - y \right\|$, $\forall y \in X$, *then we have* $s(x) = 0$ *and* $s(y) = \|y - x\|$, $\forall y \in X$.

2. *If X has the weak Banach–Saks property and $x \in X$ is the weak limit of the sequence $(x_n)_n$, then there exists a subsequence $(x_{n_k})_k$ whose Cesaro norm limit is x such that if s is defined by*

   $s(y) = \limsup_m \left\| \frac{1}{m} \sum_{k=1}^{m} x_{n_k} - y \right\|$, $\forall y \in X$, *then we have* $s(x) = 0$ *and* $s(y) = \|y - x\|$, $\forall y \in X$.

Hence, due to the weak Banach–Saks property of our space, which can be deduced by the works [2, 26, 27], the above applies.

Theorem 5.4 *The set E defined as in Example 5.2 has the fixed point property for $|||.|||$-nonexpansive mappings where the norm $|||\cdot|||$ on ℓ^1 is given as follows: $|||x||| = \|x\|_1 + \|x\|_\infty$, $\forall x \in \ell^1$.*

Proof We will be using the proof steps of Goebel and Kuczumow given in detail as in Everest's PhD thesis [10], written under the supervision of Lennard. Let $T : E \to E$ be a nonexpansive mapping. Then there exists a sequence $\left(x^{(n)}\right)_{n\in\mathbb{N}} \in E$ such that $\left|\left|\left|Tx^{(n)} - x^{(n)}\right|\right|\right| \underset{n}{\to} 0$ and so $\left\|Tx^{(n)} - x^{(n)}\right\|_1 \underset{n}{\to} 0$. Without loss of generality, passing to a subsequence if necessary, there exists $z \in \ell^1$ such that $x^{(n)}$ converges to z in weak* topology. Then, by Lemma 5.3, we can define a function $s : \ell^1 \longrightarrow [0, \infty)$ by

$$s(y) = \limsup_m \left|\left|\left| \frac{1}{m} \sum_{k=1}^{m} x^{(k)} - y \right|\right|\right| \ , \ \forall y \in \ell^1$$

and so

$$s(y) = |||y - z||| \ , \ \forall y \in \ell^1.$$

Next, define

$$W := \overline{E}^{w^*} = \left\{ \sum_{n=1}^{\infty} t_n\, f_n : \text{ each } t_n \geq 0 \text{ and } \sum_{n=1}^{\infty} t_n \leq 1 \right\}.$$

Case 1: $z \in E$.

Then we have $s(Tz) = |||Tz - z|||$ and

$$\begin{aligned} s(Tz) &= \limsup_m \left|\left|\left| Tz - \frac{1}{m}\sum_{k=1}^{m} x^{(k)} \right|\right|\right| \\ &\leq \limsup_m \left|\left|\left| Tz - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right) \right|\right|\right| + \limsup_m \left|\left|\left| \frac{1}{m}\sum_{k=1}^{m} x^{(k)} - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right) \right|\right|\right|. \end{aligned}$$

Then, since T is affine,

$$\begin{aligned} s(Tz) &\leq \limsup_m \left|\left|\left| Tz - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right) \right|\right|\right| + \limsup_m \left|\left|\left| \frac{1}{m}\sum_{k=1}^{m} x^{(k)} - \frac{1}{m}\sum_{k=1}^{m} Tx^{(k)} \right|\right|\right| \\ &\leq \limsup_m \left|\left|\left| z - \frac{1}{m}\sum_{k=1}^{m} x^{(k)} \right|\right|\right| \\ &= s(z). \end{aligned}$$

Therefore, $|||z - Tz||| \leq 0$ and so $Tz = z$.

Case 2: $z \in W \setminus E$.

Then z is of the form $\sum\limits_{n=1}^{\infty} \gamma_n f_n$ such that $\sum\limits_{n=1}^{\infty} \gamma_n < 1$ and $\gamma_n \geq 0, \ \forall n \in \mathbb{N}$.

$$\text{Define } \delta := 1 - \sum_{n=1}^{\infty} \gamma_n \text{ and next define}$$

$$h_\lambda := (\gamma_1 + \lambda\delta) f_1 + (\gamma_2 + (1-\lambda)\delta) f_2 + \sum_{n=3}^{\infty} \gamma_n f_n.$$

We want h_λ to be in E, so we restrict values of λ to be in $\left[-\frac{\gamma_1}{\delta}, \ \frac{\gamma_2}{\delta} + 1\right]$, and then

$$\begin{aligned} |||h_\lambda - z||| &= |||\lambda\delta f_1 + (1-\lambda)\delta f_2||| \\ &= |||(\lambda\delta b, (1-\lambda)\delta b, 0, 0, \cdots)||| \\ &= b\delta \max\{|\lambda|, |1-\lambda|\} + b\delta|\lambda| + b\delta|1-\lambda| \\ &= \max \begin{cases} 2b\delta - 3b\delta\lambda & \text{if } \lambda \in \left[-\frac{\gamma_1}{\delta}, \ 0\right), \\ 2b\delta - b\delta\lambda & \text{if } \lambda \in \left[0, \ \frac{1}{2}\right), \\ b\delta(1+\lambda) & \text{if } \lambda \in \left(\frac{1}{2}, \ 1\right], \\ 3b\delta\lambda - b\delta & \text{if } \lambda \in \left(1, \ \frac{\gamma_2}{\delta} + 1\right]. \end{cases} \end{aligned}$$

Define

$$\Gamma := \min_{\lambda \in \left[-\frac{\gamma_1}{\delta},\, \frac{\gamma_2}{\delta}+1\right]} |||h_\lambda - z||| \,.$$

Therefore, $|||h_\lambda - z|||$ is minimized when $\lambda \in [0,1]$ with unique minimizer such that its minimum value would be $\Gamma = \frac{3b\delta}{2}$.

Now fix $y \in E$ of the form $\sum_{n=1}^{\infty} t_n f_n$ such that $\sum_{n=1}^{\infty} t_n \ = 1$ with $t_n \geq 0$, $\forall n \in \mathbb{N}$.

Then,

$$\begin{aligned}
|||y - z||| &= \left|\left|\left|\sum_{k=1}^{\infty} t_k f_k - \sum_{k=1}^{\infty} \gamma_k f_k\right|\right|\right| \\
&= |||(t_1-\gamma_1)be_1 + (t_2-\gamma_2)be_2 + (t_3-\gamma_3)e_3 + (t_4-\gamma_4)e_4 + \cdots||| \\
&= \max\left\{\begin{array}{l}
2b|t_1-\gamma_1| + b|t_2-\gamma_2| + |t_3-\gamma_3| \\
+|t_4-\gamma_4| + |t_5-\gamma_5| + \cdots\ , \\
2b|t_2-\gamma_2| + b|t_1-\gamma_1| + |t_3-\gamma_3| \\
+|t_4-\gamma_4| + |t_5-\gamma_5| + \cdots\ , \\
2|t_3-\gamma_3| + b|t_1-\gamma_1| + b|t_2-\gamma_2| \\
+|t_4-\gamma_4| + |t_5-\gamma_5| + \cdots\ , \\
2|t_4-\gamma_4| + b|t_1-\gamma_1| + b|t_2-\gamma_2| \\
+|t_3-\gamma_3| + |t_5-\gamma_5| + \cdots\ , \\
2|t_5-\gamma_5| + b|t_1-\gamma_1| + b|t_2-\gamma_2| + |t_3-\gamma_3| \\
+|t_4-\gamma_4| + |t_6-\gamma_6| + \cdots\ , \\
\cdots\cdots\cdots\cdots
\end{array}\right\}.
\end{aligned}$$

Subcase 2.1: $|t_1-\gamma_1| \geq |t_2-\gamma_2|$ and $b|t_1-\gamma_1| \geq |t_k-\gamma_k|$, $\forall k \geq 3$.

Then,

$$\begin{aligned}
|||y - z||| &= b|t_1-\gamma_1| + b\sum_{k=1}^{\infty} |t_k-\gamma_k| + (1-b)\sum_{k=3}^{\infty} |t_k-\gamma_k| \\
&\geq b\delta + b|t_1-\gamma_1| + (1-b)\sum_{k=3}^{\infty} |t_k-\gamma_k| \\
&\geq b\delta + b|t_1-\gamma_1| + (1-b)|\delta - (t_1-\gamma_1) - (t_2-\gamma_2)| \\
&\geq b\delta + b|t_1-\gamma_1| + (1-b)\delta - 2(1-b)|t_1-\gamma_1| \\
&= \delta - (2-3b)|t_1-\gamma_1|.
\end{aligned}$$

Subcase 2.1.1: Assume $\frac{\delta}{2} \geq |t_1-\gamma_1|$.

Then clearly the last inequality from above says that $|||y-z||| \geq \frac{3b\delta}{2}$.

Subcase 2.1.2: Assume $\frac{\delta}{2} < |t_1-\gamma_1|$.

Then $|||y-z||| \geq b\delta + b|t_1-\gamma_1| + (1-b)\sum_{k=3}^{\infty} |t_k-\gamma_k| \geq \frac{3b\delta}{2}$.

Subcase 2.2: $|t_2-\gamma_2| \geq |t_1-\gamma_1|$ and $b|t_2-\gamma_2| \geq |t_k-\gamma_k|$, $\forall k \geq 3$.

Then,

$$\begin{aligned}
|||y-z||| &= b|t_2-\gamma_2| + b\sum_{k=1}^{\infty}|t_k-\gamma_k| + (1-b)\sum_{k=3}^{\infty}|t_k-\gamma_k| \\
&\geq b\delta + b|t_2-\gamma_2| + (1-b)\sum_{k=3}^{\infty}|t_k-\gamma_k| \\
&\geq b\delta + b|t_2-\gamma_2| + (1-b)|\delta-(t_1-\gamma_1)-(t_2-\gamma_2)| \ .
\end{aligned}$$

Therefore,

$$\begin{aligned}
|||y-z||| &\geq b\delta + b|t_2-\gamma_2| + (1-b)\delta - 2(1-b)|t_2-\gamma_2| \\
&\geq \delta-(2-3b)|t_2-\gamma_2|.
\end{aligned}$$

Subcase 2.2.1: Assume $\frac{\delta}{2} \geq |t_2-\gamma_2|$.

Then clearly the last inequality from above says that $|||y-z||| \geq \frac{3b\delta}{2}$.

Subcase 2.2.2: Assume $\frac{\delta}{2} < |t_2-\gamma_2|$.

Then $|||y-z||| \geq b\delta + b|t_2-\gamma_2| + (1-b)\sum_{k=3}^{\infty}|t_k-\gamma_k| \geq \frac{3b\delta}{2}$.

Subcase 2.3: $|t_3-\gamma_3| \geq b|t_1-\gamma_1|$, $|t_3-\gamma_3| \geq b|t_2-\gamma_2|$, and $|t_3-\gamma_3| \geq |t_k-\gamma_k|$, $\forall k \geq 4$.

Then,

$$\begin{aligned}
|||y-z||| &= |t_3-\gamma_3| + b\sum_{k=1}^{\infty}|t_k-\gamma_k| + (1-b)\sum_{k=3}^{\infty}|t_k-\gamma_k| \\
&\geq b\delta + |t_3-\gamma_3| + (1-b)\sum_{k=3}^{\infty}|t_k-\gamma_k| \\
&\geq b\delta + |t_3-\gamma_3| + (1-b)|\delta-(t_1-\gamma_1)-(t_2-\gamma_2)| \\
&\geq b\delta + |t_3-\gamma_3| + (1-b)\delta - \frac{2(1-b)}{b}|t_3-\gamma_3| \\
&= \delta - \frac{2-3b}{b}|t_3-\gamma_3|.
\end{aligned}$$

Subcase 2.3.1: Assume $\frac{b\delta}{2} \geq |t_3-\gamma_3|$.

Then clearly the last inequality from above says that $|||y-z||| \geq \frac{3b\delta}{2}$.

Subcase 2.3.2: Assume $\frac{b\delta}{2} < |t_3-\gamma_3|$.

Then $|||y-z||| \geq b\delta + |t_3-\gamma_3| + (1-b)\sum_{k=3}^{\infty}|t_k-\gamma_k| \geq \frac{3b\delta}{2}$.

Subcase 2.4: $|t_4-\gamma_4| \geq b|t_1-\gamma_1|$, $|t_4-\gamma_4| \geq b|t_2-\gamma_2|$, and $|t_4-\gamma_4| \geq |t_k-\gamma_k|$, $\forall k \geq 5$ and for $k=3$.

Then,

$$\begin{aligned}
|||y-z||| &= |t_4-\gamma_4| + b\sum_{k=1}^{\infty}|t_k-\gamma_k| + (1-b)\sum_{k=3}^{\infty}|t_k-\gamma_k| \\
&\geq b\delta + |t_4-\gamma_4| + (1-b)\sum_{k=3}^{\infty}|t_k-\gamma_k| \\
&\geq b\delta + |t_4-\gamma_4| + (1-b)|\delta-(t_1-\gamma_1)-(t_2-\gamma_2)| \\
&\geq b\delta + |t_4-\gamma_4| + (1-b)\delta - \frac{2(1-b)}{b}|t_4-\gamma_4| \\
&= \delta - \frac{2-3b}{b}|t_4-\gamma_4|.
\end{aligned}$$

Subcase 2.4.1: Assume $\frac{b\delta}{2} \geq |t_4-\gamma_4|$.

Then clearly the last inequality from above says that $|||y-z||| \geq \frac{3b\delta}{2}$.

Subcase 2.4.2: Assume $\frac{b\delta}{2} < |t_4-\gamma_4|$.

Then $|||y-z||| \geq b\delta + |t_4-\gamma_4| + (1-b)\sum\limits_{k=3}^{\infty}|t_k-\gamma_k| \geq \frac{3b\delta}{2}$.

Thus, we continue in this way and see that $|||y-z||| \geq \frac{3b\delta}{2}$ from all cases.

Therefore, when λ is choosen to be in $[0,1]$, for any $y \in E$ and for $z \in W \setminus E$, $|||y-z||| \geq \Gamma$ such that there exists unique $\lambda_0 \in [0,1]$ with $|||h_{\lambda_0}-z|||=\Gamma$.

Now define a subset in our set by $\Lambda := \{y \,:\, |||y-z||| \leq \Gamma\}$. Note that $\Lambda \subseteq E$ is a nonempty compact convex subset such that for any $h \in \Lambda$,

$$\begin{aligned}
s(Th) &= \limsup_m \left|\left|\left| Th - \frac{1}{m}\sum_{k=1}^{m} x^{(k)} \right|\right|\right| \\
&\leq \limsup_m \left|\left|\left| Th - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right) \right|\right|\right| + \limsup_m \left|\left|\left| \frac{1}{m}\sum_{k=1}^{m} x^{(k)} - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right) \right|\right|\right| .
\end{aligned}$$

However, since T is affine,

$$\begin{aligned}
s(Th) &\leq \limsup_m \left|\left|\left| Th - T\left(\frac{1}{m}\sum_{k=1}^{m} x^{(k)}\right) \right|\right|\right| + \limsup_m \left|\left|\left| \frac{1}{m}\sum_{k=1}^{m} x^{(k)} - \frac{1}{m}\sum_{k=1}^{m} Tx^{(k)} \right|\right|\right| \\
&\leq \limsup_m \left|\left|\left| h - \frac{1}{m}\sum_{k=1}^{m} x^{(k)} \right|\right|\right| \\
&= s(h).
\end{aligned}$$

Also, $s(Th) = |||z-Th|||$ and $s(h) = |||z-h|||$. Hence,

$$\begin{aligned}
|||z-Th||| \leq |||z-h||| &\implies |||z-Th||| = |||z-h||| \\
&\implies Th \in \Lambda.
\end{aligned}$$

Therefore, $T(\Lambda) \subseteq \Lambda$, and since T is continuous, Schauder's fixed point theorem [28] tells us that T has a fixed point such that $h = h_{\lambda_0}$ is the unique minimizer of $|||y - z|||\ :\ y \in E$ and $Th = h$.

Therefore, E has fpp(ne) as desired. □

Remark 5.5 *Generalizing the sets as Goebel and Kuczumow [13] or Everest [10] did, one can obtain larger classes with fixed point property for nonexpansive mappings, which could be considered as a future project for other researchers in the field and for us as well.*

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