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On the n-strong Drazin invertibility in rings

Honglin ZOU[1*], Dijana MOSIĆ[2], Kezheng ZUO[1], Yinlan CHEN[1]
[1]School of Mathematics and Statistics, Hubei Normal University, Huangshi, China
[2]Faculty of Sciences and Mathematics, University of Niš, Niš, Serbia

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Abstract: Let R be a ring and n be a positive integer. In this paper, further results on the n-strong Drazin inverse are obtained in a ring. We prove that $a \in R$ is n-strongly Drazin invertible if and only if $a - a^{n+1}$ is nilpotent. In terms of this characterization, the extensions of Cline's formula and Jacobson's lemma for this inverse are proved. Moreover, the n-strong Drazin invertibility for the sums of two elements is considered. We prove that $a, b \in R$ are n-strongly Drazin invertible if and only if $a + b$ is n-strongly Drazin invertible, under the condition $ab = 0$. As applications for the additive results, we obtain some equivalent conditions of the n-strong Drazin invertibility of matrices over a ring.

Key words: Strong Drazin inverse, Hirano inverse, n-strong Drazin inverse, Drazin inverse ring

1. Introduction

Let R^D denote the set of all Drazin invertible elements in a ring R. It is well known that if $a, b \in R$, then

$$ab \in R^D \Longleftrightarrow ba \in R^D.$$

In this case, $(ba)^D = b((ab)^D)^2 a$ [4]. This formula is called Cline's formula for the Drazin inverse. Many researchers considered Cline's formula for various types of generalized inverses, such as (b, c)-inverse [10], Mary inverse [27], Hirano inverse [2], pseudo-Drazin inverse [20], generalized Drazin inverse [13, 14, 16, 23, 24]. In [23], Zeng et al. extended Cline's formula for the (pseudo, generalized) Drazin inverse to more general case. Namely, if $a, b, c, d \in R$ satisfy $acd = dbd$ and $dba = aca$, then

$$ac \in R^D \Longleftrightarrow bd \in R^D.$$

In this case, $(bd)^D = b((ac)^D)^2 d$ and $(ac)^D = d((bd)^D)^3 bac$. Corresponding to Cline's formula, many researchers paid attention to Jacobson's lemma, that is

$$1 - ab \in R^{-1} \Longleftrightarrow 1 - ba \in R^{-1}.$$

In this case, $(1-ba)^{-1} = 1 + b(1-ab)^{-1}a$. They investigated Jacobson's lemma for different generalized inverses in different settings [1, 2, 5, 6, 17, 18, 25].

The topic for generalized inverses of the sums was studied by many authors. In 1958, Drazin [9] proved that $a + b \in R^D$ with $(a+b)^D = a^D + b^D$ under the condition $a, b \in R^D$ and $ab = ba = 0$. For $a, b \in \mathbb{C}^{n \times n}$,

*Correspondence: honglinzou@163.com
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Hartwig et al. [12] obtained a formula for $(a+b)^D$ under the one-sided condition $ab=0$, which was extended to the additive category by Chen et al. [3]. In addition, the problem of generalized inverses of $a+b$ was also studied under the condition $ab=ba$. For example, Wei and Deng [21] gave the relations of Drazin inverses of $a+b$ and $1+a^Db$, where $a,b\in\mathbb{C}^{n\times n}$. Later, Zhuang et al. [26] extended the result of [21] to the ring case. The generalized Drazin invertibility and strong Drazin invertibility of the sums under the commutative condition were also investigated [7, 8, 19].

All results mentioned above were the motivation for further consideration of the n-strong Drazin inverse in a ring. This article consists of five sections. In Section 2, we recall the definitions of some generalized inverses and give related notations. In Section 3, characterizations of n-strongly Drazin invertible elements are given in terms of the nilpotency. Then, we investigate Cline's formula and Jacobson's lemma for the n-strong Drazin inverse in a ring. In Section 4, we obtain some equivalent conditions for the n-strong Drazin invertibility of the sum $a+b$ under the hypothesis $ab=0$ (or $a^2b=aba, ab^2=bab$). In Section 5, as applications of the previous additive results, we mainly consider the n-strong Drazin invertibility of matrices over a ring. We remark that some results presented in this paper are different from those of Drazin inverses.

2. Preliminaries

Throughout this paper, R denotes a ring with unity 1. R^{nil} and $\mathbb{N}$ stand for the sets of all nilpotent elements in R and positive integers, respectively. Denote by $\binom{n}{k}$ the binomial coefficient $\frac{n!}{k!(n-k)!}$ $(0\le k\le n)$.

For the readers' convenience, we first recall the definitions of some generalized inverses. The Drazin inverse [9] of $a\in R$ is the element $x\in R$ which satisfies

$$xax=x,\quad ax=xa,\quad \text{and}\quad a-a^2x\in R^{nil}.$$

The element x above is unique if it exists and is denoted by a^D. The power of nilpotency of $a-a^2a^D$ is called the index of a, and will be denoted by $\mathrm{ind}(a)$. Drazin [9] proved that $a\in R$ is Drazin invertible if and only if a is both right π-regular (i.e. $a^m\in a^{m+1}R$, for some $m\in\mathbb{N}$) and left π-regular ($a^m\in Ra^{m+1}$, for some $m\in\mathbb{N}$), namely a is strongly π-regular.

In 2017, Wang [19] gave the notion of the strong Drazin inverse in a ring. An element $a\in R$ is said to be strongly Drazin invertible [19] if there exists $x\in R$ such that

$$xax=x,\quad ax=xa,\quad \text{and}\quad a-ax\in R^{nil}.$$

In this case, x is unique if it exists and is called the strong Drazin inverse of a. We will denote the strong Drazin inverse of a by a^{sD}. The strongly Drazin invertible elements are exactly the ones which are strongly nil-clean (see [19, Lemma 2.2]). Let $a\in R$, then a^D exists if and only if there exists $x\in R$ such that

$$x\in aR\cap Ra,\ ax=xa,\ \text{and}\ a-ax\in R^{nil}.$$

Suppose that a^D exists. Then, let $x=aa^D$. Obviously, x satisfies $x\in aR\cap Ra$, $ax=xa$, and $a-ax\in R^{nil}$. On the contrary, we have $(a-ax)^m=0$ for some $m\in\mathbb{N}$. Hence, $a^m(1-x)^m=a^m(1+\sum_{i=1}^{m}(-1)^i\binom{m}{i}x^i)=0$, which implies that $a^m=a^mxu=uxa^m$, for some $u\in R$. Observe that $x=as=ta$, where $s,t\in R$. Hence, we deduce that $a^m=a^{m+1}su=uta^{m+1}\in a^{m+1}R\cap Ra^{m+1}$. Hence, a^D exists.

The definition of the Hirano inverse [2] was introduced by Chen and Sheibani in 2017. The Hirano inverse of $a \in R$ is the unique element x (written $x = a^H$) satisfying

$$xax = x, \quad ax = xa, \quad \text{and} \quad a^2 - ax \in R^{nil}.$$

It is interesting that the Hirano inverse is related to tripotent elements (see [2, Theorem 3.3]). In addition, they obtained the relations of the above three kinds of generalized inverses, that is $R^{sD} \subsetneqq R^H \subsetneqq R^D$, where R^{sD} and R^H mean the sets of all strongly Drazin invertible and Hirano invertible elements in R, respectively.

Recently, motivated by the concepts of the strong Drazin inverse and Hirano inverse, Mosić [15] introduced the notion of the n-strong Drazin inverse in a ring. Let $n \in \mathbb{N}$. An element $x \in R$ is called the n-strong Drazin inverse of $a \in R$ if it satisfies

$$xax = x, \quad ax = xa, \quad \text{and} \quad a^n - ax \in R^{nil}.$$

The previous x is unique if such element exists, and we denote it by a^{nsD}. Clearly, the n-strong Drazin inverse covers the strong Drazin inverse and Hirano inverse, that is, $a^{1sD} = a^{sD}$ and $a^{2sD} = a^H$. The power of nilpotency of $a^n - aa^{nsD}$ is called the n-strong Drazin index of a, denoted by n-ind(a). The symbol R^{nsD} denote the set of all n-strongly Drazin invertible elements in R. We note that $R^{nil} \subseteq R^{nsD}$. Indeed, $a \in R^{nil}$ if and only if $a \in R^{nsD}$ with $a^{nsD} = 0$. In addition, $R^{-1} \nsubseteq R^{nsD}$. For exampe, let $R = \mathbb{C}$. Then, $3 \in R^{-1}$, but $3 \notin R^{nsD}$.

Next, we introduce two known lemmas, which are related to the nilpotency.

Lemma 2.1 *Let $a, b \in R$ with $ab = ba$. Then,*

(1) *If $a \in R^{nil}$ (or $b \in R^{nil}$), then $ab \in R^{nil}$.*

(2) *If $a, b \in R^{nil}$, then $a + b \in R^{nil}$.*

Lemma 2.2 [22, Lemma 3.5] *Let $a \in R$. If $a^2 - a \in R^{nil}$, then there exists a monic polynomial $\theta(t) \in \mathbb{Z}[t]$ such that $\theta(a) = \theta(a)^2$ and $a - \theta(a)$ is nilpotent.*

3. Cline's formula and Jacobson's lemma

In this section, we give an existence criterion for the n-strong Drazin inverse in a ring. Then, by this characterization we prove Cline's formula and Jacobson's lemma for the n-strong Drazin inverse. The results presented extend the corresponding ones of the strong Drazin inverse [19] and Hirano inverse [2].

Firstly, we give the relationship between the n-strong Drazin inverse and Drazin inverse. The proof of the following proposition is similar to that of [15, Lemma 2.1].

Proposition 3.1 *Let $n \in \mathbb{N}$. If $a \in R^{nsD}$ with n-ind$(a) = m$, then $a \in R^D$ and $a^D = a^{nsD}$. Moreover,* ind$(a) \leq nm$.

Proof Assume that $a \in R^{nsD}$ with n-ind$(a) = m$. Let $x = a^{nsD}$. Then we have $xax = x$, $ax = xa$, and $(a^n - ax)^m = 0$, which yield

$$(a - a^2x)^{nm} = (a^n - a^{n+1}x)^m = (a^n - ax)^m(1 - ax)^m = 0.$$

Hence, $a - a^2x \in R^{nil}$. Hence, $a \in R^D$ and $a^D = x$. Moreover, $\mathrm{ind}(a) \leq nm$.

Inspired by [2, Theorem 3.1], we obtain a characterization for the n-strong Drazin invertibility in a ring, which plays an important role in the sequel.

Theorem 3.2 *Let $n \in \mathbb{N}$. Then $a \in R^{nsD}$ if and only if $a - a^{n+1} \in R^{nil}$.*

Proof Suppose that $a \in R^{nsD}$ and $x = a^{nsD}$, i.e. $xax = x$, $ax = xa$, and $a^n - ax \in R^{nil}$. Then we deduce that

$$a^n - a^{2n} = (a^n - ax)(1 - ax - a^n) \in R^{nil},$$

which yields that

$$(a - a^{n+1})^n = (a - a^{n+1})a^{n-1}(1 - a^n)^{n-1} = (a^n - a^{2n})(1 - a^n)^{n-1} \in R^{nil}.$$

Hence, $a - a^{n+1} \in R^{nil}$.

On the contrary, since $a - a^{n+1} \in R^{nil}$, we conclude that $(a^n)^2 - a^n = a^{n-1}(a^{n+1} - a) \in R^{nil}$. By Lemma 2.2, there exists a monic polynomial $\theta(t) \in \mathbb{Z}[t]$ such that $\theta(a^n) = \theta(a^n)^2$ and $a^n - \theta(a^n) \in R^{nil}$. Take $e = \theta(a^n)$. Then we have $e = e^2$, $ea = ae$ and $a^n - e \in R^{nil}$. Hence, we obtain $1 + a^n - e \in R^{-1}$. Let $x = (1 + a^n - e)^{-1}a^{n-1}e$. Next, we show that $a^{nsD} = x$ by the definition of the n-strong Drazin inverse. Obviously, $ax = xa$. Note that $a^n e = (1 + a^n - e)e = e(1 + a^n - e)$. Then, we obtain

$$\begin{aligned} xax &= (1 + a^n - e)^{-1}a^n e(1 + a^n - e)^{-1}a^{n-1}e \\ &= (1 + a^n - e)^{-1}(1 + a^n - e)e(1 + a^n - e)^{-1}a^{n-1}e \\ &= (1 + a^n - e)^{-1}a^{n-1}e \\ &= x \end{aligned}$$

and

$$a^n - ax = a^n - (1 + a^n - e)^{-1}a^n e = a^n - e \in R^{nil}.$$

Therefore, $a \in R^{nsD}$ with $a^{nsD} = x$.

Remark 3.3 *(1) Let $A \in \mathbb{C}^{m\times m}$ ($\mathrm{rank}\ A = r > 0$) have the Jordan form*

$$A = P \begin{bmatrix} D & 0 \\ 0 & N \end{bmatrix} P^{-1},$$

where D is invertible and N is nilpotent. Then, by Theorem 3.2 we have

$$\begin{aligned} A \in (\mathbb{C}^{m\times m})^{nsD} &\iff I - D^n \in (\mathbb{C}^{r\times r})^{nil} \\ &\iff \sigma(A) \subseteq \{0, 1, \varepsilon, \varepsilon^2, \cdots, \varepsilon^{n-1}\}, \end{aligned}$$

where $\sigma(A)$ denotes the spectrum of A and $\varepsilon = cos\frac{2\pi}{n} + isin\frac{2\pi}{n}$.

(2) We have the following special case,

$$a \in R^{nsD} \text{ and } n\text{-ind}(a) = 1 \iff a^n = a^{2n}.$$

The necessity is obvious. In fact, from $(a^n - aa^{nsD})^1 = 0$ *it follows that* $a^n = aa^{nsD}$ *and consequently* $a^{2n} = aa^{nsD}aa^{nsD} = aa^{nsD} = a^n$. *On the contrary, suppose that* $a^n = a^{2n}$. *Let* $x = a^{2n-1}$. *Then, we have* $xax = a^{2n}a^{2n-1} = a^n a^{2n-1} = a^{2n}a^{n-1} = a^{2n-1} = x$. *In addition, it is clear that* $ax = xa$ *and* $a^n - ax = 0$. *Hence, we get* $a \in R^{nsD}$ *and* $a^{nsD} = x$. *Moreover,* $n\text{-ind}(a) = 1$.

Applying Theorem 3.2, we get some properties of the n-strong Drazin inverse in a ring.

Corollary 3.4 *Let* $n, k \in \mathbb{N}$. *If* $a \in R^{nsD}$, *then* $a^k \in R^{nsD}$ *and* $(a^k)^{nsD} = (a^{nsD})^k$.

Proof Since $a \in R^{nsD}$, by Theorem 3.2 we have $a - a^{n+1} \in R^{nil}$, which yields

$$a^k - (a^k)^{n+1} = a^k - (a^{n+1})^k = (a - a^{n+1}) \sum_{i=0}^{k-1} a^{ni+k-1} \in R^{nil}.$$

Hence, we obtain $a^k \in R^{nsD}$. In view of Proposition 3.1 and [9, Theorem 2], one can see that $(a^k)^{nsD} = (a^k)^D = (a^D)^k = (a^{nsD})^k$.

Corollary 3.5 *Let* $n \in \mathbb{N}$. *If* $a \in R^{nsD}$, *then* $a^{nsD} \in R^{nsD}$ *and* $(a^{nsD})^{nsD} = a^2 a^{nsD}$.

Proof Let $x = a^{nsD}$. Then we get $xax = x$, $ax = xa$, and $a^n - ax \in R^{nil}$. Hence,

$$x - x^{n+1} = x^{n+1}(a^n - ax) \in R^{nil}.$$

Hence, by Theorem 3.2 we obtain $x \in R^{nsD}$. From [9, Theorem 3], it follows that

$$x^{nsD} = x^D = (a^D)^D = a^2 a^D = a^2 a^{nsD}.$$

Corollary 3.6 *Let* $n \in \mathbb{N}$. *Then,*

(1) *If* $a \in R^{sD}$, *then* $a \in R^{nsD}$ *and* $a^{nsD} = a^{sD} = a^D$.

(2) *If* $a \in R^H$, *then* $a \in R^{2nsD}$ *and* $a^{2nsD} = a^H = a^D$.

(3) *If* $a \in R^{nsD}$, *then* $a \in R^{2nsD}$ *and* $a^{2nsD} = a^{nsD} = a^D$.

Proof (1) Since $a \in R^{sD}$, by Theorem 3.2 we have $a - a^2 \in R^{nil}$, which gives

$$a - a^{n+1} = a(1 - a^n) = a(1 - a) \sum_{i=0}^{n-1} a^i = (a - a^2) \sum_{i=0}^{n-1} a^i \in R^{nil}.$$

Hence, $a - a^{n+1} \in R^{nil}$, i.e. $a \in R^{nsD}$. In addition, $a^{nsD} = a^D = a^{sD}$.

(2) can be proved in the same way as the item (1).

(3) follows directly by the equality $a - a^{2n+1} = (a - a^{n+1})(1 + a^n)$.

In terms of Theorem 3.2, we are now in the position to prove the extension of Cline's formula for the n-strong Drazin inverse when $acd = dbd$ and $dba = aca$.

Theorem 3.7 *Let $a, b, c, d \in R$ and $n \in \mathbb{N}$. If $acd = dbd$ and $dba = aca$, then*

$$ac \in R^{nsD} \Longleftrightarrow bd \in R^{nsD}.$$

In this case, $(bd)^{nsD} = b((ac)^{nsD})^2 d$ and $(ac)^{nsD} = d((bd)^{nsD})^3 bac$.

Proof It will suffice to prove the sufficiency, since the necessity can be proved similarly. From $dba = aca$, it follows that $(ac)^i = (db)^{i-1}ac$ for any $i \in \mathbb{N}$. Now, we show that

$$(ac - (ac)^{n+1})^{m+1} = d(bd - (bd)^{n+1})^{m-1}(b - (bd)^n b)(ac - (ac)^{n+1})$$

by induction on positive integer m.

For $m = 1$, we have

$$\begin{aligned} (ac - (ac)^{n+1})^2 &= (ac - (ac)^{n+1})(ac - (ac)^{n+1}) \\ &= ((ac)^2 - (ac)^{n+2})(1 - (ac)^n) \\ &= (dbac - (db)^{n+1}ac)(1 - (ac)^n) \\ &= (db - (db)^{n+1})(ac - (ac)^{n+1}) \\ &= d(b - (bd)^n b)(ac - (ac)^{n+1}). \end{aligned}$$

Assume that the conclusion holds for positive integer $m = l$. Now, we check it for $m = l + 1$ as follows:

$$\begin{aligned} (ac - (ac)^{n+1})^{l+2} &= (ac - (ac)^{n+1})^{l+1}(ac - (ac)^{n+1}) \\ &= d(bd - (bd)^{n+1})^{l-1}(b - (bd)^n b)(ac - (ac)^{n+1})^2 \\ &= d(bd - (bd)^{n+1})^{l-1}(b - b(db)^n)d(b - (bd)^n b)(ac - (ac)^{n+1}) \\ &= d(bd - (bd)^{n+1})^{l}(b - (bd)^n b)(ac - (ac)^{n+1}). \end{aligned}$$

Note that $bd \in R^{nsD}$, i.e. $bd - (bd)^{n+1} \in R^{nil}$. Hence, $ac - (ac)^{n+1} \in R^{nil}$, i.e. $ac \in R^{nsD}$. By Proposition 3.1 and the formula of [23, Theorem 2.1], we obtain $(ac)^{nsD} = d((bd)^{nsD})^3 bac$.

In Theorem 3.7, let $d = a$ and $c = b$, then it is reduced as the following.

Corollary 3.8 *Let $a, b \in R$ and $n \in \mathbb{N}$. Then,*

$$ab \in R^{nsD} \Longleftrightarrow ba \in R^{nsD}.$$

In this case, $(ba)^{nsD} = b((ab)^{nsD})^2 a$.

Corollary 3.9 *Let $a, b \in R$ and $n, k \in \mathbb{N}$. If $(ab)^k \in R^{nsD}$, then $(ba)^k \in R^{nsD}$.*

Proof Since $a((ba)^{k-1}b) = (ab)^k \in R^{nsD}$, by Corollary 3.8 we deduce that $(ba)^k \in R^{nsD}$.

Under the same hypotheses $acd = dbd$ and $dba = aca$, Jacobson's lemma for the n-strong Drazin inverse is investigated as follows.

Theorem 3.10 *Let $a, b, c, d \in R$ and $n \in \mathbb{N}$. If $acd = dbd$ and $dba = aca$, then*

$$1 - ac \in R^{nsD} \Longleftrightarrow 1 - bd \in R^{nsD}.$$

Proof Suppose that $1-ac \in R^{nsD}$. Then $(1-ac)-(1-ac)^{n+1} \in R^{nil}$. Now, by mathematical induction we prove that

$$((1-bd)-(1-bd)^{n+1})^{m+1} = -b((1-ac)-(1-ac)^{n+1})^m d(1+\sum_{i=1}^{n+1}(-1)^i\binom{n+1}{i}(bd)^{i-1}),$$

for any $m \in \mathbb{N}$.

Assume that $m=1$. Since $acd = dbd$, we deduce that $(db)^i d = (ac)^i d$ for any $i \in \mathbb{N}$. Then we have

$$\begin{aligned}
((1-bd)-(1-bd)^{n+1})^2 &= (bd+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^i)(bd+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^i) \\
&= (bdbd+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}b(db)^i d)(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^{i-1}) \\
&= b(ac+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^i)d(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^{i-1}) \\
&= -b((1-ac)-(1-ac)^{n+1})d(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^{i-1}).
\end{aligned}$$

Assume that the conclusion holds for positive integer $m=l$. Now, we verify it for $m=l+1$. One can see that

$$\begin{aligned}
& ((1-bd)-(1-bd)^{n+1})^{l+2} \\
= & ((1-bd)-(1-bd)^{n+1})^{l+1}((1-bd)-(1-bd)^{n+1}) \\
= & b((1-ac)-(1-ac)^{n+1})^l d(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^{i-1})(bd+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^i) \\
= & b((1-ac)-(1-ac)^{n+1})^l(dbd+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(db)^i d)(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^{i-1}) \\
= & b((1-ac)-(1-ac)^{n+1})^l(acd+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^i d)(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^{i-1}) \\
= & -b((1-ac)-(1-ac)^{n+1})^{l+1}d(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^{i-1}).
\end{aligned}$$

Note that $(1-ac)-(1-ac)^{n+1} \in R^{nil}$. Then $(1-bd)-(1-bd)^{n+1} \in R^{nil}$, i.e. $1-bd \in R^{nsD}$.

Conversely, assume that $1-bd \in R^{nsD}$. In order to prove $1-ac \in R^{nsD}$, we will prove the following equality

$$((1-ac)-(1-ac)^{n+1})^{m+2} = d((1-bd)-(1-bd)^{n+1})^m bac(1+\sum_{i=1}^{n+1}(-1)^i\binom{n+1}{i}(ac)^{i-1})^2,$$

for any $m \in \mathbb{N}$.

For the case $m=1$. Note that $dba = aca$. Then we obtain $a(ca)^i = (db)^i a$, for any $i \in \mathbb{N}$. Hence, we deduce that

$$\begin{aligned}
((1-ac)-(1-ac)^{n+1})^3 &= -(ac+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^i)(ac+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^i)^2\\
&= -((ac)^3+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i+2})(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2\\
&= -(a(ca)^2c+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}a(ca)^{i+1}c)(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2\\
&= -((db)^2ac+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(db)^{i+1}ac)(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2\\
&= -(d(bd)bac+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}d(bd)^ibac)(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2\\
&= -d(bd+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^i)bac(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2\\
&= d((1-bd)-(1-bd)^{n+1})bac(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2.
\end{aligned}$$

Assume that the conclusion holds for positive integer $m=l$. Then, for the case $m=l+1$, we get

$$\begin{aligned}
&((1-ac)-(1-ac)^{n+1})^{l+3}\\
=\;& -d((1-bd)-(1-bd)^{n+1})^lbac(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2(ac+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^i)\\
=\;& -d((1-bd)-(1-bd)^{n+1})^lb(ac)^2(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2\\
=\;& -d((1-bd)-(1-bd)^{n+1})^l(bacac+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}ba(ca)^ic)(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2\\
=\;& -d((1-bd)-(1-bd)^{n+1})^l(bdbac+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}b(db)^iac)(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2\\
=\;& -d((1-bd)-(1-bd)^{n+1})^l(bdbac+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^ibac)(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2\\
=\;& -d((1-bd)-(1-bd)^{n+1})^l(bd+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(bd)^i)bac(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2\\
=\;& d((1-bd)-(1-bd)^{n+1})^{l+1}bac(1+\sum_{i=1}^{n+1}(-1)^i\tbinom{n+1}{i}(ac)^{i-1})^2.
\end{aligned}$$

Observe that $(1-bd)-(1-bd)^{n+1}\in R^{nil}$. Hence, $(1-ac)-(1-ac)^{n+1}\in R^{nil}$, as required.

Corollary 3.11 *Let $a,b\in R$ and $n\in\mathbb{N}$. Then,*

$$1-ab\in R^{nsD}\Longleftrightarrow 1-ba\in R^{nsD}.$$

4. The n-strong Drazin invertibility of the sum

Let $p\in R$ be an idempotent ($p^2=p$). Then we can represent element $a\in R$ as

$$a=pap+pa(1-p)+(1-p)ap+(1-p)a(1-p)$$

or in the matrix form

$$a=\begin{bmatrix} a_1 & a_3\\ a_4 & a_2\end{bmatrix}_p,$$

where $a_1 = pap$, $a_2 = (1-p)a(1-p)$, $a_3 = pa(1-p)$ and $a_4 = (1-p)ap$. For

$$x = \begin{bmatrix} x_1 & x_3 \\ x_4 & x_2 \end{bmatrix}_p \text{ and } y = \begin{bmatrix} y_1 & y_3 \\ y_4 & y_2 \end{bmatrix}_p,$$

one can use usual matrix rules to obtain matrix forms of the sum $x+y$ and the product xy.

Remark that if $a \in R^{nsD}$, then we have the following matrix representations relative to $p = aa^{nsD}$:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \text{ and } a^{nsD} = \begin{bmatrix} a_1^{-1} & 0 \\ 0 & 0 \end{bmatrix}_p = \begin{bmatrix} a_1^{nsD} & 0 \\ 0 & 0 \end{bmatrix}_p,$$

where $a_1 \in (pRp)^{-1} \cap (pRp)^{nsD}$ and $a_2 \in ((1-p)R(1-p))^{nil}$.

In this section, our purpose is to investigate the n-strong Drazin invertibility of the sum of two elements in a ring. We start will a crucial auxiliary lemma.

Lemma 4.1 *Let $a, b \in R$ be such that $ab = 0$. Then,*

$$a, b \in R^{nil} \Longleftrightarrow a + b \in R^{nil}.$$

Proof Since $ab = 0$, we have

$$(a+b)^m = a^m + ba^{m-1} + b^2a^{m-2} + \cdots + b^m$$

for any $m \in \mathbb{N}$.

Suppose that $a, b \in R^{nil}$. Choose $k_1 \in \mathbb{N}$ satisfying $a^{k_1} = b^{k_1} = 0$. Then, we have $(a+b)^{2k_1} = 0$, which gives $a + b \in R^{nil}$.

On the contrary, assume that $(a+b)^{k_2} = 0$, for some $k_2 \in \mathbb{N}$. Then,

$$a^{k_2} + ba^{k_2-1} + b^2a^{k_2-2} + \cdots + b^{k_2} = 0.$$

Multiplying the preceding equality by a from the left side (resp. by b from the right side), we obtain $a^{k_2+1} = 0$ (resp. $b^{k_2+1} = 0$). Hence, we have $a, b \in R^{nil}$.

Now, we state the relationship between the n-strong Drazin invertibility of the elements a, b and that of the sum $a+b$, under the condition $ab = 0$.

Theorem 4.2 *Let $n \in \mathbb{N}$ and $a, b \in R$ be such that $ab = 0$. Then,*

$$a, b \in R^{nsD} \Longleftrightarrow a + b \in R^{nsD}.$$

Proof By the hypothesis $ab = 0$, we have

$$\begin{aligned} x &:= (a+b) - (a+b)^{n+1} \\ &= (a - a^{n+1}) + (b - b^{n+1}) - (ba^n + b^2a^{n-1} + \cdots + b^na) \\ &:= x_1 + x_2 - x_3. \end{aligned}$$

Note that $x_1(x_2 - x_3) = 0$, $x_3x_2 = 0$ and $x_3^2 = 0$. In view of Lemma 4.1, we get

$$\begin{aligned} x \in R^{nil} &\iff x_1 \in R^{nil} \text{ and } x_2 - x_3 \in R^{nil} \\ &\iff x_1 \in R^{nil} \text{ and } x_2 \in R^{nil}. \end{aligned}$$

Then, by Theorem 3.2 we obtain $a, b \in R^{nsD}$ if and only if $a + b \in R^{nsD}$.

Remark 4.3 *(1) For the Drazin invertibility, we have*

$$a, b \in R^D \iff a + b \in R^D,$$

under the condition $ab = ba = 0$*. In fact, the necessity can be seen from [9, Corollary 1]. Now, suppose that* $a + b \in R^D$*. Then* $(a+b)^m = (a+b)^{m+1}R \cap R(a+b)^{m+1}$*, for some* $m \in \mathbb{N}$*. Hence, we have* $a^m + b^m = (a^{m+1} + b^{m+1})u = v(a^{m+1} + b^{m+1})$*, for some* $u, v \in R$*. Multiplying the previous equality by* a *from the left side and right side respectively, we have* $a^{m+1} = a^{m+2}u = va^{m+2}$*. Hence,* $a \in R^D$*. Similarly, we can obtain* $b \in R^D$*.*

(2) By [3, Theorem 2.1], one can see that

$$a, b \in R^D \Longrightarrow a + b \in R^D,$$

under the condition $ab = 0$*. Now, we consider its converse. Assume that* $a + b \in R^D$*. Then, we can obtain that* a *is right* π*-regular and* b *is left* π*-regular. Is* a *left* π*-regular? Is* b *right* π*-regular?*

Next, we will consider the n-strong Drazin invertibility of the sum $a + b$ under another new condition $a^2b = aba$ and $ab^2 = bab$, which is weaker than $ab = ba$. Indeed, it is obvious that $ab = ba$ imply $a^2b = aba$ and $ab^2 = bab$. However, the converse does not hold in general, which can be seen from the following example:

Example 4.4 *Let* $R = M_2(\mathbb{Z}_2)$*, where* $\mathbb{Z}_2$ *denote the residue class ring modulo* 2*. Take* $a = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ *and* $b = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$*. Clearly,* $a^2b = aba$ *and* $ab^2 = bab$*. However,* $ab \neq ba$*.*

In order to prove our main result, we need the following lemmas.

Lemma 4.5 *Let* $a, b \in R^{nil}$*. If* $a^2b = aba$ *and* $ab^2 = bab$*, then* $a + b \in R^{nil}$*.*

Proof By the hypothesis $a, b \in R^{nil}$, there exists $m \in \mathbb{N}$ such that $a^m = 0$ and $b^m = 0$. Since $a^2b = aba$ and $ab^2 = bab$, we can see that each of the monomials in the expansion of $(a+b)^{3m}$ is either $a^{k_1}b^{k_2}a^{k_3}$ or $b^{l_1}a^{l_2}b^{l_3}$, where $k_1 + k_2 + k_3 = l_1 + l_2 + l_3 = 3m$. Hence, $(a+b)^{3m} = 0$, which means $a + b \in R^{nil}$.

Lemma 4.6 *Let* $x \in R$ *and* $p^2 = p \in R$*. If* x *has the representation* $x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p$*, then*

$$a \in (pRp)^{nil} \text{ and } b \in ((1-p)R(1-p))^{nil} \iff x \in R^{nil}.$$

Proof Assume that $a \in (pRp)^{nil}$ and $b \in ((1-p)R(1-p))^{nil}$. By a simple computation, we obtain $x^k = \begin{bmatrix} a^k & f_k \\ 0 & b^k \end{bmatrix}_p$, for any $k \in \mathbb{N}$, where $f_k = \sum\limits_{i=0}^{k-1} a^i c b^{k-i-1}$. Let $a^{t_1} = 0$ and $b^{t_2} = 0$, where $t_1, t_2 \in \mathbb{N}$. Then, we have $x^{t_1+t_2} = 0$, i.e. $x \in R^{nil}$. Conversely, it is clear.

Lemma 4.7 *Let $n \in \mathbb{N}$ and $p^2 = p, x, y \in R$. If x and y have the representations*

$$x = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p \quad \text{and} \quad y = \begin{bmatrix} b & 0 \\ c & a \end{bmatrix}_{1-p},$$

then

$$a \in (pRp)^{nsD} \text{ and } b \in ((1-p)R(1-p))^{nsD} \Longleftrightarrow x \in R^{nsD} \text{ (resp. } y \in R^{nsD}).$$

Proof Observe that

$$x - x^{n+1} = \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p - \begin{bmatrix} a & c \\ 0 & b \end{bmatrix}_p^{n+1} = \begin{bmatrix} a - a^{n+1} & * \\ 0 & b - b^{n+1} \end{bmatrix}_p.$$

By Lemma 4.6, it follows that

$$a - a^{n+1} \in (pRp)^{nil} \text{ and } b - b^{n+1} \in ((1-p)R(1-p))^{nil} \Longleftrightarrow x - x^{n+1} \in R^{nil}.$$

Using Theorem 3.2, we complete the proof.

Lemma 4.8 *Let $n \in \mathbb{N}$ and $a, b \in R^{nsD}$ be such that $a^2 b = aba$. Then $ab \in R^{nsD}$.*

Proof Since $a^2 b = aba$, by induction we can obtain $(ab)^m = a^m b^m$ for any $m \in \mathbb{N}$. Applying Theorem 3.2, we only need to prove

$$ab - (ab)^{n+1} = (a - a^{n+1})b + a^{n+1}(b - b^{n+1}) := x + y \in R^{nil}.$$

Note that

$$\begin{aligned} yx &= a^{n+1}(b - b^{n+1})(a - a^{n+1})b \\ &= (a^n(ab) - (ab)^{n+1})(a - a^{n+1})b \\ &= (a - a^{n+1})(a^n(ab) - (ab)^{n+1})b \\ &= a^{n+1}(a - a^{n+1})b(b - b^{n+1}) \\ &= (a - a^{n+1})ba^{n+1}(b - b^{n+1}) \\ &= xy. \end{aligned}$$

In addition, we can check that $x^m = (a - a^{n+1})^m b^m$ and $y^m = (a^{n+1})^m (b - b^{n+1})^m$ for any $m \in \mathbb{N}$. Note that $a - a^{n+1} \in R^{nil}$ and $b - b^{n+1} \in R^{nil}$, which imply $x \in R^{nil}$ and $y \in R^{nil}$. Hence, $x + y \in R^{nil}$ by Lemma 2.1(2).

Remark 4.9 *In view of Lemma 4.8 and [26, Lemma 2], one can see that if $n \in \mathbb{N}$ and $a, b \in R^{nsD}$ be such that $ab = ba$, then $ab \in R^{nsD}$ and $(ab)^{nsD} = b^{nsD} a^{nsD}$.*

Now, we state our main result in this section as follows.

Theorem 4.10 *Let $n \in \mathbb{N}$ and $a, b \in R^{nsD}$ be such that $a^2 b = aba$ and $ab^2 = bab$. Then,*

$$1 + a^{nsD} b \in R^{nsD} \Longleftrightarrow a + b \in R^{nsD}.$$

Proof We consider the matrix representations of a and b relative to the idempotent $p = aa^{nsD}$:

$$a = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}_p \text{ and } b = \begin{bmatrix} b_1 & b_3 \\ b_4 & b_2 \end{bmatrix}_p,$$

where $a_1 \in (pRp)^{-1} \cap (pRp)^{nsD}$ and $a_2 \in ((1-p)R(1-p))^{nil}$.

The condition $a^2b = aba$ expressed in matrix form yields

$$\begin{bmatrix} a_1^2 b_1 & a_1^2 b_3 \\ a_2^2 b_4 & a_2^2 b_2 \end{bmatrix}_p = a^2 b = aba = \begin{bmatrix} a_1 b_1 a_1 & a_1 b_3 a_2 \\ a_2 b_4 a_1 & a_2 b_2 a_2 \end{bmatrix}_p.$$

Thus, we have $a_1^2 b_3 = a_1 b_3 a_2$, i.e. $b_3 = a_1^{-1} b_3 a_2$, which implies $b_3 = a_1^{-m} b_3 a_2^m$ for any $m \in \mathbb{N}$. Since $a_2 \in ((1-p)R(1-p))^{nil}$, we have $b_3 = 0$. Moreover, we can get $a_1 b_1 = b_1 a_1$ and $a_2^2 b_2 = a_2 b_2 a_2$. Similarly, by $ab^2 = bab$ we obtain $a_2 b_2^2 = b_2 a_2 b_2$. Therefore, we have

$$b = \begin{bmatrix} b_1 & 0 \\ b_4 & b_2 \end{bmatrix}_p \text{ and } a + b = \begin{bmatrix} a_1 + b_1 & 0 \\ b_4 & a_2 + b_2 \end{bmatrix}_p.$$

Now, we prove that $a_2 + b_2 \in ((1-p)R(1-p))^{nsD}$. Since $b \in R^{nsD}$, by Lemma 4.7 we have $b_2 \in ((1-p)R(1-p))^{nsD}$. Let $p' = b_2 b_2^{nsD}$. We consider the matrix representations of b_2 and a_2 relative to the idempotent p':

$$b_2 = \begin{bmatrix} b_1' & 0 \\ 0 & b_2' \end{bmatrix}_{p'} \text{ and } a_2 = \begin{bmatrix} a_1' & a_3' \\ a_4' & a_2' \end{bmatrix}_{p'},$$

where $b_1' \in (p'Rp')^{-1} \cap (p'Rp')^{nsD}$ and $b_2' \in ((1-p')R(1-p'))^{nil}$. Note that $a_2^2 b_2 = a_2 b_2 a_2$ and $a_2 b_2^2 = b_2 a_2 b_2$. Then, we can obtain that $a_4' = 0$, $a_1' b_1' = b_1' a_1'$, $a_2'(b_2')^2 = b_2' a_2' b_2'$ and $(a_2')^2 b_2' = a_2' b_2' a_2'$. Hence,

$$a_2 = \begin{bmatrix} a_1' & a_3' \\ 0 & a_2' \end{bmatrix}_{p'} \text{ and } a_2 + b_2 = \begin{bmatrix} a_1' + b_1' & a_3' \\ 0 & a_2' + b_2' \end{bmatrix}_{p'}.$$

In order to show that $a_1' + b_1' = (p' + a_1'(b_1')^{-1}) b_1' \in (p'Rp')^{nsD}$, by Lemma 4.8 we only need to prove $p' + a_1'(b_1')^{-1} \in (p'Rp')^{nsD}$. Since $a_2 \in ((1-p)R(1-p))^{nil}$, by Lemma 4.6 we obtain $a_1' \in (p'Rp')^{nil}$, which yields

$$\begin{aligned} & (p' + a_1'(b_1')^{-1}) - (p' + a_1'(b_1')^{-1})^{n+1} \\ = \;& a_1'(b_1')^{-1}\Big(p' - \sum_{i=1}^{n+1} \tbinom{n+1}{i} (a_1')^{i-1} (b_1')^{1-i}\Big) \in (p'Rp')^{nil}. \end{aligned}$$

Hence, $p' + a_1'(b_1')^{-1} \in (p'Rp')^{nsD}$. Applying Lemma 4.5 to the nilpotent elements a_2' and b_2', we conclude that $a_2' + b_2' \in ((1-p')R(1-p'))^{nil}$, which implies $a_2' + b_2' \in ((1-p')R(1-p'))^{nsD}$. In view of Lemma 4.7, we obtain $a_2 + b_2 \in ((1-p)R(1-p))^{nsD}$. Then, by Lemma 4.7 again, it follows that $a + b \in R^{nsD}$ is equivalent to $a_1 + b_1 \in (pRp)^{nsD}$.

Note that

$$1 + a^{nsD} b = \begin{bmatrix} p + a_1^{-1} b_1 & 0 \\ 0 & 1 - p \end{bmatrix}_p.$$

Since $1-p \in ((1-p)R(1-p))^{nsD}$, then $1+a^{nsD}b \in R^{nsD}$ is equivalent to $p+a_1^{-1}b_1 \in (pRp)^{nsD}$. Note that $a_1 \in (pRp)^{nsD}$. Applying Corollary 3.5 we obtain $a_1^{-1} = a_1^{nsD} \in (pRp)^{nsD}$. Hence, $a_1+b_1 = a_1(p+a_1^{-1}b_1) \in (pRp)^{nsD}$ is identical to $p+a_1^{-1}b_1 \in (pRp)^{nsD}$ by Lemma 4.8. Hence, we conclude that $a+b \in R^{nsD}$ if and only if $1+a^{nsD}b \in R^{nsD}$.

Remark 4.11 *In the proof of the necessity of Theorem 4.10, the condition $ab^2 = bab$ was not used. However, if we drop it, then the sufficiency is not true in general, which will be shown in the next example:*

Example 4.12 *Let $n = 1 \in \mathbb{N}$ and $R = M_3(\mathbb{C})$. Choose*

$$a = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } b = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Then, we can check that $a^2b = aba$, $a^2 = 0$ and $b^3 = 0$. Hence, $a, b, 1+a^{sD}b = 1 \in R^{sD}$. Note that the eigenvalues of $(a+b)-(a+b)^2$ are 0, $\sqrt{3}i$ and $-\sqrt{3}i$. Hence, $(a+b)-(a+b)^2 \notin R^{nil}$, which yields that $a+b \notin R^{sD}$. In addition, this example also illustrates that the condition $ab^2 = bab$ of Lemma 4.5 cannot be dropped.

The following corollary can be directly derived from Theorem 4.10.

Corollary 4.13 *Let $a, b \in R^{nsD}$ be such that $ab = ba$. Then,*

$$a+b \in R^{nsD} \Longleftrightarrow 1+a^{nsD}b \in R^{nsD}.$$

5. The n-strong Drazin invertibility of the matrix

In this section, as applications for our additive results of Section 4, we obtain some equivalent conditions for the n-strong Drazin invertibility of the matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ over a ring. For the convenience of expressions, we assume that $\sum_{i=j}^{k} s(i) = 0$ if $k < j$, where $s(i)$ is a function on i, and $a^0 = 1$ for $a \in R$. For any nonnegative integer k, by $\lfloor k/2 \rfloor$ we denote the integer part of $k/2$.

Firstly, we investigate the n-strong Drazin invertibility of some special antitriangular matrices over a ring.

Proposition 5.1 *Let $n \in \mathbb{N}$ and $M = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in M_2(R)$ be such that $ab = b$, $a = a^2$. Then,*

(1) *$bc \in R^{nil}$ if and only if $M \in M_2(R)^{sD}$.*

(2) *If $bc \in R^{nil}$, then $M \in M_2(R)^{nsD}$.*

Proof (1) Since $ab = b$ and $a = a^2$, we have

$$M - M^2 = \begin{bmatrix} -bc & 0 \\ c-ca & -cb \end{bmatrix}.$$

Hence, $M - M^2 \in M_2(R)^{nil}$ is identical to $bc \in R^{nil}$. From Theorem 3.2, it follows that $bc \in R^{nil}$ if and only if $M \in M_2(R)^{sD}$.

(2) This follows from item (1) and Corollary 3.6(1) directly.

Remark 5.2 *(1) In Proposition 5.1(1), if we change the condition "$bc \in R^{nil}$" to "$bc \in R^{sD}$", then the conclusion does not hold in general. For example, take* $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \in M_2(\mathbb{C})$. *Obviously,* $1 \in \mathbb{C}^{sD}$. *However,* $M \notin M_2(\mathbb{C})^{sD}$, *since* $M - M^2 \notin M_2(\mathbb{C})^{nil}$.

(2) The converse of Proposition 5.1(2) is not true for $n \geq 2$ in general, which will be illustrated by the following example:

Example 5.3 *Let* $R = M_2(\mathbb{Z}_3)$. *Choose* $a = b = 1 \in R$ *and* $c = \begin{bmatrix} 2 & 2 \\ 0 & 0 \end{bmatrix} \in R$. *Then, it is easy to see* $bc \notin R^{nil}$. *However,* $M = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in M_2(R)^{2sD}$, *since we can check that* $(M - M^3)^2 = 0$.

Theorem 5.4 *Let* $n \in \mathbb{N}$ *and* $a, b \in R$ *be such that* $aba = 0$. *Then,*

$$M = \begin{bmatrix} a & a \\ b & 0 \end{bmatrix} \in M_2(R)^{nsD} \Longleftrightarrow a \in R^{nsD} \Longleftrightarrow M' = \begin{bmatrix} a & b \\ a & 0 \end{bmatrix} \in M_2(R)^{nsD}.$$

Proof We will only prove that $a \in R^{nsD}$ is equivalent to $M \in M_2(R)^{nsD}$, since the case for M' is similar.

Suppose that $n = 1$. Then, by the condition $aba = 0$ we have

$$X := M - M^2 = \begin{bmatrix} a - a^2 - ab & a - a^2 \\ b - ba & -ba \end{bmatrix}$$

and

$$X^{m+1} = \begin{bmatrix} (a - a^2)^m (a - a^2 - ab + b) & (a - a^2)^{m+1} \\ (b - ba)(a - a^2)^{m-1}(a - a^2 - ab + b) & (b - ba)(a - a^2)^m \end{bmatrix}$$

for any $m \geq 2$. Hence, $X \in M_2(R)^{nil}$ if and only if $a - a^2 \in R^{nil}$. Applying Theorem 3.2, we claim that $a \in R^{sD}$ is equivalent to $M \in M_2(R)^{sD}$.

The result for $n \geq 2$ follows analogously.

Next, we present an existence criterion for the n-strong Drazin inverse of the anti-triangular $\begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix}$, which will be used later.

Theorem 5.5 *Let* $n \in \mathbb{N}$ *and* $M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \in M_2(R)$ *be such that* $ab = 0$. *Then,*

$$a \in R^{nsD} \text{ and } N = \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} \in M_2(R)^{nil} \Longleftrightarrow M \in M_2(R)^{nsD},$$

where

$n_{11} = n_{22} = \frac{1-(-1)^n}{2}(b^{\lfloor \frac{n+1}{2} \rfloor})^2 + (1 - \frac{1+(-1)^n}{2} b^{\lfloor \frac{n}{2} \rfloor})(b - \frac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor})$,
$n_{12} = ((-1)^n - 1) b^{\lfloor \frac{n+1}{2} \rfloor}$,
$n_{21} = ((-1)^n - 1) b^{\lfloor \frac{n+1}{2} \rfloor + 1}$.

Proof Since $ab = 0$, then by induction we have

$$\begin{aligned} X &:= M - M^{n+1} \\ &= \begin{bmatrix} (a - a^{n+1}) - (t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}) & (1 - a^{2\lfloor \frac{n}{2} \rfloor + \frac{1-(-1)^n}{2}}) - (t_3 + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n}{2} \rfloor}) \\ b - (t_2 + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor}) & -(t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}) \end{bmatrix}, \end{aligned}$$

where

$$\begin{aligned} t_1 &= \sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} b^i a^{2(\lfloor \frac{n+1}{2} \rfloor - i) + \frac{1-(-1)^{n+1}}{2}}, \\ t_2 &= \sum_{i=1}^{\lfloor \frac{n+1}{2} \rfloor} b^i a^{2(\lfloor \frac{n+1}{2} \rfloor - i) + \frac{3-(-1)^{n+1}}{2}}, \\ t_3 &= \sum_{i=1}^{\lfloor \frac{n-1}{2} \rfloor} b^i a^{2(\lfloor \frac{n}{2} \rfloor - i) + \frac{1-(-1)^n}{2}}. \end{aligned}$$

By a computation, we obtain $X^2 = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$, where

$$\begin{aligned} u_{11} = {} & (a - a^{n+1})^2 - (t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor})(a - a^{n+1}) + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}(t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}) \\ & + (1 - \frac{1+(-1)^n}{2} b^{\lfloor \frac{n}{2} \rfloor})(b - (t_2 + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor})), \end{aligned}$$

$$\begin{aligned} u_{12} = {} & (a - a^{n+1})(1 - a^{2\lfloor \frac{n}{2} \rfloor + \frac{1-(-1)^n}{2}}) - (t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor})(1 - a^{2\lfloor \frac{n}{2} \rfloor + \frac{1-(-1)^n}{2}}) \\ & - (t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}) + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n}{2} \rfloor}(t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}) + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}(t_3 + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n}{2} \rfloor}), \end{aligned}$$

$$\begin{aligned} u_{21} = {} & b(a - a^{n+1}) - b(t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}) - (t_2 + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor})(a - a^{n+1}) \\ & + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor}(t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}) - \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}(b - (t_2 + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor})), \end{aligned}$$

$$\begin{aligned} u_{22} = {} & b(1 - a^{2\lfloor \frac{n}{2} \rfloor + \frac{1-(-1)^n}{2}}) - b(t_3 + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n}{2} \rfloor}) - (t_2 + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor})(1 - a^{2\lfloor \frac{n}{2} \rfloor + \frac{1-(-1)^n}{2}}) \\ & + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor}(t_3 + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n}{2} \rfloor}) + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}(t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}). \end{aligned}$$

Consider the following splitting:

$$X^2 = \begin{bmatrix} (a - a^{n+1})^2 & p_{12} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} + \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix} := P + Q + N,$$

where

$p_{12} = (a - a^{n+1})(1 - a^{2\lfloor \frac{n}{2} \rfloor + \frac{1-(-1)^n}{2}})$,
$q_{11} = -(t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor})(a - a^{n+1}) + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor} t_1 - (1 - \frac{1+(-1)^n}{2} b^{\lfloor \frac{n}{2} \rfloor}) t_2$,
$q_{12} = -2t_1 + (t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor}) a^{2\lfloor \frac{n}{2} \rfloor + \frac{1-(-1)^n}{2}} + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n}{2} \rfloor} t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor} t_3$,
$q_{21} = b(a - a^{n+1}) - bt_1 - (t_2 + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor})(a - a^{n+1}) + \frac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor} t_1 + \frac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor} t_2$,

$$q_{22} = -bt_3 - t_2 + (t_2 + \tfrac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor} - b) a^{2\lfloor \frac{n}{2} \rfloor + \frac{1-(-1)^n}{2}} + \tfrac{1+(-1)^n}{2} b^{\lfloor \frac{n+2}{2} \rfloor} t_3 + \tfrac{1-(-1)^n}{2} b^{\lfloor \frac{n+1}{2} \rfloor} t_1.$$

Note that $P(Q+N)=0$, $QN=0$ and $Q^2=0$. In view of Lemma 4.1, we have

$$\begin{aligned} X \in M_2(R)^{nil} &\iff X^2 \in M_2(R)^{nil} \\ &\iff P \in M_2(R)^{nil} \text{ and } Q+N \in M_2(R)^{nil} \\ &\iff P \in M_2(R)^{nil} \text{ and } N \in M_2(R)^{nil} \\ &\iff a - a^{n+1} \in R^{nil} \text{ and } N \in M_2(R)^{nil}. \end{aligned}$$

In view of Theorem 3.2, we can conclude that $a \in R^{nsD}$ and $N \in M_2(R)^{nil}$ if and only if $M \in M_2(R)^{nsD}$.

Now, we state a special case of Theorem 5.5.

Corollary 5.6 *Let $n = 2k$ $(k \in \mathbb{N})$ and let $a, b \in R$ be such that $ab = 0$. Then,*

$$a \in R^{nsD} \text{ and } b \in R^{ksD} \iff M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \in M_2(R)^{nsD}.$$

Proof Let $n = 2k$ in Theorem 5.5, we have $N = \begin{bmatrix} b(1-b^k)^2 & 0 \\ 0 & b(1-b^k)^2 \end{bmatrix}$. Then, one can see that

$$N \in M_2(R)^{nil} \Leftrightarrow b(1-b^k)^2 \in R^{nil} \Leftrightarrow (b(1-b^k))^2 \in R^{nil} \Leftrightarrow b - b^{k+1} \in R^{nil}.$$

Hence, we have that $M \in M_2(R)^{nsD}$ is equivalent to $a \in R^{nsD}$ *and* $b \in R^{ksD}$.

Remark 5.7 *By Corollary 5.6 and Corollary 3.6(3), we can see that*

$$M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \in M_2(R)^{nsD} \Longrightarrow a \in R^{nsD} \text{ and } b \in R^{nsD},$$

under the condition $ab = 0$ and n is one even number. However, the converse does not hold in general, which can be seen in the next example:

Example 5.8 *Let $R = M_2(\mathbb{C})$ and $n = 2 \in \mathbb{N}$. Setting $M = \begin{bmatrix} 0 & 1 \\ b & 0 \end{bmatrix} \in M_2(R)$, where $b = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in R$. Obviously, $b - b^3 = 0$, which yields $b \in R^{2sD}$. However, we can check that $M - M^3 \notin M_2(R)^{nil}$, so we have $M \notin M_2(R)^{2sD}$.*

Following the same strategy as in the proof of Theorem 5.5, we derive the equivalent condition for the n-strong Drazin invertibility of the transpose of the matrix M as follows:

Theorem 5.9 *Let $n \in \mathbb{N}$ and $M' = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \in M_2(R)$ be such that $ab = 0$. Then,*

$$a \in R^{nsD} \text{ and } N' = \begin{bmatrix} n_{11} & n_{21} \\ n_{12} & n_{22} \end{bmatrix} \in M_2(R)^{nil} \iff M' \in M_2(R)^{nsD},$$

where n_{11}, n_{12}, n_{21}, and n_{22} are defined as in Theorem 5.5.

Combining Theorem 5.5 and Theorem 5.9, together with the equality $(N')^m = (N^m)'$ for any $m \in \mathbb{N}$, we obtain the relationship between the n-strong Drazin invertibility of the matrix M and that of its transpose M'.

Corollary 5.10 *Let $n \in \mathbb{N}$ and let $a, b \in R$ be such that $ab = 0$. Then,*

$$M = \begin{bmatrix} a & 1 \\ b & 0 \end{bmatrix} \in M_2(R)^{nsD} \Longleftrightarrow M' = \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \in M_2(R)^{nsD}.$$

In the rest of this section, applying the previous results we obtain some characterizations for the n-strong Drazin invertibility of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, under some conditions.

Theorem 5.11 *Let $n \in \mathbb{N}$ and $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$ be such that $abc = 0$ and $bd = 0$. Then,*

$$a, d \in R^{nsD} \text{ and } T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \in M_2(R)^{nil} \Longleftrightarrow M \in M_2(R)^{nsD},$$

where

$$t_{11} = t_{22} = \tfrac{1-(-1)^n}{2}((bc)^{\lfloor \frac{n+1}{2} \rfloor})^2 + (1 - \tfrac{1+(-1)^n}{2}(bc)^{\lfloor \frac{n}{2} \rfloor})(bc - \tfrac{1+(-1)^n}{2}(bc)^{\lfloor \frac{n+2}{2} \rfloor}),$$
$$t_{12} = ((-1)^n - 1)(bc)^{\lfloor \frac{n+1}{2} \rfloor + 1},$$
$$t_{21} = ((-1)^n - 1)(bc)^{\lfloor \frac{n+1}{2} \rfloor}.$$

Proof We write $M = P + Q$, where

$$P = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}.$$

The condition $bd = 0$ ensures $PQ = 0$. Note that $Q \in M_2(R)^{nsD}$ if and only if $d \in R^{nsD}$. In view of Theorem 4.2, we obtain $M \in M_2(R)^{nsD}$ is equivalent to $P \in M_2(R)^{nsD}$ and $d \in R^{nsD}$.

Since

$$P = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix},$$

by Corollary 3.8 we have

$$P \in M_2(R)^{nsD} \Longleftrightarrow P' := \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c \end{bmatrix} = \begin{bmatrix} a & bc \\ 1 & 0 \end{bmatrix} \in M_2(R)^{nsD}.$$

Since $abc = 0$, by Theorem 5.9 we obtain

$$a \in R^{nsD} \text{ and } T = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \in M_2(R)^{nil} \Longleftrightarrow P' \in M_2(R)^{nsD},$$

as required.

Now, we can derive some special cases of Theorem 5.11.

Corollary 5.12 *Let* $n = 2k$ $(k \in \mathbb{N})$ *and* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$ *be such that* $abc = 0$ *and* $bd = 0$. *Then,*

$$a, d \in R^{nsD} \text{ and } bc \in R^{ksD} \Longleftrightarrow M \in M_2(R)^{nsD}.$$

Let $k = 1$ in Corollary 5.12, we have

Corollary 5.13 *Let* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$ *be such that* $abc = 0$ *and* $bd = 0$. *Then,*

$$a, d \in R^{H} \text{ and } bc \in R^{sD} \Longleftrightarrow M \in M_2(R)^{H}.$$

Corollary 5.14 *Let* $n \in \mathbb{N}$ *and* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$ *be such that* $abc = 0$, $bd = 0$ *and* $bc = bca$. *Then,*

$$a, d \in R^{nsD} \Longleftrightarrow M \in M_2(R)^{nsD}.$$

Proof Using the condition $abc = 0$ and $bc = bca$, we have $T^2 = 0$, where T is defined as in Theorem 5.11.

Remark 5.15 *Let* $n \in \mathbb{N}$ *and* $a, b, c, d \in R$ *be such that* $abc = 0$ *and* $bd = 0$. *Then*

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)^{nsD} \Longrightarrow M' = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \in M_2(R)^{nsD}$$

does not hold in general, even if $d = 0$. *For example:*

Example 5.16 *Let* $R = M_2(\mathbb{Z}_2)$ *and* $n = 1 \in \mathbb{N}$. *Choose* $M = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix} \in M_2(R)$, *where*

$$a = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \; b = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } c = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

Then, we can check that $(M - M^2)^2 = 0$ *and* $M' - (M')^2 \notin M_2(R)^{nil}$. *Hence,* $M \in M_2(R)^{sD}$. *However,* $M' \notin M_2(R)^{sD}$.

Similar to the proof of Lemma 4.7 and using the representations of [11, Theorem 1], we can obtain the following lemma.

Lemma 5.17 *Let* $n \in \mathbb{N}$ *and* $M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \in M_2(R)$. *Then*

$$a \in R^{nsD} \text{ and } d \in R^{nsD} \Longleftrightarrow M \in M_2(R)^{nsD}.$$

In this case,

$$M^{nsD} = \begin{bmatrix} a^{nsD} & z \\ 0 & d^{nsD} \end{bmatrix},$$

where

$$z = \sum_{i=0}^{\mathrm{ind}(d)-1} (a^{nsD})^{i+2} b d^i (1 - dd^{nsD}) + \sum_{i=0}^{\mathrm{ind}(a)-1} (1 - aa^{nsD}) a^i b (d^{nsD})^{i+2} - a^{nsD} b d^{nsD}.$$

Applying Lemma 5.17, we may now state the following result.

Theorem 5.18 *Let* $n \in \mathbb{N}$ *and* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$ *be such that* $ca = 0$ *and* $cb = 0$. *Then,*

$$a, d \in R^{nsD} \Longleftrightarrow M \in M_2(R)^{nsD}.$$

Proof The matrix M can be split as

$$M = \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} := P + Q.$$

The conditions $ca = 0$ and $cb = 0$ imply $PQ = 0$. Note that $P^2 = 0$. By Theorem 4.2 and Lemma 5.17, we conclude that $a, d \in R^{nsD}$ if and only if $M \in M_2(R)^{nsD}$.

The next theorem presents new conditions under which we give a characterization for the n-strong Drazin invertibility of the matrix M over a ring.

Theorem 5.19 *Let* $n \in \mathbb{N}$ *and* $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$ *be such that* $bc = cb = 0$ *and* $ca = dc$. *Then,*

$$a, d \in R^{nsD} \Longleftrightarrow M \in M_2(R)^{nsD}.$$

Proof Suppose that $a, d \in R^{nsD}$. Now, we consider the following splitting

$$M = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} := P + Q.$$

The conditions $bc = cb = 0$ and $ca = dc$ imply $PQ = QP$. In view of Lemma 5.17, we deduce that $P \in M_2(R)^{nsD}$ and

$$1 + P^{nsD}Q = \begin{bmatrix} 1 + zc & 0 \\ d^{nsD}c & 1 \end{bmatrix},$$

where z is defined as in Lemma 5.17. Since $ca = dc$, by [10, Theorem 2] we obtain

$$ca^{nsD} = ca^D = d^D c = d^{nsD}c.$$

Hence, for any $m \in \mathbb{N}$, we have

$$b(d^{nsD})^m c = bc(a^{nsD})^m = 0,$$

and

$$bd^m(1 - dd^{nsD})c = bd^m c - bd^{m+1}d^{nsD}c = bca^m - bca^{m+1}a^{nsD} = 0.$$

In addition, $a^{nsD}bd^{nsD}c = a^{nsD}bca^{nsD} = 0$. Hence, $zc = 0$. Hence, $1 + P^{nsD}Q \in M_2(R)^{nsD}$. Applying Corollary 4.13, we deduce that $M \in M_2(R)^{nsD}$.

Conversely, suppose that $M \in M_2(R)^{nsD}$, i.e. $X := M - M^{n+1} \in M_2(R)^{nil}$. By induction, we can obtain

$$X = \begin{bmatrix} a - a^{n+1} & b - \sum\limits_{i=0}^{n} a^{n-i}bd^i \\ c - (n+1)d^n c & d - d^{n+1} \end{bmatrix}.$$

By a computation, we have

$$X^m = \begin{bmatrix} (a - a^{n+1})^m & * \\ * & (d - d^{n+1})^m \end{bmatrix},$$

for any $m \in \mathbb{N}$. Therefore, we conclude that $a - a^{n+1} \in R^{nil}$ and $d - d^{n+1} \in R^{nil}$, which yield $a, d \in R^{nsD}$, as required.

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