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On composition factors in modules over some group rings

Martyn Russell DIXON[1,*], **Leonid Andreevich KURDACHENKO**[2], **Igor Yakov SUBBOTIN**[3]

[1]Department of Mathematics, University of Alabama, Tuscaloosa, AL, USA
[2]Department of Algebra and Geometry, National University of Dnepropetrovsk, Dnipro, Ukraine
[3]Department of Mathematics, National University, Los Angeles, CA, USA

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Abstract: The aim of this paper is to prove the following result: Let G be an FC-hypercentral group and let A have a finite FG-composition series. Then A contains two FG-submodules B, C such that $A = B \oplus C$, where each FG-composition factor of B has finite F-dimension and each FG-composition factor of C has infinite F-dimension. This has consequences for FG-modules whose proper submodules all have finite F-dimension and for those FG-modules whose proper quotients all have finite F-dimension.

Key words: FC-hypercentral, module, FG-composition series, simple FG-module, quasifinite module, just infinite dimensional module

1. Introduction

Let R be a ring, let G be a group, and let A be a (left) module over the group ring RG, notation that we shall use throughout with little further mention. We shall also let F denote a field. Suppose that A has a finite RG-composition series. Some factors in this series may have similar properties (for example, some may be RG-central); however, their relative positions in the composition series may be quite diverse. Therefore, it is natural to consider questions regarding the positioning of RG-composition factors having similar properties. The roots of this problem date to the famous Fitting lemma, a slight generalization of which can be stated as follows:

> Let G be a finite nilpotent group. If A has finite composition length, then $A = Z \oplus E$, where each RG-composition factor U/V of Z (respectively of E) satisfies the condition $G = C_G(U/V)$ (respectively $G \neq C_G(U/V)$).

This result served as a starting point for many generalizations that have found applications in various fields of group theory (see [1, Chapters 1, 7, and 8], for example). In turn, these results were also generalized to factors associated with various formations, a topic that will not be pursued further here, but as references see the book [5, Chapter 10] and the survey [2]. In this paper, we consider a different side of this issue.

Suppose that A has a finite FG-composition series. The factors of this series are simple FG-modules, some of which will have finite F-dimension and some infinite F-dimension. The question naturally arises whether all the finite dimensional factors can be made consecutive and whether all infinite dimensional factors can be made consecutive. Clearly, the answer will generally depend on the properties of the group G.

*Correspondence: mdixon@ua.edu

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If a simple FG-factor U/V has finite F-dimension k, we can consider $G/C_G(U/V)$ as an irreducible subgroup of the finite dimensional linear group $GL_k(F)$. There is a lot of information concerning such groups. On the other hand, if the simple FG-factor U/V has infinite F-dimension, then information concerned with the structure of $G/C_G(U/V)$ is rather limited. Such information is available in the book [4, Chapters 2 and 3], where the structure of irreducible hypercentral and FC-hypercentral groups was given.

It is appropriate here to recall the following notation and definitions.

For the group G, if $x \in G$, then let

$$x^G = \{x^g = g^{-1}xg | g \in G\} \text{ and } \mathbf{FC}(G) = \{x \in G | x^G \text{ is finite}\},$$

this latter characteristic subgroup of G being the FC-center of G.

Beginning with the FC-center, we may construct the upper FC-central series of the group G, the series of characteristic subgroups of G,

$$1 = C_0 \leq C_1 \leq \ldots C_\alpha \leq C_{\alpha+1} \leq \ldots C_\gamma,$$

defined by

$$C_1 = \mathbf{FC}(G),$$
$$C_{\alpha+1}/C_\alpha = \mathbf{FC}(G/C_\alpha) \text{ for ordinals } \alpha,$$
$$C_\lambda = \bigcup_{\beta<\lambda} C_\beta \text{ for limit ordinals } \lambda.$$

The last term C_γ of this series is called the upper FC-hypercenter of the group G, and if $G = C_\gamma$, then G is called FC-hypercentral. In the case when γ is finite and $G = C_\gamma$ the group G is called FC-nilpotent.

The main result of this paper is the following:

Theorem A *Let G be an FC-hypercentral group and let A have a finite FG-composition series. Then A contains two FG-submodules B, C satisfying the following conditions:*

(i) each FG-composition factor of B has finite F-dimension;

(ii) each FG-composition factor of C has infinite F-dimension;

(iii) $A = B \oplus C$.

We note some consequences of this theorem. Suppose that the module A has the property that A is not a finitely generated RG-module but every proper RG-submodule of A is finitely generated as an RG-submodule.

In this case, one of the following situations arises:

(i) A is the ascending union of its proper RG-submodules.

(ii) A contains a proper RG-submodule B such that A/B is a simple RG-module.

An RG-module satisfying condition (i) is called a quasifinite RG-module. Quasifinite FG-modules have been considered in the papers [7, 8] of the last two authors of this paper and these results were generalized by Kurdachenko [3] to the case of RG-modules when R is a Dedekind domain.

Corollary A *Let G be an FC-hypercentral group and let A be an FG-module having infinite F-dimension. If every proper FG-submodule of A has finite F-dimension, then either A is a quasifinite FG-module or A is a simple FG-module.*

We may also consider the dual situation. Suppose that A has infinite F-dimension but every proper FG-quotient module of A has finite F-dimension. (The quotient module A/B is proper if the submodule B is nonzero.)

In this case, one of the following situations arises:

(i) The intersection of all nonzero FG-submodules of A is zero, so A is a nonmonolithic FG-module.

(ii) A contains a proper nonzero simple FG-submodule B (the FG-monolith of A) such that the factor module A/B has finite F-dimension.

The FG-module satisfying condition (i) is called a just-infinite dimensional FG-module. Just-infinite dimensional FG-modules were discussed in the papers [9, 10] of the last two authors. It is also worth mentioning that an RG-module A is called just-non-Artinian if A is not Artinian but each proper factor of A is. Such modules were discussed in the papers [11, 12] again in the case where R is Dedekind.

Corollary B *Let G be an FC-hypercentral group and let A be an FG-module having infinite F-dimension. If every proper FG-quotient module of A has finite F-dimension, then either A is a just-infinite dimensional FG-module or A is a simple FG-module.*

2. Extensions of finite dimensional modules by infinite dimensional simple modules

We begin with the following result.

Lemma 2.1 *Let A be an FG-module such that $C_G(A) = 1$. Suppose that A contains an FG-submodule B such that B and A/B are simple FG-modules, B has finite F-dimension, and A/B has infinite dimension. If $C_G(B) \cap \boldsymbol{FC}(G) \neq 1$, then there is an FG-submodule L of A such that $A = B \oplus L$.*

Proof Let z be a nontrivial element of $C_G(B) \cap \mathbf{FC}(G)$. Since the conjugacy class of z is finite the normal subgroup $V = C_G(\langle z\rangle^G)$ has finite index in G. Let $\varphi : A \longrightarrow A$ be the mapping defined by $\varphi(a) = (z-1)a = [z, a]$ for all $a \in A$. If $v \in V$, then

$$v(z-1)a = (z-1)va = (z-1)(va) \in [z, A],$$

which shows that $[z, A] = \mathbf{Im}(\varphi)$ is an FV-module. Likewise, $\mathbf{ker}(\varphi) = C_A(z)$ is an FV-module. Clearly $B \leq C_A(z)$ and $C_A(z) \neq A$ since $C_G(A) = 1$. Since A/B is a simple FG-module, it follows from [5, Theorem 5.5] that A/B is a direct sum of finitely many FV-submodules, each of which has infinite F-dimension. Since A/B is a semisimple FV-module its FV-submodule $C_A(z)/B$ has an FV-complement E/B (see [5, Corollary 4.3]). Again by [5, Corollary 4.3] both E/B and $C_A(z)/B$ are semisimple FV-modules, each of which is a direct sum of finitely many simple factors having infinite F-dimension. However,

$$E/B \cong_{FV} (A/B)/(C_A(z)/B) \cong_{FV} A/C_A(z) \cong_{FV} [z, A],$$

so $[z,A]$ is a semisimple FV-module, each of whose simple FV-factors has infinite F-dimension. Let Y be one of these infinite-dimensional simple FV-submodules of $[z,A]$. Let $\{g_1,\dots,g_l\}$ be a transversal to V in G and let $L = g_1Y+\cdots+g_lY$. Then L is a semisimple FV-module, by [5, Corollary 4.3] again, whose FV-factors are infinite-dimensional. Thus, L is an FG-submodule of infinite F-dimenson. Since A/B is a simple FG-module and B has finite F-dimension it follows that $A = L + B$. Moreover, if $L \cap B \neq 0$, then $B \leq L$ since B is simple, but then $L = A$ is a semisimple FV-module whose FV-factors have infinite F-dimension. This contradicts the fact that B is finite-dimensional. Hence, $L \cap B = 0$ and $A = B \oplus L$. □

Lemma 2.2 *Let G be an FC-hypercentral group and let A be a simple FG-module such that $C_G(A) = 1$. If A has finite F-dimension, then G is abelian-by-finite.*

Proof It follows from [13, Theorem 2] that G is nilpotent-by-finite and then that G is abelian-by-finite follows from [1, Theorem 1.4.11], for example. □

We shall require the following technical result. As usual we let the annihilator of a subset Y of A be

$$\mathbf{Ann}_R(Y) = \{r \in R | ry = 0 \text{ for all } y \in Y\}.$$

Assume now that A is an FG-module and that $C_G(A) = 1$. For each $1 \neq g \in \zeta(G)$ we let $X(g) = \langle x(g)\rangle$ denote an infinite cyclic group and let $D(g) = FX(g)$ denote the corresponding group algebra. We note that $D(g)$ is a principal ideal domain. We can define an action of $x(g)$ on A by setting

$$x(g)a = ga \text{ for all } a \in A.$$

This action can be extended in a natural way to an action of $D(g)$ on A and hence we may consider A as a left $D(g)G$-module.

Let R be an integral domain and let A be an R-module. Let

$$\mathbf{Tor}_R(A) = \{a \in A | \mathbf{Ann}_R(a) \neq 0\}.$$

It is easy to see that $\mathbf{Tor}_R(A)$ is an R-submodule of A called the R-periodic part of A. We say that A is periodic as an R-module or, simply, that A is R-periodic, if $\mathbf{Tor}_R(A) = A$ so that $\mathbf{Ann}_R(a) \neq 0$, for each element $a \in A$. We say that A is R-torsion-free if $\mathbf{Tor}_R(A) = 0$.

Analogous to the annihilator of a subset of A is the notion of the annihilator of a subset X of R. In this case, we have

$$\mathbf{Ann}_A(X) = \{a \in A | xa = 0 \text{ for all } x \in X\}.$$

We define the R-assassinator of A to be the set

$$\mathbf{Ass}_R(A) = \{P | P \text{ is a prime ideal of } R \text{ such that } \mathbf{Ann}_A(P) \neq 0\}.$$

If U is an ideal of R, then we set

$$A_U = \{a \in A | U^n a = 0 \text{ for some natural number } n\}.$$

It is easy to see that A_U is an R-submodule of A called the U-component of A. If $A = A_U$, then A is called a U-module. Furthermore, let

$$\Omega_{U,k}(A) = \{a \in A | U^k a = 0 \text{ for this fixed } k\}.$$

It is easy to see that $\Omega_{U,k}(A)$ is an R-submodule, that

$$\Omega_{U,1}(A) \leq \Omega_{U,2}(A) \leq \cdots \leq \Omega_{U,k}(A) \leq \dots, \text{ and that}$$
$$A_U = \bigcup_{k\in\mathbb{N}} \Omega_{U,k}(A).$$

We refer the reader to [6], which has further details.

Lemma 2.3 *Let G be an abelian group and A an FG-module. Suppose that A contains an FG-submodule B such that B and A/B are simple. If B has finite F-dimension and A/B has infinite F-dimension, then A contains an FG-submodule C such that $A = B \oplus C$.*

Proof Without loss of generality we may assume that $C_G(A) = 1$. Note that if A contains a nonzero FG-submodule C such that $C \cap B = 0$, then $(C + B)/B$ is a nonzero FG-submodule of A/B. Since A/B is simple it follows that $A = B + C = B \oplus C$.

We may therefore suppose that every nonzero FG-submodule of A contains B, so B is then the FG-monolith of A. Using the notation introduced above, A is a left $D(g)$-module for each element $g \in G$.

Let $b \in B$ so that $D(g)b \leq B$. We note that $D(g)$ has infinite F-dimension. Since $D(g)b \cong D(g)/\mathbf{Ann}_{D(g)}(b)$ and $\mathbf{dim}_F(B)$ is finite it follows that b has nonzero annihilator in the ring $D(g)$. Hence, the $D(g)$-periodic part of A is nonzero also. Let $\mathbf{Tor}_{D(g)}(A) = T$ so that $B \leq T$. Also, $\mathbf{Ann}_{D(g)}(b) = \mathbf{Ann}_{D(g)}(B) = P$ is a maximal ideal of $D(g)$. Since G is abelian, T is an FG-submodule and since A/B is simple we have $T = B$ or $T = A$.

Suppose first that $T = B$. Then there is a $D(g)$-submodule U of A such that $A = B \oplus U$ by [6, Proposition 8.9]. Then U is $D(g)$-torsion-free so that $PU \neq 0$. Since $PA = PU \leq U$ we have

$$PA \cap B = PA \cap T = 0.$$

However, G is abelian so PA is an FG-submodule and hence $B \leq PA$, a contradiction, which shows that $T = A$. Therefore, A is a periodic $D(g)$-module.

In this case $A = \oplus_{Q\in\pi} A_Q$ where A_Q is a non-zero Q-component of A and $\pi = \mathbf{Ass}_{D(g)}(A)$ (see [6, Corollary 3.8], for example). Since G is abelian, A_Q is an FG-submodule of A and since A is FG-monolithic with $B \leq A_P$ we have $A = A_P$.

Suppose that $\mathbf{\Omega}_{P,1}(A) \neq A$. Then $\mathbf{\Omega}_{P,1}(A) = B$ since $\mathbf{\Omega}_{P,1}(A)$ is an FG-module. Since $\mathbf{\Omega}_{P,2}(A)$ is also an FG-module it then follows that $\mathbf{\Omega}_{P,2}(A) = A$. Since $D(g)$ is a principal ideal domain there exists $y \in P$ such that $P = yD(g)$ so $y^2a = 0$, since $\mathbf{\Omega}_{P,2}(A) = A$, for each element $a \in A$. Therefore, $yA \leq B$ so yA has finite dimension. Since G is abelian the map $\rho : A \longrightarrow A$ defined by $\rho(a) = ya$ for $a \in A$ is an FG-endomorphism of A and $yA = \mathbf{Im}(\rho) \leq B$. Clearly $\mathbf{ker}(\rho) = \mathbf{\Omega}_{P,1}(A) = B$ so $yA \cong A/B$, a contradiction since yA has finite dimension whereas A/B has infinite F-dimension. Therefore, $\mathbf{\Omega}_{P,1}(A) = A$. Hence, $\mathbf{Ann}_{D(g)}(A) = P$ is a maximal ideal of $D(g)$; this is true for all $g \in G$.

Let $d \in A \setminus B$ and let $D = (FG)d$, the FG-submodule generated by d. Clearly $B \leq D$ since A is FG-monolithic so there exists $u \in FG$ such that $0 \neq ud \in B$. Let

$$u = \alpha_1 g_1 + \cdots + \alpha_m g_m$$

for certain $\alpha_i \in F, g_i \in G$, $1 \leq i \leq m$. From what we proved above, $\mathbf{Ann}_{D(g_1)}(A)$ is a maximal ideal of $D(g_1)$. Let F_1 be the field $D(g_1)/\mathbf{Ann}_{D(g_1)}(A)$, so we can consider A as an F_1G-module. This module is again F_1G-monolithic and B is its F_1G-monolith. By the arguments used above A is annihilated by some maximal ideal of the group ring $F_1\langle g_2\rangle$ and it follows that $\mathbf{Ann}_{F\langle g_1,g_2\rangle}(A)$ is a maximal ideal of the ring $F\langle g_1, g_2\rangle$. Using these arguments we see that after finitely many steps $\mathbf{Ann}_{F\langle g_1,\dots,g_m\rangle}(A)$ is a maximal ideal of the ring $F\langle g_1,\dots,g_m\rangle$. Then $F\langle g_1,\dots,g_m\rangle/\mathbf{Ann}_{F\langle g_1,\dots,g_m\rangle}(d)$ is simple, so $F\langle g_1,\dots,g_m\rangle d$ is simple. As B is an FG-module it is an $F\langle g_1,\dots,g_m\rangle$-module and hence $(F\langle g_1,\dots,g_m\rangle)d \cap B = 0$, since $d \notin B$. On the other hand,

$$0 \neq ud \in (F\langle g_1,\dots,g_m\rangle)d \cap B,$$

and we obtain a contradiction, which proves the lemma. □

We next extend this result to the abelian-by-finite case.

Lemma 2.4 *Let G be an abelian-by-finite group and A an FG-module. Suppose that A contains a simple FG-submodule B such that A/B is also simple. If B has finite F-dimension and A/B has infinite F-dimension, then A contains an FG-submodule C such that $A = B \oplus C$.*

Proof As in the proof of Lemma 2.3 we may assume that $C_G(A) = 1$ and that every nonzero FG-submodule of A contains B. We shall obtain a contradiction in this case.

Let H be a normal abelian subgroup of G of finite index. Then both B and A/B contain simple FH-submodules, denoted by C and D/B, respectively, such that $B = \oplus_{1\leq j\leq n} x_jC$ and $A/B = \oplus_{1\leq m\leq k} y_mD/B$ for certain $x_j, y_j \in A$ (see [5, Theorem 5.5], for example). We note that C has finite F-dimension and that D/B has infinite F-dimension. It follows that A has a finite series of FH-submodules:

$$0 = B_0 \leq C = B_1 \leq \cdots \leq B_n = B \leq D = B_{n+1} \leq \cdots \leq B_{n+k} = A,$$

whose factors are simple FH-modules, where B_j/B_{j-1} has finite dimension for $1 \leq j \leq n$ and B_m/B_{m-1} has infinite F-dimension for $n+1 \leq m \leq n+k$. Let $E = E_0 = B_{n-1}$.

By Lemma 2.3 B_{n+1}/E contains an FH-submodule E_1/E such that $B_{n+1}/E = B_n/E \oplus E_1/E$. In particular, B_{n+1}/E_1 is a simple FH-module of finite F-dimension. Suppose, inductively, that for some $r \geq 0$ we have constructed a proper FH-submodule E_r such that B_{n+r}/E_r is a simple FH-module of finite F-dimension. Then B_{n+r+1}/E_r satisfies the hypotheses of Lemma 2.3 and we see that there is an FH-submodule E_{r+1}/E_r such that $B_{n+r+1}/E_r = B_{n+r}/E_r \oplus E_{r+1}/E_r$. Then B_{n+r+1}/E_{r+1} is a simple FH-module of finite F-dimension. This argument implies the existence of a proper FH-submodule E_k such that A/E_k has finite F-dimension.

Let $\{g_1,\dots,g_t\}$ be a transversal to H in G. Since $A/E_k \cong_F A/g_jE_k$ for each j with $1 \leq j \leq t$, it follows that A/g_jE_k has finite F-dimension. Hence, the FG-submodule $L = \bigcap_{1\leq j\leq t} g_jE_k$ has finite codimension. Since $L \neq 0$ we have $B \leq L$ and L/B is a proper FG-submodule of A/B, a contradiction since A/B is simple. This proves the result. □

Proposition 2.5 *Let G be an FC-hypercentral group and A an FG-module. Suppose that A contains an FG-submodule B such that B and A/B are simple FG-modules. If B has finite F-dimension and A/B has infinite F-dimension, then A contains an FG-submodule C such that $A = B \oplus C$.*

Proof Without loss of generality we may suppose that $C_G(A)$ is trivial. If $C_G(B) \neq 1$, then $C_G(B) \cap \mathbf{FC}(G) \neq 1$ also (see [5, Corollary 3.16], for example), in which case the result follows from Lemma 2.1. On the other hand, if $C_G(B) = 1$, then we may think of G as a subgroup of $GL_n(F)$, where n is the F-dimension of B. By Lemma 2.2 G is then abelian-by-finite and Lemma 2.4 gives the result. □

Corollary 2.6 *Let G be FC-hypercentral and let A be an FG-module. Suppose that A contains a finite dimensional FG-submodule B such that A/B is an infinite dimensional simple FG-module. Then A contains an FG-submodule C such that $A = B \oplus C$.*

Proof Since B has finite F-dimension it has a finite series

$$0 = B_0 \leq B_1 \leq \cdots \leq B_n = B$$

whose factors are simple FG-modules. We use induction on n, the case $n = 1$ being covered by Proposition 2.5.

Suppose that $n > 1$ and that the result is true for A/B_1. Then A/B_1 contains an FG-submodule D/B_1 such that $A/B_1 = B/B_1 \oplus D/B_1$. Then D/B_1 is a simple infinite-dimensional FG-module and since B_1 is a simple finite-dimensional FG-module, Proposition 2.5 implies that D contains an FG-submodule C such that $D = B_1 \oplus C$. Then we have

$$A = B + D = B + B_1 + C = B + C$$

and $B \cap C = 0$ since $D \cap B = B_1$. Hence, $A = B \oplus C$, as required. □

To conclude this section we note that Corollary 2.6 is no longer true when we step outside the realm of FC-hypercentral groups even in the case when G is soluble. In his paper [14], Zaitsev gave an example of an $\mathbb{F}_2G$-module A over the Charin $\{2,3\}$-group G, which has a finite $\mathbb{F}_2G$-submodule B such that A/B is an infinite simple $\mathbb{F}_2G$-module and B has no direct $\mathbb{F}_2G$-complement in A.

3. Extensions of infinite-dimensional simple modules by finite-dimensional modules

In this section we consider the dual situation to that occurring in Section 2; many of the results in this section are analogous to those occurring in that last section.

Lemma 3.1 *Let A be an FG-module such that $C_G(A) = 1$. Suppose that A contains an FG-submodule B such that B and A/B are simple FG-modules, B has infinite F-dimension, and A/B has finite dimension. If $C_G(A/B) \cap \boldsymbol{FC}(G) \neq 1$, then there is an FG-submodule L of A such that $A = B \oplus L$.*

Proof Let $1 \neq z \in C_G(A/B) \cap \mathbf{FC}(G)$. The conjugacy class of z is finite so $V = C_G(\langle z \rangle^G)$ has finite index in G. Let $\varphi : A \longrightarrow A$ be defined by $\varphi(a) = (z-1)a = [z,a]$ for all $a \in A$, as in Lemma 2.1. Then $[z,A] = \mathbf{Im}(\varphi)$ and $\mathbf{ker}(\varphi) = C_A(z)$ are FV-submodules and $[z,A] \leq B$. Since $C_G(A) = 1$, $[z,A] \neq 0$. It follows from [5, Theorem 5.5] that the simple FG-module B has a simple FV-submodule C such that $B = \oplus_{1 \leq j \leq n} x_j C$ is a direct sum of finitely many simple FV-submodules each having infinite F-dimension. Since $[z,A] \leq B$, [5, Corollary 4.4] implies that $[z,A]$ is also a semisimple FV-submodule such that its simple FV-factors have infinite F-dimension. However,

$$[z,A] = \mathbf{Im}(\varphi) \cong A/\mathbf{ker}(\varphi) = A/C_A(z)$$

by the first isomorphism theorem so A has a proper FV-submodule Y such that A/Y is a simple FV-module of infinite F-dimension. Let $\{g_1, \ldots, g_k\}$ be a transversal to V in G and let $L = g_1Y \cap \cdots \cap g_kY$, an FG-module. By Remak's theorem we obtain an embedding:

$$A/L \longrightarrow \oplus_{1\leq j\leq k} A/g_jY.$$

Since $A/g_jY = g_jA/g_jY \cong A/Y$ for each j it follows that A/g_jY is a simple FV-module of infinite F-dimension. Thus, A/L is a semisimple FV-module and each of its FV-factors has infinite F-dimension (see [5, Corollary 4.4]). Furthermore, if $L = 0$, then A is a semisimple FV-module each of whose FV-factors have infinite F-dimension. This implies that A/B cannot be a semisimple FV-module of finite F-dimension. Hence, $L \neq 0$.

Since B is a simple FG-submodule, either $B \cap L = 0$ or $B \leq L$. Suppose that $B \leq L$. Since A/B is a simple FG-module and L is a proper FG-submodule of A we deduce that $L = B$. However, B has finite codimension and L has infinite codimension, a contradiction that proves that $B \cap L = 0$. Since A/B is a simple FG-module we have $A = B + L = B \oplus L$, as required. □

Lemma 3.2 *Let G be an abelian group and let A be an FG-module. Suppose that B is an FG-submodule of A such that B and A/B are simple FG-modules, B has infinite F-dimension, and A/B has finite dimension. Then there is an FG-submodule L of A such that $A = B \oplus L$.*

Proof As in the proof of Lemma 2.3 we may assume that $C_G(A) = 1$ and that every nonzero FG-submodule of A contains B. We shall obtain a contradiction in this case.

We may therefore suppose that every nonzero FG-submodule of A contains B so that B is the FG-monolith of A. We again consider A as a $D(g)$-module for an arbitrary element $g \in G$.

Let $a \in A \setminus B$ and note that $D(g)$ has infinite F-dimension. Then $D(g)(a+B) \cong D(g)/\mathbf{Ann}_{D(g)}(a+B)$ and since $\mathbf{dim}_F(A/B)$ is finite it follows that $a+B$ has nonzero annihilator in the ring $D(g)$. Thus, A/B is $D(g)$-periodic and A/B has finite dimension, so it is a periodic finitely generated module over the principal ideal domain $D(g)$. Hence, there is an element $y \in D(g)$ such that $yA \leq B$.

Suppose that A is $D(g)$-torsion-free. Since G is abelian, the map $\rho : A \longrightarrow A$ defined by $\rho(a) = ya$ is an FG-endomorphism of A and $\mathbf{Im}(\rho) = yA$ is an FG-submodule of A. Since A is $D(g)$-torsion-free, we have $\mathbf{ker}(\rho) = 0$, so $yA = B$ since B is simple. However,

$$yA = \mathbf{Im}(\rho) \cong A/\mathbf{ker}(\rho) \cong A,$$

a contradiction since yA is simple.

This shows that A is not $D(g)$-torsion-free so that $T = \mathbf{Tor}_{D(g)}(A) \neq 0$ and since B is the FG-monolith of A we have $B \leq T$. Since G is abelian T is an FG-submodule and since A/B is simple either $T = B$ or $T = A$. However, if $T = B$, then we have $yA \leq T$, so that A is a $D(g)$-periodic module in this case and hence $A = T$. Hence, we may suppose that A is $D(g)$-periodic.

Then $A = \oplus_{Q\in\pi} A_Q$, where A_Q is the nonzero Q-component of A and $\pi = \mathbf{Ass}_{D(g)}(A)$ (see [6, Corollary 3.8], for example). Each Q-component A_Q is an FG-submodule of A so $B = A_P$ for some maximal ideal P of $D(g)$, since A is FG-monolithic. Also, $\mathbf{\Omega}_{P,1}(A)$ is a nonzero FG-submodule and if $\mathbf{\Omega}_{P,1}(A) \neq A$, then $\mathbf{\Omega}_{P,1}(A) = B$. Since $\mathbf{\Omega}_{P,2}(A)$ is an FG-submodule, it then follows that $\mathbf{\Omega}_{P,2}(A) = A$. We know that $D(g)$

is a principal ideal domain so it contains an element z such that $P = zD(g)$ and we deduce that $z^2a = 0$ for each element $a \in A$. Therefore, defining rho in a similar manner to that above we see that $\mathbf{Im}(\rho) = zA \leq B$, but in this case $\mathbf{ker}(\rho) = \mathbf{\Omega}_{P,1}(A)$ so that

$$zA \cong A/\mathbf{\Omega}_{P,1}(A).$$

Thus, zA has finite F-dimension so $zA \neq B$. On the other hand, $A \neq \mathbf{\Omega}_{P,1}(A)$ so zA is nonzero, contradicting the fact that B is simple. It therefore follows that $\mathbf{\Omega}_{P,1}(A) = A$ and hence $\mathbf{Ann}_{D(g)}(A) = P$ is a maximal ideal of $D(g)$, which is true for all $g \in G$.

The last part of the proof now follows almost verbatim the last part of the proof of Lemma 2.3, so we omit it. □

We now extend this to the abelian-by-finite case.

Lemma 3.3 *Let G be an abelian-by-finite group. Suppose that B is an FG-submodule of A such that B and A/B are simple. If B has infinite F-dimension and A/B has finite F-dimension, then A contains an FG-submodule C such that $A = B \oplus C$.*

Proof Let H be a normal abelian subgroup of G of finite index. Then both B and A/B contain minimal H-invariant subspaces, denoted by C and D/B, respectively, such that $B = \oplus_{1 \leq j \leq n} x_j C$ and $A/B = \oplus_{1 \leq m \leq k} y_m D/B$ (see [5, Theorem 5.5], for example) for certain $x_j, y_j \in A$. We note that C has infinite F-dimension and that D/B has finite F-dimension. It follows that A has a finite series

$$0 = B_0 \leq C = B_1 \leq \cdots \leq B_n = B \leq D = B_{n+1} \leq \cdots \leq B_{n+k} = A$$

of FH-submodules whose factors are simple, the factors B_j/B_{j-1} have infinite dimension for $1 \leq j \leq n$, and the factors B_m/B_{m-1} have finite dimension for $n+1 \leq m \leq n+k$.

Let $E = E_0 = B_{n-1}$ and consider D/E. Using Lemma 3.2 we see that D/E contains an FH-submodule E_1/E such that $D/E = B/E \oplus E_1/E$. Thus, $E_1/E \cong_{FH} D/B$ is a simple FH-module of finite F-dimension.

Suppose, inductively, that we have constructed for some $r \geq 0$ an FH-submodule E_r such that E_r/B_{n-r} is a simple FH-module of finite F-dimension. Then, using Lemma 3.2, we see that E_r/B_{n-r-1} contains an FH-submodule E_{r+1}/B_{n-r-1} such that $E_r/B_{n-r-1} = E_{r+1}/B_{n-r-1} \oplus B_{n-r}/B_{n-r-1}$ so E_{r+1}/B_{n-r-1} is simple of finite F-dimension. We obtain a nonzero FH-submodule E_k, which has finite F-dimension. Let $\{g_1, \ldots, g_t\}$ be a transversal to H in G. Then $E_k \cong_F g_j E_k$ for each j with $1 \leq j \leq t$ and hence $g_j E_k$ is finite-dimensional. Hence, $L = \sum_{1 \leq j \leq t} g_j E_k$ is a finite-dimensional FG-module. Since $E_k \neq 0$, we have $L \neq 0$. However, B is simple and has infinite F-dimension, so $L \cap B = 0$. Since A/B is a simple FG-module, $A/B = (L+B)/B$, so $A = L + B = L \oplus B$ as required. □

Proposition 3.4 *Let G be an FC-hypercentral group and let A be an FG-module. Suppose that B is an FG-submodule of A such that B and A/B are simple FG-modules. If B has infinite F-dimension and A/B has finite F-dimension, then A contains an FG-submodule C such that $A = B \oplus C$.*

Proof Without loss of generality we may assume that $C_G(A)$ is trivial. Suppose first that $C_G(A/B) \neq 0$. Then $C_G(A/B) \cap \mathbf{FC}(G) \neq 0$ also (see [5, Corollary 3.16], for example) and the result follows from Lemma 3.1.

Suppose that $C_G(A/B) = 1$. In this case we may think of G as a subgroup of $GL_n(F)$, where $n = \mathbf{dim}_F(A/B)$. Using Lemma 2.2 we deduce that G is abelian-by-finite and then the result follows from Lemma 3.3. □

Corollary 3.5 *Let G be an FC-hypercentral group and let A be an FG-module. Suppose that B is a simple FG-submodule of A of infinite F-dimension such that A/B has finite F-dimension. Then A contains an FG-submodule C such that $A = B \oplus C$.*

Proof Since B has finite codimension over F it follows that A has a finite series

$$0 = B_0 \leq B_1 = B \leq B_2 \leq \cdots \leq B_{n+1} = A$$

whose factors are simple FG-modules and the factors B_j/B_{j-1} have finite F-dimension for $2 \leq j \leq n+1$. We use induction on n. If $n = 1$, the result follows from Proposition 3.4.

Suppose that $n > 1$ and that the result is true for B_n. Thus, B_n contains an FG-submodule D such that $B_n = B \oplus D$. Since $D \cong B_n/B$ it follows that D has finite F-dimension. The simple FG-module $B_n/D \cong B$ has infinite dimension and A/B_n is a simple FG-module of finite F-dimension. By Proposition 3.4 we deduce that A/D contains an FG-submodule C/D of finite F-dimension such that $A/D = B_n/D \oplus C/D$. We have

$$A = B_n + C = (B + D) + C = B + C.$$

Since C has finite dimension and B is a simple FG-submodule, $C \cap B = 0$, so $A = B \oplus C$. □

We note that just as with Corollary 2.6, Corollary 3.5 is no longer true for arbitrary groups G, even for soluble groups; a counterexample appears in the paper [14] of Zaitsev.

Proof of Theorem A

Let

$$0 = A_0 \leq A_1 \leq \cdots \leq A_{n-1} \leq A_n = A$$

be an FG-composition series of A. We remark that the number n is an invariant of the module A.

We use induction on the length n of the FG-composition series of A. If $n = 1$, then there is nothing to prove. Suppose that $n > 1$ and that our statement has been proved for modules having a composition series of length strictly less than n. We apply the induction hypothesis to A/A_1 so $A/A_1 = D/A_1 \oplus E/A_1$ where D, E are FG-submodules of A, each FG-composition factor of D/A_1 has finite F-dimension, and each FG-composition factor of E/A_1 has infinite F-dimension.

Suppose first that A_1 has infinite F-dimension. If $E = A_1$, the result follows from Corollary 3.5. Therefore, suppose that $E/A_1 \neq 0$. Then the FG-composition length of D is less than the FG-composition length of A. By induction hypothesis D contains an FG-submodule B such that $D = B \oplus A_1$ and every FG-composition factor of B has finite dimension. We have

$$A = D + E = B + A_1 + E = B + E.$$

The choice of B shows that

$$B \cap E \leq B \cap D = A_1, \text{ so that } B \cap E \leq B \cap A_1 = 0,$$

which shows that $A = B \oplus E$.

Suppose now that A_1 has finite F-dimension. If $E = A_1$, then each FG-composition factor of A has finite F-dimension and the result follows. Therefore, suppose that $E/A_1 \neq 0$. Then A/A_1 contains a simple FG-submodule S/A_1 of infinite F-dimension. By Proposition 2.5 S contains an FG-submodule Y such that $S = A_1 \oplus Y$. In particular, A contains a simple FG-submodule of infinite F-dimension. The FG-composition length of A/Y is less than the FG-composition length of A, a situation that was considered above. The result follows. □

Proof of Corollary A

Suppose that A contains a nonzero proper FG-submodule, but A is not FG-quasifinite. Then A contains a nonzero proper FG-submodule B such that A/B is a simple FG-module. Being proper, B has finite F-dimension, but then Corollary 2.6 shows that A contains an FG-submodule C such that $A = B \oplus C$. Since $B \neq 0$ it follows that C is a proper FG-submodule and we obtain a contradiction, because C has infinite F-dimension. This contradiction proves the result. □

Proof of Corollary B

Suppose that A is a monolithic FG-module and let B be its FG-monolith. Then B is a nonzero simple FG-submodule of finite codimension over F. Since A has infinite dimension, $\mathbf{dim}_F(B)$ is infinite. Then Corollary 3.5 shows that A contains an FG-submodule C such that $A = B \oplus C$. In particular, C has finite F-dimension and we obtain a contradiction since C is nonzero. This contradiction proves the result. □

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