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The relation between b-weakly compact operator and KB-operator

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Abstract: Our aim is to solve the problem asked by Bahramnezhad and Azar in "KB-operators on Banach lattices and their relationships with Dunford-Pettis and order weakly compact operators". We show that a continuous operator R from a Banach lattice N into a Banach space M is a b-weakly compact operator if and only if R is a KB-operator.

Key words: b-weakly compact operator, KB-operator

1. Introduction

In [6], Bahramnezhad and Azar defined a new classes of operators, named KB-operator and they have examined some of their properties and asked the following problem;

Problem 1.1 ([6], *Problem 2.27*) *Give an operator R from a Banach lattice N into a Banach space M which is a KB-operator but is not b-weakly compact.*

We answer the question in the negative. A lot of properties and results on b-weakly compact operators were given in $[2--5,7]$. Now, we recall the definitions of b-weakly compact operator and KB-operator.

Definition 1.2 *Let R be a continuous operator from a Banach lattice N into a Banach space M.*

(i) R is called KB-operator if $R(x_n)$ has a norm convergent subsequence in M for every positive increasing sequence (x_n) of the closed unit ball B_N of N.

(ii) R is called b-weakly compact if $R(x_n)$ is norm convergent for every positive increasing sequence (x_n) of the closed unit ball B_N of N.

For the basic theory on vector lattices and for unexplained terminology we refer to $[1,8]$.

2. Section

We will prove that the classes of KB-operators and the b-weakly compact operators are the same.

Theorem 2.1 *Let R be an operator from a Banach lattice N into a Banach space M. R is a b-weakly compact operator if and only if R is a KB-operator.*

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Proof It is clear that if R is a b-weakly compact then R is a KB-operator. Let (x_n) be a sequence in N such that $0 \leq x_n \uparrow$ and $\|x_n\| \leq 1$. For an arbitrary subsequence (x_k) of (x_n), let us define

$$\Psi : N'_+ \to \mathbb{R}_+,\ f \to \Psi(f) = \sup f(x_k).$$

For each k and $f \in N'_+$ we have

$$f(x_k) = |f(x_k)| \leq \|f\| \, \|x_k\| \leq \|f\| \, .$$

Then, $\sup f(x_k) \in \mathbb{R}_+$. We claim that Ψ is additive. To see this, let $f, g \in N'_+$.

$$\Psi(f+g) = \sup\left[(f+g)(x_k)\right] = \sup\left[f(x_k) + g(x_k)\right] \leq \sup f(x_k) + \sup g(x_k) = \Psi(f) + \Psi(g).$$

On the other hand, if $x_m, x_t \in (x_k)$, then pick $x_l \in (x_k)$ with $x_m \leq x_l$ and $x_t \leq x_l$, and note that

$$f(x_m) + g(x_t) \leq f(x_l) + g(x_l) \leq \sup\left[(f+g)(x_l)\right] = \Psi(f+g).$$

Using a well-known technique (e.g., $[1, p.14]$), we have $\Psi(f) + \Psi(g) \leq \Psi(f+g)$. Therefore, Ψ is additive and by Theorem 1.7 in [1] Ψ extends uniquely to a positive operator from N' into $\mathbb{R}$ (we call Ψ again). Hence, $\Psi \in N''$. It is easy to see that Ψ is an upper bound of $(x_k)''$ in N'' (where $(x_k)''$ is the image of (x_k) under the well known canonical emdedding of N into to the bidual N''). There exists G in N'' with $(x_k)'' \uparrow G$ as N'' is Dedekind complete. Since $(x_k)''(f) \to G(f)$ for each $f \in N'_+$, we have $(x_k)'' \to G$ with respect to $\sigma(N'', N')$. Thus, all subsequences of (x_n) are convergent to the same limit G with respect to $\sigma(N'', N')$. By the hypothesis, there exists a subsequence (x_{n_k}) of (x_n) and $y \in M$ such that $R(x_{n_k}) \to y$ with respect to norm-topology. This leads to $[R(x_{n_k})]'' \to y''$ in M'' with respect to norm-topology which implies $[R(x_{n_k})]'' \to y''$ in M'' with respect to $\sigma(M'', M')$. The continuity of $R^{''} : (N'', \sigma(N'', N')) \to (M'', \sigma(M'', M'))$ yields $R^{''}[(x_k)''] \to R^{''}(G)$ and $[R(x_{n_k})]'' \to y''$ with respect to $\sigma(M'', M')$. Since $R^{''}[(x_{n_k})''] = [R(x_{n_k})]''$, we have $R^{''}(G) = y''$. So, this means that every norm convergent a subsequence of $R(x_n)$ has the same norm limit. Now, we will show that $R(x_n) \to y$ in M with respect to norm-topology. We assume that $R(x_n)$ does not convergence to y. Thus, there exist $\varepsilon > 0$ and a subsequence (x_m) of (x_n) such that $\|R(x_m) - y\| > \varepsilon$ for all m. By the hypothesis and the above conclusion there exists a subsequence (x_{m_k}) of (x_m) such that $R(x_{m_k}) \to y$ with respect to norm-topology, which is a contradiction. $\square$

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