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A circulant functional equation for the additive function and its stability

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Abstract: A general solution of a matrix functional equation involving circulant matrices of the additive function is determined, and its stability is established.

Key words: Circulant, additive function, stability

1. Introduction

The classical additive functional equation

$$f(x+y) = f(x) + f(y) \tag{A}$$

was first studied by Cauchy in 1821 for continuous function $f : \mathbb{R} \to \mathbb{R}$; the solution takes the form $f(x) = ax$ for some $a \in \mathbb{R}$ (see e.g. [3, Chapter 1]). This result of Cauchy has been extended in various directions. In particular, for $f : \mathbb{C} \to \mathbb{C}$, we have the following result taken from the book [3, Theorem 1.21]: the general solution $f : \mathbb{C} \to \mathbb{C}$ of (A) is given by $f(z) = A_1(x) + iA_2(x) + A_3(y) + iA_4(y)$, where $z = x + iy$ $(x, y \in \mathbb{R})$, and $A_j : \mathbb{R} \to \mathbb{R}$ $(j = 1, 2, 3, 4)$ are real additive functions, i.e. satisfying (A). In 1903, Pexider solved the functional equation

$$f(x+y) = g(x) + h(y), \tag{PA}$$

referred to as the pexiderized additive functional equation, for unknown functions $f, g, h : \mathbb{C} \to \mathbb{C}$, and found that the solutions of (PA) are given by $f(x) = a + A(x) + b$, $g(x) = a + A(x)$, $h(x) = A(x) + b$, where $A : \mathbb{C} \to \mathbb{C}$ is a complex additive function, and a, b are complex constants (see [3, Chapter 1]).

The classical concepts of odd and even functions have been extended to that of type-j functions by Schwaiger in [6]. These are components f_j of a function f defined by

$$f_j(x) = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{-kj} f(\omega^k x) \qquad (j = 0, 1, \ldots, n-1)$$

and satisfy $f_j(\omega x) = \omega^j f_j(x)$, $f = \sum_{j=0}^{n-1} f_j$, where $\omega := \exp(2\pi i/n)$ is a primitive n^{th} root of unity. Schwaiger adopted this concept and solved the following system of functional equations satisfied by components of the

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exponential function

$$f_j(x+\omega^m y) = \sum_{\ell=0}^{j} \omega^{(j-\ell)m} f_\ell(x) f_{j-\ell}(y) + \sum_{\ell=j+1}^{n-1} \omega^{(n+j-\ell)m} f_\ell(x) f_{n+j-\ell}(y) \quad (j=0,1,\ldots,n-1) \tag{1.1}$$

where $m \in \{0,1,\ldots,n-1\}$ is fixed. The stability of the system (1.1) was established one year later by Förg-Rob and Schwaiger in [2]. The results in [6] and [2] were simplified and systematized by Muldoon [4] through the use of circulant matrices. In [5], Laohakosol and Ponpetch used Muldoon's approach [4] to solve a circulant functional equation for the quadratic function, i.e., a system of functional equations satisfied by components of the quadratic function derived via their corresponding circulant matrices, and established its stability.

In this work, through the concept of type- j function, we derive a circulant matrix functional equation for the additive function. Given such a circulant matrix functional equation, their component solutions are determined, and the stability of such system is established.

2. Preliminaries

This section consists of two parts. The first part lays out the notation and terminology that will be kept fixed throughout, while the second part lists (without proofs) basic results needed later.

2.1. Notation

For a fixed integer $n \geq 2$, let $\omega = \exp(2\pi i/n)$ be a primitive n^{th} root of unity. The $n \times n$ (symmetric) Fourier matrix and its conjugate matrix are defined, respectively, by

$$\mathscr{F} := \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \cdots & \omega^{-(n-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \cdots & \omega^{-2(n-1)} \\ \vdots & \vdots & \ddots & & \vdots \\ 1 & \omega^{-(n-1)} & \omega^{-2(n-1)} & \cdots & \omega^{-(n-1)^2} \end{bmatrix}, \quad \mathscr{F}^* := \frac{1}{\sqrt{n}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^{2} & \cdots & \omega^{n-1} \\ 1 & \omega^{2} & \omega^{4} & \cdots & \omega^{2(n-1)} \\ \vdots & \vdots & \ddots & & \vdots \\ 1 & \omega^{n-1} & \omega^{2(n-1)} & \cdots & \omega^{(n-1)^2} \end{bmatrix}.$$

Note that $\mathscr{F}$ is unitary, i.e., $\mathscr{F}\mathscr{F}^* = I = \mathscr{F}^*\mathscr{F}$, where I denotes the $n \times n$ identity matrix.

The diagonal matrix Ω is defined by $\Omega = \mathrm{diag}(1,\omega,\omega^2,\ldots,\omega^{n-1}) := \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & \omega & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \omega^{n-1} \end{bmatrix}$.

Given a sequence $\{a_0,\ldots,a_{n-1}\} \subset \mathbb{C}$, its *circulant matrix* is defined by

$$\mathrm{circ}(a_0,a_1,\ldots,a_{n-1}) := \begin{bmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{bmatrix};$$

its diagonal matrix is defined by $\operatorname{diag}(a_0, a_1, \ldots, a_{n-1}) := \begin{bmatrix} a_0 & 0 & \cdots & 0 \\ 0 & a_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n-1} \end{bmatrix}$.

The *circulant matrix corresponding to the sequence* $\{0, 1, 0, \ldots, 0\}$ is

$$\pi := \operatorname{circ}(0, 1, 0, \ldots, 0) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

The *circulant matrix corresponding to a function* f, with j-component f_j, is defined by

$$\mathbf{F}(x) := \operatorname{circ}(f_0(x), f_1(x), \ldots, f_{n-1}(x)) = \begin{bmatrix} f_0(x) & f_1(x) & \cdots & f_{n-1}(x) \\ f_{n-1}(x) & f_0(x) & \cdots & f_{n-2}(x) \\ \vdots & \vdots & \ddots & \vdots \\ f_1(x) & f_2(x) & \cdots & f_0(x) \end{bmatrix}.$$

2.2. Basic results

The following results are taken from [4, 5].

1) If $A = \operatorname{circ}(a_0, a_1, \ldots, a_{n-1})$, then $\mathscr{F} A \mathscr{F}^* = \sqrt{n}\ \operatorname{diag}(\mathscr{F}^*\, \bar{a})^T$, $\bar{a} := [a_0, \ldots, a_{n-1}]^T$, T denoting transpose.

2) For a nonnegative integer m, if A is a circulant matrix, then $\mathscr{F}(\Omega^{-m} A \Omega^m)\mathscr{F}^* = \pi^m (\mathscr{F} A \mathscr{F}^*)\pi^{-m}$.

3) Any $f : \mathbb{C} \to \mathbb{C}$ can be written uniquely as $f(x) = f_0(x) + f_1(x) + \cdots + f_{n-1}(x)$, where each j-component f_j $(j = 0, 1, \ldots, n-1)$ is a type-j function that can be obtained from

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}} \mathscr{F} \begin{bmatrix} f(x) \\ f(\omega x) \\ \vdots \\ f(\omega^{n-1} x) \end{bmatrix} = \frac{1}{\sqrt{n}} \mathscr{F}^* \begin{bmatrix} f(x) \\ f(\omega^{-1} x) \\ \vdots \\ f(\omega^{-(n-1)} x) \end{bmatrix}.$$

4) The circulant matrix $\mathbf{F}(x)$ corresponding to $f : \mathbb{C} \to \mathbb{C}$ satisfies

$$\mathscr{F}\mathbf{F}(x)\mathscr{F}^* = \operatorname{diag}\left(f(x), f(\omega x), \ldots, f(\omega^{n-1} x)\right) \tag{2.1}$$

$$\mathbf{F}(\omega^m x) = \Omega^{-m}\mathbf{F}(x)\Omega^m \quad (m \in \mathbb{N}). \tag{2.2}$$

5) For a nonnegative integer m, if $B = \operatorname{diag}(b_0, b_1, \ldots, b_{n-1})$, then $\pi^m B \pi^{-m} = \operatorname{diag}(b_m, b_{m+1}, \ldots, b_{m+n-1})$, where suffixes are taken modulo n.

6) If $B = \operatorname{diag}(b_0, b_1, \ldots, b_{n-1})$ is a diagonal matrix, then $\mathscr{F}^* B \mathscr{F}$ is a circulant matrix, i.e. $\mathscr{F}^* B \mathscr{F} = \operatorname{circ}(d_0, d_1, \ldots, d_{n-1})$, where $d_j = (1/n)\sum_{k=0}^{n-1} \omega^{n-kj} b_k$ $(j = 0, 1, \ldots, n-1)$.

7) For $m \in \{0, 1, \ldots, n-1\}$, $d = \gcd(n, m)$, we have $s + tm \not\equiv u + vm \pmod{n}$, for every $s, u \in \{0, 1, \ldots, d-1\}$ and $t, v \in \{0, 1, \ldots, n/d - 1\}$, except when $s = u$, $t = v$.

3. A circulant functional equation

We begin by deriving a circulant equation for the additive function.

Theorem 3.1 *If $f, g, h : \mathbb{C} \to \mathbb{C}$ satisfy (PA), then their circulant matrices $\mathbf{F}, \mathbf{G}, \mathbf{H}$ satisfy*

$$\mathbf{F}(x+y) = \mathbf{G}(x) + \mathbf{H}(y) \quad (x, y \in \mathbb{C}).$$

More generally, we have

$$\mathbf{F}(x+\omega^m y) = \mathbf{G}(x) + \Omega^{-m}\mathbf{H}(y)\Omega^m \quad (x, y \in \mathbb{C};\ m \in \{0, \ldots, n-1\}). \tag{3.1}$$

Proof From the basic result (2.1) of property 4) in the last section, we have

$$\begin{aligned}
\mathbf{G}(x) + \mathbf{H}(y) &= \mathscr{F}^*\mathrm{diag}\left(g(x), g(\omega x), \ldots, g(\omega^{n-1}x)\right)\mathscr{F} + \mathscr{F}^*\mathrm{diag}\left(h(y), h(\omega y), \ldots, h(\omega^{n-1}y)\right)\mathscr{F} \\
&= \mathscr{F}^*\mathrm{diag}\left(g(x)+h(y), g(\omega x)+h(\omega y), \ldots, g(\omega^{n-1}x)+h(\omega^{n-1}y)\right)\mathscr{F} \\
&= \mathscr{F}^*\mathrm{diag}\left(f(x+y), f(\omega(x+y)), \ldots, f(\omega^{n-1}(x+y))\right)\mathscr{F} = \mathbf{F}(x+y).
\end{aligned}$$

Using this last relation together with the basic result (2.2) of property 4), the last assertion is immediate. □

Our next step is to solve the circulant equation (3.1)

Theorem 3.2 *Let $\mathbf{F}, \mathbf{G}, \mathbf{H}$ be circulant matrices whose first rows are, respectively,*

$$(f_0(x), f_1(x), \ldots, f_{n-1}(x)),\ (g_0(x), g_1(x), \ldots, g_{n-1}(x)),\ (h_0(x), h_1(x), \ldots, h_{n-1}(x)),$$

where $f_i, g_i, h_i : \mathbb{C} \to \mathbb{C}$ are arbitrary functions which need not be components of any functions. Let $m \in \{0, 1, \ldots, n-1\}$, $d := \gcd(n, m)$. If $\mathbf{F}, \mathbf{G}, \mathbf{H}$ satisfy

$$\mathbf{F}(x+\omega^m y) = \mathbf{G}(x) + \Omega^{-m}\mathbf{H}(y)\Omega^m, \tag{3.2}$$

then

- *when $d = n$, we have*

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}}\mathscr{F}\begin{bmatrix} A_0(x)+w_0(0)+v_0(0) \\ A_1(x)+w_1(0)+v_1(0) \\ \vdots \\ A_{n-1}(x)+w_{n-1}(0)+v_{n-1}(0) \end{bmatrix} \tag{3.3}$$

$$\begin{bmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}}\mathscr{F}\begin{bmatrix} A_0(x)+v_0(0) \\ A_1(x)+v_1(0) \\ \vdots \\ A_{n-1}(x)+v_{n-1}(0) \end{bmatrix} \tag{3.4}$$

$$\begin{bmatrix} h_0(x) \\ h_1(x) \\ \vdots \\ h_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}}\mathscr{F}\begin{bmatrix} A_0(x)+w_0(0) \\ A_1(x)+w_1(0) \\ \vdots \\ A_{n-1}(x)+w_{n-1}(0) \end{bmatrix}, \tag{3.5}$$

where $A_i : \mathbb{C} \to \mathbb{C}$ $(i = 0, 1, \ldots, n-1)$ *are additive functions and*

$$v_i(x) := \sum_{\ell=0}^{n-1} \omega^{i\ell} g_\ell(x), \quad w_i(x) := \sum_{\ell=0}^{n-1} \omega^{i\ell} h_\ell(x) \quad (i = 0, \ldots, n-1); \tag{3.6}$$

- *when* $1 \le d < n$, *we have*

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}} \mathscr{F} \begin{bmatrix} \alpha_0(x) + C_m(0) + B_0(0) \\ \alpha_m(x) + C_{2m}(0) + B_m(0) \\ \vdots \\ \alpha_{(n/d-1)m}(x) + C_{(n/d)m}(0) + B_{(n/d-1)m}(0) \end{bmatrix} \tag{3.7}$$

$$\begin{bmatrix} g_0(x) \\ g_1(x) \\ \vdots \\ g_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}} \mathscr{F} \begin{bmatrix} \alpha_0(x) + B_0(0) \\ \alpha_m(x) + B_m(0) \\ \vdots \\ \alpha_{(n/d-1)m}(x) + B_{(n/d-1)m}(0) \end{bmatrix} \tag{3.8}$$

$$\begin{bmatrix} h_0(x) \\ h_1(x) \\ \vdots \\ h_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}} \mathscr{F} \begin{bmatrix} \alpha_0(\omega^m x) + C_m(0) \\ \alpha_m(\omega^m x) + C_{2m}(0) \\ \vdots \\ \alpha_{(n/d-1)m}(\omega^m x) + C_{(n/d)m}(0) \end{bmatrix} \tag{3.9}$$

where, for $j = 0, 1, \ldots, n/d - 1$,

$$\alpha_{jm}(x) = \begin{bmatrix} A_{0+jm}(x) \\ A_{1+jm}(x) \\ \vdots \\ A_{d-1+jm}(x) \end{bmatrix}, \quad B_{jm}(0) = \begin{bmatrix} v_{0+jm}(0) \\ v_{1+jm}(0) \\ \vdots \\ v_{d-1+jm}(0) \end{bmatrix}, \quad C_{(j+1)m}(0) = \begin{bmatrix} w_{0+(j+1)m}(0) \\ w_{1+(j+1)m}(0) \\ \vdots \\ w_{d-1+(j+1)m}(0) \end{bmatrix}$$

and $A_{k+jm} : \mathbb{C} \to \mathbb{C}$ $(k = 0, 1, \ldots, d-1)$ *are additive functions.*

Proof From (3.2), we have

$$\mathscr{F}\mathbf{F}(x + \omega^m y)\mathscr{F}^* = \mathscr{F}\mathbf{G}(x)\mathscr{F}^* + \mathscr{F}\Omega^{-m}\mathbf{H}(y)\Omega^m\mathscr{F}^*.$$

Using the basic results 1) and 2), this relation takes the form

$$\mathbf{U}(x + \omega^m y) = \mathbf{V}(x) + \mathbf{W}_m(y), \tag{3.10}$$

where

$$\mathrm{diag}(u_0(x), u_1(x), \ldots, u_{n-1}(x)) = \mathbf{U}(x) = \mathscr{F}\mathbf{F}(x)\mathscr{F}^* = \sqrt{n}\,\mathrm{diag}(\mathscr{F}^*\bar{f}(x))^T \tag{3.11}$$

$$\mathrm{diag}(v_0(x), v_1(x), \ldots, v_{n-1}(x)) = \mathbf{V}(x) = \mathscr{F}\mathbf{G}(x)\mathscr{F}^* = \sqrt{n}\,\mathrm{diag}(\mathscr{F}^*\bar{g}(x))^T \tag{3.12}$$

$$\mathrm{diag}(w_0(x), w_1(x), \ldots, w_{n-1}(x)) = \mathbf{W}(x) = \mathscr{F}\mathbf{H}(x)\mathscr{F}^* = \sqrt{n}\,\mathrm{diag}(\mathscr{F}^*\bar{h}(x))^T \tag{3.13}$$

$$\mathbf{W}_m(x) = \pi^m\mathbf{W}(x)\pi^{-m}. \tag{3.14}$$

Equation (3.10) and the basic result 5) yield the following system of n equations

$$u_0(x+\omega^m y) = v_0(x) + w_m(y)$$

$$\vdots$$

$$u_{n-1}(x+\omega^m y) = v_{n-1}(x) + w_{m+n-1}(y).$$

Using the basic result 7), we subdivide these n equations into d different classes, each with n/d equations, briefly written as

$$u_{k+jm}(x+\omega^m y) = v_{k+jm}(x) + w_{k+(j+1)m}(y) \quad (j=0,1,\ldots,n/d-1; k=0,1,\ldots,d-1). \tag{3.15}$$

For each $j \in \{0,1,\ldots,n/d-1\}$ and $k \in \{0,1,\ldots,d-1\}$, replacing $y=0$ in (3.15), we have

$$u_{k+jm}(x) = v_{k+jm}(x) + w_{k+(j+1)m}(0), \tag{3.16}$$

while replacing $x=0$ in (3.15), we have

$$u_{k+jm}(\omega^m y) = v_{k+jm}(0) + w_{k+(j+1)m}(y). \tag{3.17}$$

Substituting (3.16) and (3.17) into (3.15), we get

$$u_{k+jm}(x+\omega^m y) = u_{k+jm}(x) - w_{k+(j+1)m}(0) + u_{k+jm}(\omega^m y) - v_{k+jm}(0).$$

Replacing y by $\omega^{-m}y$ in the last equation and adding $-w_{k+(j+1)m}(0) - v_{k+jm}(0)$ to both sides, we obtain

$$\begin{aligned} &u_{k+jm}(x+y) - w_{k+(j+1)m}(0) - v_{k+jm}(0) \\ &= u_{k+jm}(x) - w_{k+(j+1)m}(0) - v_{k+jm}(0) + u_{k+jm}(y) - w_{k+(j+1)m}(0) - v_{k+jm}(0). \end{aligned} \tag{3.18}$$

Defining

$$A_{k+jm}(x) := u_{k+jm}(x) - w_{k+(j+1)m}(0) - v_{k+jm}(0) \qquad (j \in \{0,1,\ldots,n/d-1\},\ k \in \{0,1,\ldots,d-1\}),$$

the relation (3.18) shows that it is an additive function. Substituting back into (3.16) and (3.17), we have, for $j=0,1,\ldots,n/d-1,\ k=0,1,\ldots,d-1$,

$$u_{k+jm}(x) = A_{k+jm}(x) + w_{k+(j+1)m}(0) + v_{k+jm}(0), \tag{3.19}$$

$$v_{k+jm}(x) = A_{k+jm}(x) + v_{k+jm}(0), \tag{3.20}$$

$$w_{k+(j+1)m}(x) = A_{k+jm}(\omega^m x) + w_{k+(j+1)m}(0), \tag{3.21}$$

where A_{k+jm} are additive functions. If $d=n$, then $m=0$ and from (3.19), (3.20), and (3.21), we get, for $k=0,1,\ldots,n-1$,

$$u_k(x) = A_k(x) + w_k(0) + v_k(0), \quad v_k(x) = A_k(x) + v_k(0), \quad w_k(x) = A_k(x) + w_k(0). \tag{3.22}$$

From (3.11), (3.12), (3.13), and (3.22), we get the three solution matrices (3.3), (3.4), (3.5).

If $1 \le d < n$, then using (3.11)–(3.13), (3.19)–(3.21), we similarly get the three solution matrices (3.7)–(3.9). □

Remark. If the functions f_i, g_i, h_i $(i = 0, 1, \ldots, n-1)$ in Theorem 3.2 are identical, then (3.2) becomes

$$\mathbf{F}(x + \omega^m y) = \mathbf{F}(x) + \Omega^{-m}\mathbf{F}(y)\Omega^m.$$

The solutions thus take the form:

- when $d = n$,

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}}\mathscr{F} \begin{bmatrix} A_0(x) \\ A_1(x) \\ \vdots \\ A_{n-1}(x) \end{bmatrix} \tag{3.23}$$

  where $A_i : \mathbb{C} \to \mathbb{C}$ $(i = 0, 1, \ldots, n-1)$ are additive functions;

- when $1 \le d < n$,

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ \vdots \\ f_{n-1}(x) \end{bmatrix} = \frac{1}{\sqrt{n}}\mathscr{F} \begin{bmatrix} \alpha(x) \\ \alpha(\omega^m x) \\ \vdots \\ \alpha(\omega^{(n/d-1)m} x) \end{bmatrix} \tag{3.24}$$

  where $\alpha(x) = [A_0(x), A_1(x), \ldots, A_{d-1}(x)]^T$ and $A_k : \mathbb{C} \to \mathbb{C}$ $(k = 0, 1, \ldots, d-1)$ are additive functions.

4. Stability

In this section, we establish the stability of the circulant equation (3.1). As in [4], we use the usual 1-norm for a square matrix $A = (a_{i,j})_{0 \le i,j \le n-1}$ defined by

$$\|A\| = \max_{0 \le i \le n-1} \sum_{j=0}^{n-1} |a_{i,j}|.$$

Throughout this section, let $\phi : \mathbb{C} \times \mathbb{C} \to [0, \infty)$ be a mapping satisfying the condition that the sum $\sum_{\ell=0}^{\infty} t^{-(\ell+1)} \left(\phi(t^\ell x, t^\ell y) + \phi(t^\ell x, 0) + \phi(0, t^\ell y)\right)$ converges for all $x, y \in \mathbb{C}$ and for all integers $t \ge 2$.

Theorem 4.1 *Let* $\mathbf{F}, \mathbf{G}, \mathbf{H}$ *be circulant matrices whose first rows are, respectively,*

$$(f_0(x), f_1(x), \ldots, f_{n-1}(x)),\ (g_0(x), g_1(x), \ldots, g_{n-1}(x)),\ (h_0(x), h_1(x), \ldots, h_{n-1}(x)),$$

where $f_i, g_i, h_i : \mathbb{C} \to \mathbb{C}$ *are arbitrary functions which need not be components of any functions, let* $m \in \{0, 1, \ldots, n-1\}$. *If* $\mathbf{F}, \mathbf{G}, \mathbf{H}$ *satisfy*

$$\|\mathbf{F}(x + \omega^m y) - \mathbf{G}(x) - \Omega^{-m}\mathbf{H}(y)\Omega^m\| \le \phi(x, y), \tag{4.1}$$

then there exists a circulant matrix $\mathbf{A}(x) := \mathrm{circ}(a_0(x), a_1(x), \ldots, a_{n-1}(x))$, $a_j : \mathbb{C} \to \mathbb{C}$ $(j = 0, 1, \ldots, n-1)$ *satisfying the matrix additive equation*

$$\mathbf{A}(x + y) = \mathbf{A}(x) + \mathbf{A}(y) \tag{4.2}$$

and

$$\|\mathbf{F}(x)-\mathbf{A}(x)\| \le n\sum_{\ell=1}^{t-1}\Phi_t(x,\ell x) \tag{4.3}$$

$$\|\mathbf{G}(x)-\mathbf{A}(x)\| \le n^2\phi(x,0)+n\sum_{\ell=1}^{t-1}\Phi_t(x,\ell x)+\|\mathbf{H}(0)\| \tag{4.4}$$

$$\|\mathbf{H}(x)-\bar{\mathbf{A}}(x)\| \le n^2\phi(0,x)+n\sum_{\ell=1}^{t-1}\Phi_t(\omega^m x,\ell\omega^m x)+\|\mathbf{G}(0)\|, \tag{4.5}$$

where

$$\Phi_t(x,y) := \sum_{\ell=0}^{\infty} t^{-(\ell+1)}\left\{n\left(\phi(t^\ell x, t^\ell\omega^{-m}y)+\phi(t^\ell x,0)+\phi(0,t^\ell\omega^{-m}y)\right)+\|\mathbf{V}(0)\|+\|\mathbf{W}_m(0)\|\right\}$$

with $\mathbf{V}(0)=\mathrm{diag}(v_0(0),v_1(0),\dots,v_{n-1}(0))$, $\mathbf{W}_m(0)=\mathrm{diag}(w_m(0),w_{m+1}(0),\dots,w_{m+n-1}(0))$, v_i, w_i *are as defined in* (3.6) *with suffixes taken modulo* n, *and* $\bar{\mathbf{A}}(x):=\Omega^m\mathbf{A}(\omega^m x)\Omega^{-m}$ *is a circulant matrix satisfying* (4.2), *and whose first row is* $(\bar{a}_0(x),\bar{a}_1(x),\dots,\bar{a}_{n-1}(x))$, $\bar{a}_j(x)=\omega^{(n-j)m}a_j(\omega^m x)$ $(j=0,1,\dots,n-1)$.

Proof Multiplying the expressions in (4.1) by $\|\mathscr{F}\|$ on the left side and by $\|\mathscr{F}^*\|$ on the right side, we obtain

$$\|\mathscr{F}\mathbf{F}(x+\omega^m y)\mathscr{F}^*-\mathscr{F}\mathbf{G}(x)\mathscr{F}^*-\mathscr{F}\Omega^{-m}\mathbf{H}(y)\Omega^m\mathscr{F}^*\| \le \|\mathscr{F}\|\phi(x,y)\|\mathscr{F}^*\|.$$

By the basic results 1) and 2), this last inequality is of the form

$$\|\mathbf{U}(x+\omega^m y)-\mathbf{V}(x)-\mathbf{W}_m(y)\| \le n\phi(x,y), \tag{4.6}$$

where U,V,W,W_m are as defined in (3.11)–(3.14). Substituting $y=0$ into (4.6) we have

$$\|\mathbf{V}(x)+\mathbf{W}_m(0)-\mathbf{U}(x)\| \le n\phi(x,0), \tag{4.7}$$

and similarly substituting $x=0$ into (4.6), we have

$$\|\mathbf{V}(0)+\mathbf{W}_m(y)-\mathbf{U}(\omega^m y)\| \le n\phi(0,y). \tag{4.8}$$

Using (4.6)–(4.8), we get

$$\begin{aligned}\|\mathbf{U}(x+\omega^m y)-\mathbf{U}(x)-\mathbf{U}(\omega^m y)\|-\|\mathbf{V}(0)+\mathbf{W}_m(0)\| &\le \|\mathbf{U}(x+\omega^m y)-\mathbf{U}(x)-\mathbf{U}(\omega^m y)+\mathbf{V}(0)+\mathbf{W}_m(0)\|\\ &\le n\left(\phi(x,y)+\phi(x,0)+\phi(0,y)\right).\end{aligned}$$

Replacing y by $\omega^{-m}y$, we get

$$\|\mathbf{U}(x+y)-\mathbf{U}(x)-\mathbf{U}(y)\| \le n\left(\phi(x,\omega^{-m}y)+\phi(x,0)+\phi(0,\omega^{-m}y)\right)+\|\mathbf{V}(0)\|+\|\mathbf{W}_m(0)\|.$$

Since $\mathbf{U}(x)=\mathrm{diag}(u_0(x),u_1(x),\dots,u_{n-1}(x))=(u_{i,j}(x))_{0\le i,j\le n-1}$, where $u_{i,j}(x)=\begin{cases}u_i(x) & \text{if } i=j\\ 0 & \text{if } i\ne j\end{cases}$, by the definition of norm, we get

$$\max_{0\le i\le n-1}\sum_{j=0}^{n-1}|u_{i,j}(x+y)-u_{i,j}(x)-u_{i,j}(y)| \le n\left(\phi(x,\omega^{-m}y)+\phi(x,0)+\phi(0,\omega^{-m}y)\right)+\|\mathbf{V}(0)\|+\|\mathbf{W}_m(0)\|.$$

For each $i = 0, 1, \dots, n-1$, we have

$$|u_i(x+y) - u_i(x) - u_i(y)| \le n\left(\phi(x, \omega^{-m}y) + \phi(x,0) + \phi(0, \omega^{-m}y)\right) + \|\mathbf{V}(0)\| + \|\mathbf{W}_m(0)\|.$$

By [1, Theorem 13.3, p. 131], there exist unique additive functions $s_i : \mathbb{C} \to \mathbb{C}$ satisfying (A) such that

$$|u_i(x) - s_i(x)| \le \sum_{\ell=1}^{t-1} \Phi_t(x, \ell x) \quad (i = 0, 1, \dots, n-1).$$

Let $\mathbf{S}(x) := \operatorname{diag}(s_0(x), s_1(x), \dots, s_{n-1}(x)) = (s_{i,j}(x))_{0 \le i,j \le n-1}$, where $s_{i,j}(x) = \begin{cases} s_i(x) & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$. Using the definition of norm, we have

$$\|\mathbf{U}(x) - \mathbf{S}(x)\| = \max_{0 \le i \le n-1} \sum_{j=0}^{n-1} |u_{i,j}(x) - s_{i,j}(x)| \le \sum_{\ell=1}^{t-1} \Phi_t(x, \ell x). \tag{4.9}$$

Multiplying the expressions in (4.9) by $\|\mathscr{F}^*\|$ on the left side and by $\|\mathscr{F}\|$ on the right side, and using (3.11), we get (4.3), where $\mathbf{A}(x) = \mathscr{F}^*\mathbf{S}(x)\mathscr{F}$. Since $\mathbf{S}(x)$ is a diagonal matrix, the basic result 6) shows that $\mathbf{A}(x)$ is a circulant matrix whose first row is $(a_0(x), a_1(x), \dots, a_{n-1}(x))$, where

$$a_j(x) = \frac{1}{n} \sum_{k=0}^{n-1} \omega^{n-kj} s_k(x) \quad (j = 0, 1, \dots, n-1).$$

Since each s_k satisfies (A), the function elements a_j satisfy (A), i.e. $\mathbf{A}(x)$ satisfies (4.2).

Multiplying the expressions in (4.7) by $\|\mathscr{F}^*\|$ on the left side and by $\|\mathscr{F}\|$ on the right side, and using (3.11)–(3.13), and the basic result 2), we get

$$\begin{aligned}\|\mathbf{G}(x) - \mathbf{A}(x)\| - \|\mathbf{F}(x) - \mathbf{A}(x)\| - \|\Omega^{-m}\mathbf{H}(0)\Omega^m\| &\le \|\mathbf{G}(x) + \Omega^{-m}\mathbf{H}(0)\Omega^m - \mathbf{F}(x) + \mathbf{A}(x) - \mathbf{A}(x)\| \\ &\le n^2\phi(x,0).\end{aligned}$$

and using (4.3) the assertion (4.4) follows.

Multiplying the expressions in (4.8) by $\|\mathscr{F}^*\|$ on the left side and by $\|\mathscr{F}\|$ on the right side, and using (3.11)–(3.13), and the basic result 2), we get

$$\begin{aligned}&\|\Omega^{-m}\mathbf{H}(y)\Omega^m - \mathbf{A}(\omega^m y)\| - \|\mathbf{F}(\omega^m y) - \mathbf{A}(\omega^m y)\| - \|\mathbf{G}(0)\| \\ &\le \|\mathbf{G}(0) + \Omega^{-m}\mathbf{H}(y)\Omega^m - \mathbf{F}(\omega^m y) + \mathbf{A}(\omega^m y) - \mathbf{A}(\omega^m y)\| \le n^2\phi(0,y).\end{aligned}$$

Using (4.3), we get

$$\|\Omega^{-m}\mathbf{H}(y)\Omega^m - \mathbf{A}(\omega^m y)\| \le n^2\phi(0,y) + n\sum_{m=1}^{t-1} \Phi_t(\omega^m y, \ell\omega^m y) + \|\mathbf{G}(0)\|. \tag{4.10}$$

Multiplying the expressions in (4.10) by $\|\Omega^m\|$ on the left side and by $\|\Omega^{-m}\|$ on the right side, we obtain (4.5). □

Taking $\phi(x,y) = \varepsilon$ in Theorem 4.1, we get the following

Corollary 4.2 *Let* $\varepsilon > 0$ *be fixed. If* $\mathbf{F}, \mathbf{G}, \mathbf{H}$ *satisfy*

$$\|\mathbf{F}(x+\omega^m y) - \mathbf{G}(x) - \Omega^{-m}\mathbf{H}(y)\Omega^m\| \le \varepsilon, \tag{4.11}$$

then there exists a circulant matrix $\mathbf{A}$ *satisfying* (4.2) *and*

$$\|\mathbf{F}(x) - \mathbf{A}(x)\| \le 3n^2\varepsilon + n\|\mathbf{V}(0)\| + n\|\mathbf{W}_m(0)\| \tag{4.12}$$

$$\|\mathbf{G}(x) - \mathbf{A}(x)\| \le 4n^2\varepsilon + n\|\mathbf{V}(0)\| + n\|\mathbf{W}_m(0)\| + \|\mathbf{H}(0)\| \tag{4.13}$$

$$\|\mathbf{H}(x) - \bar{\mathbf{A}}(x)\| \le 4n^2\varepsilon + n\|\mathbf{V}(0)\| + n\|\mathbf{W}_m(0)\| + \|\mathbf{G}(0)\|. \tag{4.14}$$

5. Examples

We end the paper by working out two examples, first for Theorem 3.2, and second for Theorem 4.1.

Example 1. For $n = 3$, the matrix equation (3.2) yields the following three systems of functional equations, corresponding to the three possible values of $m \in \{0, 1, 2\}$,

- $m = 0;\quad f_0(x+y) = g_0(x) + h_0(y),\quad f_1(x+y) = g_1(x) + h_1(y),\quad f_2(x+y) = g_2(x) + h_2(y)$
- $m = 1;\quad f_0(x+\omega y) = g_0(x) + h_0(y),\quad f_1(x+\omega y) = g_1(x) + \omega h_1(y),\quad f_2(x+\omega y) = g_2(x) + \omega^2 h_2(y)$
- $m = 2;\quad f_0(x+\omega^2 y) = g_0(x) + h_0(y),\quad f_1(x+\omega^2 y) = g_1(x) + \omega^2 h_1(y),\quad f_2(x+\omega^2 y) = g_2(x) + \omega h_2(y).$

(Observe that these systems are equivalent in the sense that each one of them can be transformed to (PA) by variable changing.)

In any case, $d = \gcd(3, m) = 1$ and the solution functions take the form

$$\begin{bmatrix} f_0(x) \\ f_1(x) \\ f_2(x) \end{bmatrix} = \frac{1}{\sqrt{3}}\mathscr{F}\begin{bmatrix} \alpha_0(x) + C_m(0) + B_0(0) \\ \alpha_m(x) + C_{2m}(0) + B_m(0) \\ \alpha_{2m}(x) + C_{3m}(0) + B_{2m}(0) \end{bmatrix}$$

$$\begin{bmatrix} g_0(x) \\ g_1(x) \\ g_2(x) \end{bmatrix} = \frac{1}{\sqrt{3}}\mathscr{F}\begin{bmatrix} \alpha_0(x) + B_0(0) \\ \alpha_m(x) + B_m(0) \\ \alpha_{2m}(x) + B_{2m}(0) \end{bmatrix}$$

$$\begin{bmatrix} h_0(x) \\ h_1(x) \\ h_2(x) \end{bmatrix} = \frac{1}{\sqrt{3}}\mathscr{F}\begin{bmatrix} \alpha_0(\omega^m x) + C_m(0) \\ \alpha_m(\omega^m x) + C_{2m}(0) \\ \alpha_{2m}(\omega^m x) + C_{3m}(0) \end{bmatrix}$$

where $\alpha_{jm}(x) = A_{jm}(x),\quad B_{jm}(0) = v_{jm}(0),\quad C_{(j+1)m}(0) = w_{(j+1)m}(0)\quad (j = 0, 1, 2)$, and $A_{jm} : \mathbb{C} \to \mathbb{C}$ are additive functions.

Our second example also treats the case $n = 3$ with constant bound in Theorem 4.1.

Example 2. Let

$$\phi(x, y) = K,\ f_i(x) = A_i(x) + a_i,\ g_i(x) = A_i(x) + b_i,\ h_i(x) = \omega^{-im}A_i(\omega^m x) + c_i \qquad (i = 1, 2, 3),$$

where A_i are additive functions and K, a_i, b_i, c_i are constants, and let

$$\mathbf{F}(x) = \operatorname{circ}(f_0(x), f_1(x), f_2(x)),\ \mathbf{G}(x) = \operatorname{circ}(g_0(x), g_1(x), g_2(x)),$$

$$\mathbf{H}(x) = \operatorname{circ}(h_0(x), h_1(x), h_2(x)),\ \mathbf{A}(x) = \operatorname{circ}(A_0(x), A_1(x), A_2(x)).$$

Clearly, $\mathbf{A}$ satisfies (4.2). The hypothesis (4.1) here takes the form

$$\|\mathbf{F}(x+\omega^m y)-\mathbf{G}(x)-\Omega^{-m}\mathbf{H}(y)\Omega^m\|\leq K,$$

which is equivalent to

$$\|\mathrm{circ}(f_0(x+\omega^m y)-g_0(x)-h_0(y), f_1(x+\omega^m y)-g_1(x)-\omega^m h_1(y), f_2(x+\omega^m y)-g_2(x)-\omega^{2m}h_2(y))\|\leq K.$$

Using the definitions of norm and f_i, g_i, h_i, this last inequality yields

$$\sum_{i=0}^{2}|a_i-b_i-c_i|=\|\mathrm{circ}(a_0-b_0-c_0, a_1-b_1-c_1, a_2-b_2-c_2)\|\leq K. \tag{5.1}$$

From (3.12) and (3.14), we see that $\mathbf{V}(0)$ and $\mathbf{W}_m(0)$ are diagonal matrices, while from their definitions, we see that $\mathbf{H}(0)$ and $\mathbf{G}(0)$ are circulant matrices. By direct computation, we get

$$\Phi_t(x,y):=\sum_{\ell=0}^{\infty}t^{-(\ell+1)}\left(9K+\|\mathbf{V}(0)\|+\|\mathbf{W}_m(0)\|\right)=\frac{1}{t-1}\left(9K+\|\mathbf{V}(0)\|+\|\mathbf{W}_m(0)\|\right),$$

$$\|\mathbf{V}(0)\|=\max_{0\leq i\leq 2}|v_i(0)|,\ \text{ where } v_i(0)=\sum_{\ell=0}^{2}\omega^{i\ell}b_i,$$

$$\|\mathbf{W}_m(0)\|=\max_{0\leq i\leq 2}|w_i(0)|,\ \text{ where } w_i(0)=\sum_{\ell=0}^{2}\omega^{i\ell}c_i,$$

$$\|\mathbf{H}(0)\|=\sum_{\ell=0}^{2}|c_i|,\ \text{and }\ \|\mathbf{G}(0)\|=\sum_{\ell=0}^{2}|b_i|.$$

Regarding the assertions in (4.3)–(4.5), we get

$$\sum_{i=0}^{2}|a_i|=\|\mathrm{circ}(f_0(x)-A_0(x), f_1(x)-A_1(x), f_2(x)-A_2(x))\|=\|\mathbf{F}(x)-\mathbf{A}(x)\|$$

$$\leq 27K+3\max_{0\leq i\leq 2}\left|\sum_{\ell=0}^{2}\omega^{i\ell}b_i\right|+3\max_{0\leq i\leq 2}\left|\sum_{\ell=0}^{2}\omega^{i\ell}c_i\right|, \tag{5.2}$$

$$\sum_{i=0}^{2}|b_i|=\|\mathrm{circ}(g_0(x)-A_0(x), g_1(x)-A_1(x), g_2(x)-A_2(x))\|=\|\mathbf{G}(x)-\mathbf{A}(x)\|$$

$$\leq 36K+3\max_{0\leq i\leq 2}\left|\sum_{\ell=0}^{2}\omega^{i\ell}b_i\right|+3\max_{0\leq i\leq 2}\left|\sum_{\ell=0}^{2}\omega^{i\ell}c_i\right|+\sum_{\ell=0}^{2}|c_i|, \tag{5.3}$$

$$\sum_{i=0}^{2}|c_i|=\|\mathrm{circ}(h_0(x)-A_0(\omega^m x), h_1(x)-\omega^{2m}A_1(\omega^m x), h_2(x)-\omega^m A_2(\omega^m x))\|=\|\mathbf{H}(x)-\bar{\mathbf{A}}(x)\|$$

$$\leq 36K+3\max_{0\leq i\leq 2}\left|\sum_{\ell=0}^{2}\omega^{i\ell}b_i\right|+3\max_{0\leq i\leq 2}\left|\sum_{\ell=0}^{2}\omega^{i\ell}c_i\right|+\sum_{\ell=0}^{2}|b_i|. \tag{5.4}$$

The hypothesis (5.1) and the assertions (5.2)–(5.4) are satisfied by choosing for example $a_i=0, b_i=c_i$ ($i=0,1,2$).

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