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Extended S-supplement submodules

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Abstract: In this paper, we introduced and studied extended S-supplement submodules. A submodule U of a module V is called extended S-supplement submodule in V if there exists a submodule T of V such that $V = T + U$ and $U \cap T$ is Goldie torsion. Extended S-supplement submodule is a dual notion of extended S-closed submodule. The class of extended S-supplement sequences is a proper class which is generated by nonsingular modules injectively. We studied coinjective objects of this class. Moreover, extended S-supplemented modules are also investigated. We present new characterizations of $Z_2(R_R)$-semiperfect rings and SI-rings by extended S-supplement submodules.

Key words: S-supplement submodules, Goldie torsion theory, coinjective modules

1. Introduction

In what follows, rings are associative with unit elements, and all modules are unitary right modules. Denote by $N \leq M$ that N is a submodule of M or M is an extension of N. Note that a submodule U of V is closed (complement) in V if U has no proper essential extension in V. A module is CS or extending if its closed submodules are direct summands [1].

There are many generalizations of closed submodules concerning various sets of submodules. In [4], a submodule A of M is called S-closed in M if M/N is nonsingular. S-closed submodules are closed but not vice-versa by [12, Lemma 2.3]. A module is called CLS-module if its S-closed submodules are direct summands. CLS-modules are recently studied in [2, 5, 6, 14, 15]. The class of S-closed short exact sequences is not a proper class, (see [3, Example 3.1]). Recall from [3], a submodule X of module V is called extended S-closed in V if there exists $S \leq V$ such that $S \cap X = 0$ and $V/(S \oplus X)$ is nonsingular. S-closed submodules are extended S-closed but the converse is not true in general. The class of extended S-closed exact sequences is the smallest proper class which is generated by the class of S-closed exact sequences. Moreover, the proper class of extended S-closed exact sequences is projectively generated by Goldie torsion modules, i.e. it is the largest proper class for which each Goldie torsion module is projective, [3]. For a proper class $\mathcal{P}$, a module A is called $\mathcal{P}$-coprojective ($\mathcal{P}$-coinjective) if every short exact sequence ending (beginning) at A belongs to $\mathcal{P}$, ([9, 11, 13]). Coprojective objects of the proper class of extended S-closed sequences is also studied in [3].

The purpose of the present paper is to introduce and study extended S-supplement submodule which is a dual notion of extended S-closed submodule. We will call a submodule X of module B extended S-supplement if there exists $S \leq B$ such that $B = S + X$ and $S \cap X$ is Goldie torsion. The class of extended S-supplement

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short exact sequences ESS is a proper class by [7, Theorem 3.1]. In Section 2, we show that the proper class ESS is injectively generated by the class of nonsingular modules, i.e. it is the largest proper class for which each nonsingular module is injective. Main properties of ESS-coinjective modules are given. Then, using properties of this kind of modules, we prove that R is right SI-ring if and only if ESS-coinjective modules are injective. It is shown that every module is ESS-coinjective if and only if R is right $Z_2(R_R)$-semiperfect if and only if projective modules are ESS-coinjective, and that a projective module X is ESS-coinjective if and only if $X/Z_2(X)$ is an injective module. In Section 3, we introduced an extented S-supplemented module. A module M is called extended S-supplemented or briefly an ESS-module if every submodule of M is an extended S-supplement. We obtain some properties of these modules. We show that every module is an ESS-module if and only if injective modules are ESS-modules if and only if R is right $Z_2(R_R)$-semiperfect if and only if, for any module T, $T = Z_2(T) \oplus N$ where N is semisimple if and only if nonsingular modules are semisimple projective. In Section 4, we studied modules whose extended S-supplement submodules are direct summands. Some characterizations of right SI-rings by these modules are given.

For a module T, $E(T)$, $Z_2(T)$, $Soc(T)$ will stand for the injective hull, the Goldie torsion submodule and the socle of T, respectively. The Jacobson radical of the ring R is denoted by $J(R)$. For a homomorphism $f : A \to B$ and a module C, the induced homomorphism $Ext^1_R(1_C, f) : Ext^1_R(C, A) \to Ext^1_R(C, B)$ will be denoted by f_*. For unexplained concepts and notations, we refer the reader to [1, 7, 8, 13].

2. Extended S-supplement submodules

In this section, we examine main properties of ESS-coinjective modules. Let us begin with the following definition.

Definition 2.1 *Let $N_1 \leq N$. N_1 is an extended S-supplement in N if there exists $S \leq N$ such that $N = S + N_1$ and $S \cap N_1$ is Goldie torsion.*

Every Goldie torsion submodule of a module is extended S-supplement. An exact sequence $0 \to M_1 \to^f M \to M_2 \to 0$ is called extended S-supplement if $f(M_1)$ is an extended S-supplement submodule of M. The class of extended S-supplement sequences ESS is a proper class by [7, Theorem 3.1]. In the next result, we show that the class ESS is injectively generated by nonsingular modules.

Proposition 2.2 *An exact sequence $\mathcal{E} : 0 \to X \to H \to Z \to 0$ is extended S-supplement if and only if $Hom(H, F) \to Hom(X, F) \to 0$ for each nonsingular module F.*

Proof ($\Rightarrow$) Let $f : X \to W$ be a homomorphism with W nonsingular. It is enough to show that $f_*(\mathcal{E}) : 0 \to W \to^g T \to Z \to 0$ is splitting. Since ESS is a proper class, $f_*(\mathcal{E}) \in$ESS; hence, there exists $S \leq T$ such that $g(W) + S = T$ and $g(W) \cap S$ is Goldie torsion. However, $g(W)$ is nonsingular; hence, $g(W) \cap S$ must be zero. Therefore, $f_*(\mathcal{E})$ is splitting, as desired. ($\Leftarrow$) By our assumption, $X/Z_2(X)$ is injective to the sequence $\mathcal{E}$. Then $(X/Z_2(X)) \oplus (T/Z_2(X)) = H/Z_2(X)$ for some $Z_2(X) \leq T \leq H$. This shows that $X + T = H$ and $X \cap T = Z_2(X)$. Thus, our claim is established. □

In the remaining part of this section, we investigated coinjective object of the proper class ESS.

Definition 2.3 *We will call a module ESS-coinjective if it is extended S-supplement in every extension.*

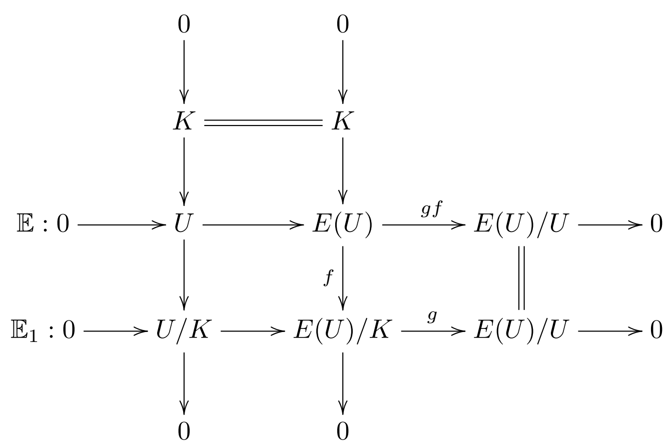
A module N is ESS-coinjective if and only if N is extended S-supplement in $E(N)$ if and only if N is extended S-supplement in any ESS-coinjective module. ESS-coinjective modules are closed under extensions and extended S-supplement submodules, (see [9, Proposition 1.7-1.8]).

Remark 2.4 *Obviously, injective modules and Goldie torsion modules are ESS-coinjective.*

R is right hereditary if and only if quotients of injective modules are injective. Next we consider when quotients of ESS-coinjective modules are ESS-coinjective.

Lemma 2.5 *ESS-coinjective modules are closed under quotients if and only if quotients of injective modules are ESS-coinjective.*

Proof ($\Rightarrow$) is clear. ($\Leftarrow$) Let U be an ESS-coinjective module and $K \leq U$. Consider the following diagram:



Since U is an ESS-coinjective module, it is extended S-supplement in $E(U)$. Then, by properties of proper classes, U/K is also extended S-supplement in $E(U)/K$. By our hypothesis, $E(U)/K$ is ESS-coinjective. Thereby, by [9, Proposition 1.8], U/K is an ESS-coinjective, as desired.

□

Corollary 2.6 *ESS-coinjective modules are closed under quotients on right hereditary rings.*

Proposition 2.7 *ESS-coinjective are closed under essential extensions.*

Proof Let N be an ESS-coinjective module and T an essential extension of N. Since N is an essential submodule of T, T/N is singular. Recall that singular modules are ESS-coinjective and extensions of ESS-coinjective modules are ESS-coinjectives. Thus, T is an ESS-coinjective. □

Proposition 2.8 *Nonsingular ESS-coinjective modules are injective.*

Proof Let U be a non-injective nonsingular ESS-coinjective module. By [9, Proposition 1.8], there exists $N_1 \leq E(U)$ such that $U + N_1 = E(U)$ and $N_1 \cap U$ is Goldie torsion. Since nonsingular modules are closed under injective hull and U is nonsingular, $E(U)$ is also nonsingular. Then $N_1 \cap U = 0$. This contradicts the essentiality of U in $E(U)$. Therefore, U must be an injective module.

□

Let I be a right ideal of R. The ring R is said to be right I-semiperfect if there exists $e^2 = e \in K$ with $(1-e)K \subseteq I$ for every right ideal K of R,(see [16]). R is right $Z_2(R_R)$-semiperfect if and only if R is right semiperfect and $J(R) = Z_2(R_R)$ by [10, Corollary 37].

Proposition 2.9 *Every module is ESS-coinjective if and only if R is right $Z_2(R_R)$-semiperfect.*

Proof $(\Rightarrow)$ By Proposition 2.8, every nonsingular right R-module is injective. Then, by [10, Theorem 49], R is right $Z_2(R_R)$-semiperfect. $(\Leftarrow)$ If every nonsingular right R-module is injective, then every exact sequence is an extended S-supplement sequence. Therefore, every right R-module is ESS-coinjective. □

Lemma 2.10 *R is right SI-ring if and only if ESS-coinjective modules are injective.*

Proof $(\Leftarrow)$ This follows by the fact that every singular module is ESS-coinjective. $(\Rightarrow)$ Let W be an ESS-coinjective module. Then there exists $U \leq E(W)$ such that $W + U = E(W)$ and $U \cap W$ is Goldie torsion. Now, consider the following diagram:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & W\cap U & = & W\cap U & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\mathbb{E}: 0 & \longrightarrow & W & \longrightarrow & E(W) & \xrightarrow{gf} & E(W)/W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow f & & \| & & \\
\mathbb{E}_1: 0 & \longrightarrow & W/W\cap U & \longrightarrow & E(W)/W\cap U & \xrightarrow{g} & E(W)/W & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

By our hypothesis, $W \cap U$ is injective, and thus be a direct summand of $E(W)$. Furthermore, $W/W \cap U$ is direct summand of $E(W)/W \cap U$. Therefore, W must be injective as it is a direct summand of $E(W)$, a contradiction.

□

R is a QF ring if and only if projective modules are injective. Next, we consider when a projective module is ESS-coinjective.

Proposition 2.11 *Let M_1, M_2 be modules and $f : M_2 \to M_1$ a monomorphism. If M_2 is ESS-coinjective module, then any nonsingular quotient of M_2 is isomorphic to a nonsingular quotient of M_1. In particular, $M_2/Z_2(M_2)$ is a quotient of an injective module.*

Proof Let F be a nonsingular quotient of M_2 and $\pi : M_2 \to F$ be epimorphism. Since M_2 is an ESS-coinjective, the map $Hom(M_1, F) \to Hom(M_2, F) \to 0$ is exact, and this implies the existence of a homomorphism $g : M_1 \to F$ with $gf = \pi$. Then g is an epimorphism and $M_1/ker(g) \cong g(M_1) = M_2$ is nonsingular. The particular case follows by taking an inclusion $f : M_2 \to E(M_2)$ and $F = M_2/Z_2(M_2)$. □

Theorem 2.12 *Let W be a projective module. W is ESS-coinjective if and only if $W/Z_2(W)$ is an injective module.*

Proof $(\Rightarrow)$ If W is projective ESS-coinjective module, then, by Proposition 2.11, $W/Z_2(W)$ is a quotient of an injective module, say E. Now, consider the sequence $0 \to ker(\alpha) \to E \to^{\alpha} W/Z_2(W) \to 0$. Recall that $W/Z_2(W)$ is nonsingular, and so $ker(\alpha)$ is closed in E by [12, Lemma 2.3]. This implies that the sequence is splitting; hence $W/Z_2(W)$ is an injective module as (isomorphic to) a direct summand of E. ($\Leftarrow$) Assume that $W/Z_2(W)$ is injective module. Let $\mathcal{E} : 0 \to W \to^g E \to A \to 0$ be a sequence with E injective, and let $f : W \to H$ be a homomorphism with H a nonsingular module. Without loss of generality, we may take f as an epimorphism. We claim that there exists a homomorphism $h : E \to H$ such that $hg = f$. Since H is a nonsingular module, $f(W) = H$ is quotient of $W/Z_2(W)$, that is, there is an epimorphism $\pi : W/Z_2(W) \to H$. Since $W/Z_2(W)$ is injective and H is nonsingular, π is a split epimorphism, and this implies that H is injective. In other words, every nonsingular module is injective with respect to $\mathcal{E}$. Therefore, W is ESS-coinjective. □

Using [12, Lemma 2.3] and following the proof of Theorem 2.12, we get:

Corollary 2.13 *A projective module P is ESS-coinjective if and only if every nonsingular quotient of P is an injective module.*

Theorem 2.14 *R is right $Z_2(R_R)$-semiperfect if and only if projective modules are ESS-coinjective*

Proof ($\Leftarrow$) It is enough to show that every nonsingular right R-module is injective by [10, Theorem 49]. Let U be a nonsingular module and $f : T \to U$ an epimorphism with T a free module. By our assumption, there exists a homomorphism $\alpha : E(T) \to U$ such that $\alpha\iota = f$, where $\iota : T \to E(T)$. Since f is epimorphism, α is also epimorphism. Since U is nonsingular, $ker(\alpha)$ is closed in $E(T)$ by [12, Lemma 2.3]. However, closed submodules of injective modules are direct summands; hence, U is injective. ($\Rightarrow$) This claim follows by Corollary 2.13 and [10, Theorem 49]. □

3. ESS-modules

A module is said to be ESS-module if all its submodules are extended S-supplements. In this section, we give some properties of an ESS-module.

Proposition 3.1 *Let T be a module and $N, V \leq T$. If N is an ESS-module and $N + V$ is an extended S-supplement in T, then V is an extended S-supplement in T.*

Proof Since $N + V$ is an extended S-supplement in T, there exists $S \leq T$ such that $S + (N + V) = T$ and $S \cap (N + V)$ is Goldie torsion. Now consider the submodule $N \cap (S + V)$ of N. By our hypothesis, N is an ESS-module; hence, there is a submodule $W \leq N$ such that $W + (N \cap (S + V)) = N$ and $W \cap (N \cap (S + V)) = W \cap (S + V)$ is Goldie torsion. Then since $T = S + (N + V)$ and $N = W + (N \cap (S + V))$, $T = S + (W + (N \cap (S + V)) + V) = S + V + W + (N \cap (S + V)) = S + V + W$. Then since $V \cap (S + W)$ is Goldie torsion as a submodule of Goldie torsion module $W \cap (S + V) + S \cap (W + V)$, V is an extended S-supplement in T. □

Corollary 3.2 *A finite sum of ESS-modules is an ESS-module.*

Proof It is enough to show that $M = M_1 + M_2$ is an ESS-module if M_1, M_2 are ESS-modules. Let N_1 be a submodule of M. Since $M_1 + (M_2 + N_1)$ has the trivial extended S-supplement in M, $M_2 + N_1$ is an extended S-supplement in M. Then, again by Proposition 3.1, N_1 is an extended S-supplement in M. □

Proposition 3.3 *Any submodule or quotient of an ESS-module is an ESS-module.*

Proof Let M_1 be an ESS-module and $L_1 \leq M_1$. We will show that L_1 and M_1/L_1 are ESS-modules. To show that L_1 is an ESS-module, let $T_1 \leq L_1$. Since M_1 is an ESS-module, there is an $N_1 \leq M_1$ such that $N_1 + T_1 = M_1$ and $N_1 \cap T_1$ is Goldie torsion. By modular law, $L_1 = T_1 + (N_1 \cap L_1)$. Moreover, $T_1 \cap N_1 \cap L_1$ is Goldie torsion as a submodule of Goldie torsion module $N_1 \cap T_1$. This shows that L_1 is an ESS-module.

Let $K_1/L_1 \leq M_1/L_1$. Since M_1 is an ESS-module, there is an $N_1 \leq M_1$ such that $N_1 + K_1 = M_1$ and $N_1 \cap K_1$ is Goldie torsion. Then $(K_1/L_1) + ((N_1 + L_1)/L_1) = M_1/L_1$ and, since Goldie torsion modules are closed under homomorphic image, $(K_1/L_1) \cap ((L_1 + N_1)/L_1) = ((K_1 \cap N_1) + L_1)/L_1 \cong (K_1 \cap N_1)/(N_1 \cap L_1)$ is Goldie torsion. Thus, $(N_1 + L_1)/L_1$ is an extended S-supplement of K_1/L_1 in M_1/L_1. So M_1/L_1 is an ESS-module. □

Lemma 3.4 *Let U be a module and $G \leq U$. If G is a Goldie torsion and U/G is an ESS-module, then U is an ESS-module.*

Proof Let U_1 be a submodule of U. Note that U/G is an ESS-module, so that $(U_1 + G)/G$ is an extended S-supplement in U/G. Then there is $T/G \leq U/G$ such that $((U_1 + G)/G) + (T/G) = U/G$ and $[(U_1 + G)/G] \cap (T/G) = ((T \cap U_1) + G)/G \cong (T \cap U_1)/(G \cap U_1)$ is Goldie torsion. Note that $U_1 \cap T$ is a Goldie torsion since Goldie torsion modules are closed under extension and $G \cap U_1$ is Goldie torsion as a submodule of G. Then $U = U_1 + T$ and $U_1 \cap T$ is Goldie torsion, implying that U_1 is an extended S-supplement in U. □

Lemma 3.5 *Let $0 \to M_1 \to X \to M_2 \to 0$ be an extended S-supplement sequence. Then, X is an ESS-module if and only if M_1 and M_2 are ESS-modules.*

Proof If X is an ESS-module, then M_1 and M_2 are ESS-modules by Proposition 3.3. Now, suppose that M_1 and M_2 are ESS-modules. Without loss of generality, we will take $M_1 \leq X$. Since M_1 is an extended S-supplement in X, there exists $S \leq X$ such that $M_1 + S = X$ and $M_1 \cap S$ is Goldie torsion. Then, $X/(M_1 \cap S) = M_1/(M_1 \cap S) \oplus S/(M_1 \cap S)$. $M_1/(M_1 \cap S)$ is an ESS-module as a quotient of M_1 by Proposition 3.3. Furthermore, $S/(M_1 \cap S) \cong X/M_1 \cong M_2$ is an ESS-module. Then $X/(M_1 \cap S)$ is an ESS-module as a sum of ESS-modules by Proposition 3.3. This implies that X is an ESS-module by Lemma 3.4. □

Proposition 3.6 *Any nonsingular ESS-module is semisimple projective.*

Proof Let Q be a nonsingular ESS-module and $N_1 \leq Q$. By our hypothesis, there exists $S \leq Q$ such that $N_1 + S = Q$ and $N_1 \cap S$ is Goldie torsion. However, Q is nonsingular; hence, $N_1 \cap S = 0$. Therefore, Q is semisimple, i.e. all submodule of Q are direct summands. Note that nonsingular simple modules are projective. Therefore Q is also projective. □

Theorem 3.7 *The following statements are equivalent.*

(1) All modules are ESS-module.

(2) All injective modules are ESS-module.

(3) All nonsingular modules are semisimple projective.

(4) For any module T, $T = Z_2(T) \oplus N$ where N is semisimple.

(5) R is right $Z_2(R_R)$-semiperfect.

Proof (1) ⇒ (2) is clear. (2) ⇒ (3) Let T be a nonsingular module. Since Goldie torsion theory is hereditary, $E(T)$ is also nonsingular. Then, by Proposition 3.6, $E(T)$ is semisimple projective, and this implies that T is also a semisimple projective. (3) ⇒ (4) For any module T, consider the sequence $0 \to Z_2(T) \to T \to T/Z_2(T) \to 0$. By (3), this exact sequence is splitting, i.e. $T = Z_2(T) \oplus N$ where $N \cong T/Z_2(T)$ is semisimple. (4) ⇔ (5) [10, Theorem 49]. (5) ⇒ (1) Let T be any module. Consider the exact sequence $0 \to Z_2(T) \to T \to T/Z_2(T) \to 0$. By [10, Theorem 49], $T/Z_2(T)$ is semisimple; hence it is an ESS-module. Since $Z_2(T)$ is an ESS-module and extended S-supplement in every extension, T is an ESS-module by Lemma 3.5.

□

4. ⊕-ESS Modules

A module is CS if its closed submodules are direct summands [1]. Inspired by CS-modules, we introduced ⊕-ESS modules. A module is called ⊕-ESS if its extended S-supplement submodules are direct summands. Note that a Goldie torsion module is ⊕-ESS if and only if it is singular and semisimple.

Proposition 4.1 *Nonsingular modules are ⊕-ESS.*

Proof Let Q be a nonsingular module and Z an extended S-supplement submodule in Q. Then there exists $S \leq Q$ such that $S + Z = Q$ and $S \cap Z$ is Goldie torsion. However, Q is nonsingular, and so $S \cap Z = 0$. Then, $S \oplus Z = Q$, as desired. □

Proposition 4.2 *Let W be an ⊕-ESS module and $Z \leq W$. If Z is an extended S-supplement in W, then Z and W/Z are ⊕-ESS modules.*

Proof Assume that Z is an extended S-supplement in W. Let T be an extended S-supplement submodule of Z. Then there exists $H \leq Z$ such that $T + H = Z$ and $T \cap H$ is Goldie torsion. Moreover, since W is ⊕-ESS, there exists $K \leq W$ such that $K \oplus Z = W$. Then $T + (K \oplus H) = W$. Note that $T \cap (K \oplus H) = T \cap H$, and so $T \cap (K \oplus H)$ is Goldie torsion. This implies that T is an extended S-supplement submodule of W; hence, there exists $T_1 \leq W$ such that $T_1 \oplus T = W$. Then, by modular law, $(Z \cap T_1) \oplus T = Z$,and this proves that Z is an ⊕-ESS module. To show that W/Z is ⊕-ESS module, let Y/Z be an extended S-supplement submodule of W/Z. Then there exists $Z \leq L \leq W$ such that $(L/Z) + (Y/Z) = W/Z$ and $(L \cap Y)/Z$ is Goldie torsion. Moreover, $(L \cap K) \oplus Z = L$; hence, $Y \cap L = (Y \cap L \cap K) \oplus Z$. Then since $(L \cap Y)/Z$ is Goldie torsion and $(L \cap Y)/Z \cong Y \cap L \cap K$, $Y \cap L \cap K$ is also Goldie torsion. Since $W = Y + L = Y + ((L \cap K) \oplus Z) = Y + (L \cap K)$, Y is an extended S-supplement in W. However, W is ⊕-ESS, and so there exists $K_1 \leq W$ such that $K_1 \oplus Y = W$. Then $(K_1/Z) \oplus (Y/Z) = W/Z$, and this shows that W/Z is ⊕-ESS module. □

Corollary 4.3 *⊕-ESS modules are closed under direct summands.*

Lemma 4.4 *A module W is an ⊕-ESS if and only if $W = Z_2(W) \oplus X$, where X is nonsingular and $Z_2(W)$ is a semisimple singular.*

Proof (⇒) Assume that a module W is an ⊕-ESS. Since $Z_2(W)$ is extended S-supplement in every extension and W is an ⊕-ESS, there exists $X \leq W$ such that $W = Z_2(W) \oplus X$. Recall that $Z_2(W)$ is the largest Goldie torsion submodule of W, and so X is nonsingular. By Proposition 4.2, $Z_2(W)$ is ⊕-ESS modules, and so it is semisimple and singular. (⇐) Let P be an extended S-supplement in W. Then there exists $F \leq W$ such that $F + P = W$ and $F \cap P$ is Goldie torsion. Since $F \cap P$ is Goldie torsion and $Z_2(W)$ is defined as the sum of all Goldie torsion submodules of W, $F \cap P \leq Z_2(W)$. Then there exists $U \leq Z_2(W)$ such that $U \oplus (F \cap P) = Z_2(W)$ since $Z_2(W)$ is semisimple. Therefore, $W = Z_2(W) \oplus X = U \oplus (F \cap P) \oplus X$; hence, by modular law, $F = F \cap (U \oplus X) \oplus (F \cap P)$ and $P = P \cap (U \oplus X) \oplus (F \cap P)$. Then $W = F + P = P \oplus (F \cap (U \oplus X))$, as desired. □

Note that singular modules, semisimple modules, and nonsingular modules are closed under direct sums. Therefore, we have the following result by Lemma 4.4.

Corollary 4.5 *A direct sum of ⊕-ESS modules is ⊕-ESS module.*

Proof Let $(A_l)_{l \in L}$ be a family of ⊕-ESS modules and denote $A = \oplus_{l \in L} A_l$. By Lemma 4.4, for each $l \in L$, $A_l = Z_2(A_l) \oplus X_l$, where X_l is nonsingular and $Z_2(A_l)$ is semisimple and singular. Then $A = \oplus_{l \in L} A_l = \oplus_{l \in L}(Z_2(A_l) \oplus X_l) = (\oplus_{l \in L} Z_2(A_l)) \oplus (\oplus_{l \in L} X_l)$. Since nonsingular modules are closed under direct sums, the module $\oplus_{l \in L} X_l$ is nonsingular. The module $\oplus_{l \in L} Z_2(A_l)$ is also semisimple singular since it is a direct sum of semisimple singular modules. Then, by Lemma 4.4, A is ⊕-ESS module, as claimed. □

Corollary 4.6 *Let W be a module with projective socle. Then W is ⊕-ESS if and only if W is nonsingular.*

Proof By Lemma 4.4, $W = Z_2(W) \oplus X$, where X is nonsingular and $Z_2(W)$ is semisimple and singular. Then $Soc(W) = Soc(Z_2(W)) \oplus Soc(X) = Z_2(W) \oplus Soc(X)$. However, $Soc(W)$ is projective, and so $Z_2(W)$ must be zero. Therefore, W is nonsingular. The converse follows by Proposition 4.1. □

Corollary 4.7 *R is right nonsingular if and only if all projective modules are ⊕-ESS.*

Corollary 4.8 *Let W be a projective module. Then W is ⊕-ESS if and only if W is nonsingular.*

Proof By Lemma 4.4, $W = Z_2(W) \oplus X$, where X is nonsingular and $Z_2(W)$ is semisimple and singular. However, W is projective, and so $Z_2(W)$ must be zero. Therefore, W is nonsingular. The converse follows by Proposition 4.1. □

Theorem 4.9 *The following statements are equivalent.*

(1) All modules are ⊕-ESS.

(2) All injective modules are ⊕-ESS

(3) R is right SI-ring

Proof (1) ⇒ (2) is clear. (2) ⇒ (3) Let Z be a singular module. Since Z is extended S-supplement in $E(Z)$ and $E(Z)$ is ⊕-ESS, Z is a direct summand of $E(Z)$; hence, it is injective. (3) ⇒ (1) Let W be any module and $U \leq W$. If U is extended S-supplement in W, then there exists $Z \leq W$ such that $Z + U = W$ and $Z \cap U$ is Goldie torsion. Since R is right SI-ring, there exists $Z_1 \leq Z$ such that $(Z \cap U) \oplus Z_1 = Z$. Then $W = U + Z = U \oplus Z_1$, as claimed. □

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