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Some properties of Riemannian geometry of the tangent bundle of Lie groups

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Abstract: We consider a bi-invariant Lie group (G, g) and we equip its tangent bundle TG with the left invariant Riemannian metric introduced in the paper of Asgari and Salimi Moghaddam. We investigate Einstein-like, Ricci soliton, and Yamabe soliton structures on TG. Then we study some geometrical tensors on TG such as Cotton, Schouten, Weyl, and Bach tensors, and we also compute projective and concircular and m-projective curvatures on TG. Finally, we compute the Szabo operator and Jacobi operator on the tangent Lie group TG.

Key words: Cotton tensor, Einstein-like structure, Lie group, soliton structure, Schouten tensor, Yamabe soliton

1. Introduction

The study of the differential geometry of tangent bundles was started in the early 1960s by Davies [10], PDombrowski [12], Ledger and Yano [20], Tachibana and Okumura [25], and Sasaki [24]. Yano and Kobayashi [27–29] developed the theory of vertical and complete lifts of tensor fields and connections to tangent bundles.

Some new interesting geometric structures on the tangent bundle of a Riemannian manifold (M, g) may be obtained by lifting the metric g from the base manifold. The natural lifts on the total space of the tangent bundle of a Riemannian manifold, introduced by Kowalski and Sekizawa [19], were intensively studied in the last decades by Abbassi, Sarih, and others (cf., for instance, [1] and [22]).

In the last years there has been increasing interest in the study of Riemannian manifolds endowed with metrics satisfying some structural equations, possibly involving curvature and some globally defined vector fields. These objects naturally arise in several different frameworks. One of the most important and well-studied examples is that of Ricci solitons [13, 18]. Other examples are, for instance, Ricci almost solitons [23] and Yamabe solitons [9, 11].

An important class of Riemannian manifolds is the family of Lie groups equipped with left invariant Riemannian metrics. A Lie group is a smooth manifold that also carries a group structure whose product and inverse operations are smooth as maps of manifolds. In the middle of the 19th century, Sophus Lie made the far-reaching discovery that techniques designed to solve particular unrelated types of ODEs, such as separable, homogeneous, and exact equations, were in fact all special cases of a general form of integration procedure based on the invariance of the differential equation under a continuous group of symmetries. One of the reasons why Lie groups are nice is that they have a differential structure, which means that the notion of tangent space makes sense at any point of the group. Furthermore, the tangent space at identity happens to have some

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algebraic structure, specifically that of a Lie algebra. Roughly speaking, the tangent space at identity provides a linearization of the Lie group, and it turns out that many properties of a Lie group are reflected in its Lie algebra. Lie groups are the natural concept for the mathematical description of symmetry in the physical world. They were initially introduced as a tool to solve or simplify ordinary and partial differential equations.

Lie group theory establishes surprising relations between many different areas of mathematics, in particular between algebra, geometry, topology, and analysis. In string theory and loop quantum gravity, which are currently the only two promising candidates for a successful unification of quantum theory and general relativity, Lie groups also play an important role.

In this paper, we consider a bi-invariant Lie group (G, g), and we equip its tangent bundle TG with a special left invariant Riemannian metric $\widetilde{g}$. It is worth mentioning that the choice of bi-invariant metrics on G is motivated by the fact that their Levi-Civita connection and consequently their curvatures of different kinds are known. This paper is divided into three parts. In the first part, we characterize conformal (resp. Killing) vertical and complete lifts vector fields and study some special Yamabe soliton structures on TG. We obtain the condition, in the second part, that the Ricci almost soliton structure in a bi-invariant Lie group is conformally invariant. Finally, in the third part we compute the Schouten and Cotton tensors of TG. In this part we introduce the concept of the Weyl tensor and we compute the Weyl tensor of TG. Moreover, by using the Weyl–Schouten theorem we conclude that the conformal flatness of TG implies the flatness of G. In this part we also introduce the Szabo operator of a Riemannian manifold (M, g) and we calculate the Szabo operator of TG. Then we show the Szabo flatness of G and the Szabo flatness of TG. Moreover, we introduce the concept of the Jacobi operator of a Riemannian manifold (M, g). Then we compute the Jacobi operator of tangent Lie group TG, and we also show that TG is nilpotent Osserman at identity when G is nilpotent Osserman at identity. This means that the nilpotency of the Jacobi operator of G implies the nilpotency of the Jacobi operator of TG.

2. Preliminaries

Let (M, g) be a Riemannian manifold and TM be its tangent bundle. If (x^i) is a local coordinate system on M and (x^i, y^i) is the induced local coordinate system on TM, then the complete lift and vertical lift of a vector field $X = X^i\partial_i$ on M are defined as follows [26]:

$$X^c = X^i\partial_i + y^a(\partial_a X^i)\dot{\partial}_i, \qquad X^v = X^i\dot{\partial}_i,$$

where $\partial_i := \frac{\partial}{\partial x^i}$ and $\dot{\partial}_i := \frac{\partial}{\partial y^i}$. Furthermore, the complete lift f^c of smooth function f on M is defined by $f^c = y^i(\frac{\partial f}{\partial x^i})$.

Consider the tangent bundle TG over a Lie group G. By using the Lie group structure of G we can construct a Lie group structure on TG as follows:

$$(x, v_x).(y, w_y) := (xy, (dl_x)(w_y) + (dr_y)(v_x)),$$

for all $x, y \in G$, $v_x \in T_xG$, and $w_y \in T_yG$, where l_x and r_y are the left and right translations of G by x and y, respectively. We can easily check that this multiplication is smooth and TG equipped with this action has a group structure, and so $(TG, .)$ is a Lie group.

The Lie brackets of vertical and complete lifts satisfy the following equations [2]:

$$[X^v, Y^v] = 0, \qquad [X^c, Y^c] = [X, Y]^c, \qquad [X^v, Y^c] = [X, Y]^v. \tag{2.1}$$

It is well known that the complete and vertical lifts of any left invariant vector fields of G are left invariant vector fields on the Lie group TG (for more details, see Proposition 1.3 of [26]). Also, for every arbitrary left invariant vector field $\widetilde{X}$ on TG, there exist two left invariant vector fields X_1, X_2 on G such that $\widetilde{X} = X_1^c + X_2^v$, where X_1^c is the complete lift of X_1 and X_2^v is the vertical lift of X_2. Moreover, if $\{X_1, \cdots, X_m\}$ is a basis for the Lie algebra $\mathfrak{g}$ of G then $\{X_1^v, \cdots, X_m^v, X_1^c, \cdots, X_m^c\}$ is a basis for the Lie algebra $\widetilde{\mathfrak{g}}$ of TG.

A metric g on a Lie group G is called left invariant (resp. right invariant) if

$$g_b(u, v) = g_{ab}((dl_a)_b u, (dl_a)_b v),$$

$$(\text{resp. } g_b(u, v) = g_{ba}((dr_a)_b u, (dr_a)_b v)),$$

for all $a, b \in G$ and all $u, v \in T_b G$. To simplify, (G, g) is called a left invariant (resp. right invariant) Lie group. A Riemannian metric that is both left and right invariant is called a bi-invariant metric. In this case, (G, g) is called a bi-invariant Lie group.

Let g be a left invariant Riemannian metric on a Lie group G. Then we consider a left invariant Riemannian metric $\widetilde{g}$ as follows [2]:

$$\widetilde{g}(X^c, Y^c) = g(X, Y), \qquad \widetilde{g}(X^v, Y^v) = g(X, Y), \qquad \widetilde{g}(X^c, Y^v) = 0, \tag{2.2}$$

where X, Y are any two left invariant vector fields on G. Throughout this paper we consider the Riemannian metric $\widetilde{g}$ on TG defined as above. Then we study the geometry of $(TG, \widetilde{g})$ and its relation with that of (G, g).

Theorem 2.1 *[2] Let ∇ and $\widetilde{\nabla}$ be the Levi-Civita connections of $(G.g)$ and $(TG, \widetilde{g})$, respectively, and X and Y be any two left invariant vector fields on G. Then we have:*

$$\widetilde{\nabla}_{X^c} Y^c = (\nabla_X Y)^c, \qquad \widetilde{\nabla}_{X^v} Y^v = (\nabla_X Y - \frac{1}{2}[X, Y])^c,$$

$$\widetilde{\nabla}_{X^c} Y^v = (\nabla_X Y + \frac{1}{2} ad^{\star}_Y X)^v, \qquad \widetilde{\nabla}_{X^v} Y^c = (\nabla_X Y + \frac{1}{2} ad^{\star}_X Y)^v,$$

where $ad^{\star}_X Y$ denotes the transpose of ad_X with respect to the inner product induced by g on $\mathfrak{g}$.

As concerns the Riemannian curvature of $(TG, \widetilde{g})$, we have the following.

Theorem 2.2 *Let (G, g) be a left invariant Lie group and $\widetilde{g}$ the left invariant Riemannian metric given by (2.2). If R and $\widetilde{R}$ denote the Riemannian curvature tensors of (G, g) and $(TG, \widetilde{g})$, respectively, then, for all left invariant vector fields X, Y, Z on G, we have the following identities:*

$$\begin{aligned}\widetilde{R}(X^c, Y^c)Z^v = (R(X, Y)Z)^v + \{&\frac{1}{2}\nabla_X(ad^*_Z Y) + \frac{1}{2} ad^*_{\nabla_Y Z + \frac{1}{2} ad^*_Z Y} X - \frac{1}{2}\nabla_Y(ad^*_Z X)\\ &- \frac{1}{2} ad^*_{\nabla_X Z + \frac{1}{2} ad^*_Z X} Y - \frac{1}{2} ad^*_Z [X, Y]\}^v,\end{aligned}$$

$$\widetilde{R}(X^c, Y^c)Z^c = (R(X, Y)Z)^c,$$

$$\widetilde{R}(X^c, Y^v)Z^v = (R(X,Y)Z)^c + \{-\frac{1}{2}\nabla_X([Y,Z]) - \frac{1}{2}\nabla_Y(ad^*_Z X) + \frac{1}{2}[Y, \nabla_X Z] + \frac{1}{4}[Y, ad^*_Z X] - \frac{1}{2}[[X,Y],Z]\}^c,$$

$$\widetilde{R}(X^v, Y^c)Z^c = (R(X,Y)Z))^v + \{\frac{1}{2}ad^*_X(\nabla_Y Z) - \frac{1}{2}\nabla_Y(ad^*_X Z) - \frac{1}{2}ad^*_{\nabla_X Z + \frac{1}{2}ad^*_X Z)}Y - \frac{1}{2}ad^*_{[X,Y]}Z\}^v,$$

$$\widetilde{R}(X^v, Y^v)Z^c = \{\nabla_X(\nabla_Y Z) + \frac{1}{2}\nabla_X(ad^*_Y Z) - \frac{1}{2}[X, \nabla_Y Z] - \frac{1}{4}[X, ad^*_Y Z] - \nabla_Y(\nabla_X Z) - \frac{1}{2}\nabla_Y(ad^*_X Z) + \frac{1}{2}[Y, \nabla_X Z] + \frac{1}{4}[Y, ad^*_X Z]\}^c,$$

$$\widetilde{R}(X^v, Y^v)Z^v = \{\nabla_X \nabla_Y Z - \frac{1}{2}\nabla_X([Y,Z]) + \frac{1}{2}ad^*_X(\nabla_Y Z - \frac{1}{2}[Y,Z]) - \nabla_Y(\nabla_X Z) + \frac{1}{2}\nabla_Y([X,Z]) - \frac{1}{2}ad^*_Y(\nabla_X Z - \frac{1}{2}[X,Z])\}^v.$$

Proof Using Theorem 2.1, we conclude that

$$\begin{aligned}
\widetilde{R}(X^c, Y^c)Z^v &= \widetilde{\nabla}_{X^c}\widetilde{\nabla}_{Y^c}Z^v - \widetilde{\nabla}_{Y^c}\widetilde{\nabla}_{X^c}Z^v - \widetilde{\nabla}_{[X^c,Y^c]}Z^v \\
&= \widetilde{\nabla}_{X^c}(\nabla_Y Z + \frac{1}{2}ad^*_Z Y)^v - \widetilde{\nabla}_{Y^c}(\nabla_X Z + \frac{1}{2}ad^*_Z X)^v - \widetilde{\nabla}_{[X,Y]^c}Z^v \\
&= \{\nabla_X(\nabla_Y Z + \frac{1}{2}ad^*_Z Y) + \frac{1}{2}ad^*_{\nabla_Y Z + \frac{1}{2}ad^*_Z Y}X\}^v \\
&- \{\nabla_Y(\nabla_X Z + \frac{1}{2}ad^*_Z X) + \frac{1}{2}ad^*_{\nabla_X Z + \frac{1}{2}ad^*_Z X}Y\}^v \\
&- \{\nabla_{[X,Y]}Z + \frac{1}{2}ad^*_Z[X,Y]\}^v \\
&= (R(X,Y)Z)^v + \{\frac{1}{2}\nabla_X(ad^*_Z Y) + \frac{1}{2}ad^*_{\nabla_Y Z + \frac{1}{2}ad^*_Z Y}X \\
&- \frac{1}{2}\nabla_Y(ad^*_Z X) - \frac{1}{2}ad^*_{\nabla_X Z + \frac{1}{2}ad^*_Z X}Y - \frac{1}{2}ad^*_Z[X,Y]\}^v.
\end{aligned}$$

In a similar way, we obtain the other relations. □

When the metric g on G is bi-invariant, then its Levi-Civita connection ∇ satisfies the identity $\nabla_X Y = \frac{1}{2}[X,Y]$ for all left invariant vector fields X and Y on G, and we also have $g(ad_X Y, Z) + g(Y, ad_X Z) = 0$, for all left invariant vector fields X, Y, and Z on G, i.e. $ad^*_X = ad_X$, for all left invariant vector fields X on G. On the other hand, the curvature R of (G,g) is given by $R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$. Using the previous identities, Theorems 2.1 and 2.2 give the following.

Corollary 2.3 *Let (G,g) be a bi-invariant Lie group and $\widetilde{g}$ the left invariant Riemannian metric on TG given by (2.2). Then, for all left invariant vector fields X, Y, Z on G, we have:*

$$(i)\quad \widetilde{\nabla}_{X^c}Y^c = \frac{1}{2}[X,Y]^c; \quad \widetilde{\nabla}_{X^c}Y^v = [X,Y]^v; \quad \widetilde{\nabla}_{X^v}Y^c = \widetilde{\nabla}_{X^v}Y^v = 0.$$

$$\begin{aligned}(ii)\quad &\widetilde{R}(X^c,Y^c)Z^c = -\frac{1}{4}[[X,Y],Z]^c,\\ &\widetilde{R}(X^c,Y^c)Z^v = \widetilde{R}(X^c,Y^v)Z^v = \widetilde{R}(X^v,Y^c)Z^c = \widetilde{R}(X^v,Y^v)Z^c = \widetilde{R}(X^v,Y^v)Z^v = 0.\end{aligned}$$

$$(iii)\quad \widetilde{Ric}(X^c,Y^c) = Ric(X,Y); \quad \widetilde{Ric}(X^c,Y^v) = \widetilde{Ric}(X^v,Y^v) = 0.$$

Hereafter, we suppose that the tangent bundle TG of any left invariant (resp. bi-invariant) Lie group (G,g) is endowed with the left invariant Riemannian metric $\widetilde{g}$ given by (2.2).

3. Conformality properties and the concepts related to it

Let (M,g) be a Riemannian manifold and X be a vector field on it. We say that X is a conformal vector field if $(L_X g)(Y,Z) = 2\rho g(Y,Z)$ for some smooth function ρ on M and every X,Y of $\chi(M)$. Moreover, X is called a Killing vector field when ρ vanishes. Furthermore, if ρ is constant we say that X is homothetic.

Let TM be the tangent bundle over M and $\widetilde{g}$ be a Riemannian metric on TM. A conformal vector field $\widetilde{X}$ on TM is said to be inessential if the scalar function $\widetilde{\rho}$ on TM in $L_{\widetilde{X}}\widetilde{g} = 2\widetilde{\rho}\widetilde{g}$ is constant on each fiber.

Theorem 3.1 *Let X be a left invariant vector field on a left invariant Lie group (G,g) and X^c be the complete lift of X. Then X is a conformal vector field on (G,g) if and only if X^c is a conformal vector field on $(TG,\widetilde{g})$.*

Proof At first it is remarkable that for left invariant vector fields X and Y on G, $g(X,Y)$ is constant, and so $Zg(X,Y) = 0$, for any vector field Z on G. Thus, direct computations give us:

$$\begin{aligned} L_{X^c}\widetilde{g}(X_j^c,X_k^c) &= X^c\widetilde{g}(X_j^c,X_k^c) - \widetilde{g}([X^c,X_j^c],X_k^c) - \widetilde{g}(X_j^c,[X^c,X_k^c])\\ &= -\widetilde{g}([X,X_j]^c,X_k^c) - \widetilde{g}(X_j^c,[X,X_k]^c)\\ &= -(g([X,X_j],X_k) + g(X_j,[X,X_k]) = L_X g(X_j,X_k), \end{aligned} \tag{3.1}$$

$$\begin{aligned} L_{X^c}\widetilde{g}(X_j^v,X_k^v) &= X^c\widetilde{g}(X_j^v,X_k^v) - \widetilde{g}([X^c,X_j^v],X_k^v) - \widetilde{g}(X_j^v,[X^c,X_k^v])\\ &= \widetilde{g}([X_j^v,X^c],X_k^v) + \widetilde{g}(X_j^v,[X_k^v,X^c])\\ &= \widetilde{g}([X_j,X]^v,X_k^v) + \widetilde{g}(X_j^v,[X_k,X]^v)\\ &= g([X_j,X],X_k) + g(X_j,[X_k,X]) = L_X g(X_j,X_k), \end{aligned} \tag{3.2}$$

$$\begin{aligned} L_{X^c}\widetilde{g}(X_j^v,X_k^c) &= X^c\widetilde{g}(X_j^v,X_k^c) - \widetilde{g}([X^c,X_j^v],X_k^c) - \widetilde{g}(X_j^v,[X^c,X_k^c])\\ &= \widetilde{g}([X_j^v,X^c],X_k^c) - \widetilde{g}(X_j^v,[X,X_k]^c)\\ &= \widetilde{g}([X_j,X]^v,X_k^c) = 0. \end{aligned} \tag{3.3}$$

Hence, X^c is a conformal vector field on TG. Conversely we have:

$$(L_X g)(X_j, X_k) = (L_{X^c}\widetilde{g})(X_j^c, X_k^c) = 2\widetilde{\rho}\widetilde{g}(X_j^c, X_k^c) = 2\widetilde{\rho} g(X_j, X_k).$$

Since the left-hand side of the above equation is a function on G, we conclude that X^c is an inessential vector field on TG and therefore $\widetilde{\rho}$ is a function on G. Hence, X is conformal. □

Here we study the conditions by which the vertical lift of a left invariant vector field on a Lie group G is Killing. Easily we get the following:

$$\begin{aligned} L_{X^v}\widetilde{g}(X_j^c, X_k^c) &= X^v\widetilde{g}(X_j^c, X_k^c) - \widetilde{g}([X^v, X_j^c], X_k^c) - \widetilde{g}(X_j^c, [X^v, X_k^c]) \\ &= -\widetilde{g}([X, X_j]^v, X_k^c) - \widetilde{g}(X_j^c, [X, X_k]^v) = 0, \end{aligned} \tag{3.4}$$

$$\begin{aligned} L_{X^v}\widetilde{g}(X_j^v, X_k^v) &= X^v\widetilde{g}(X_j^v, X_k^v) - \widetilde{g}([X^v, X_j^v], X_k^v) - \widetilde{g}(X_j^v, [X^v, X_k^v]) \\ &= -\widetilde{g}(0, X_k^v) - \widetilde{g}(X_j^v, 0) = 0, \end{aligned} \tag{3.5}$$

$$\begin{aligned} L_{X^v}\widetilde{g}(X_j^v, X_k^c) &= X^v\widetilde{g}(X_j^v, X_k^c) - \widetilde{g}([X^v, X_j^v], X_k^c) - \widetilde{g}(X_j^v, [X^v, X_k^c]) \\ &= -\widetilde{g}(0, X_k^c) - \widetilde{g}(X_j^v, [X, X_k]^v) = -g(X_j, [X, X_k]). \end{aligned} \tag{3.6}$$

The above equations imply that X^v is Killing if and only if $g(X_j, [X, X_k]) = 0$, for all $j, k = 1, \cdots n$. Obviously, if $g(X_j, [X, X_k]) = 0$ for all $j, k = 1, \cdots n$, then $(L_X g)(X_j, X_k) = 0$ for all $j, k = 1, \cdots n$, i.e. X is a Killing vector field on G. Also, the relation $g(X_j, [X, X_k]) = 0$ gives us $[X, X_k] = 0$, $\forall 1 \leq k \leq n$. These imply that X belongs to the centralizer of $\mathfrak{g}$. Therefore, we have the following theorem:

Theorem 3.2 *Let X be a left invariant vector field on a left invariant Lie group (G, g). Then the vertical lift X^v of X is a Killing vector field on $(TG, \widetilde{g})$ if and only if X belongs to the centralizer of $\mathfrak{g} = T_eG$. In this case, X is a Killing vector field on G*

Let (M, g) be a Riemannian manifold and X be a vector field on M. We say that (M, g, X) is a Yamabe soliton if the following identity holds:

$$\frac{1}{2}L_X g(Y, Z) = (S + \lambda)g(Y, Z), \tag{3.7}$$

for all vector fields Y, Z on M, where λ is a real number and S is the scalar curvature of M.

Theorem 3.3 *Let (G, g) be a bi-invariant Lie group and X be a left invariant vector field on G. If (G, g, X) is a Yamabe soliton, then $(TG, \widetilde{g}, X^c)$ is a Yamabe soliton.*

Proof Let S and $\widetilde{S}$ be the scalar curvatures of G and TG, respectively. It is known that $S = \widetilde{S}$ (see [2]). Let $\widetilde{Y}$ and $\widetilde{Z}$ be left invariant vector fields on TG. Then there exists left invariant vector fields Y_1, Y_2, Z_1, Z_2 on G such that $\widetilde{Y} = Y_1^c + Y_2^v$ and $\widetilde{Z} = Z_1^c + Z_2^v$. Now, by the definition of Yamabe solitons and by the identity (3.1), we have the following:

$$\begin{aligned}\frac{1}{2}(L_{X^c}\widetilde{g})(\widetilde{Y},\widetilde{Z}) &= \frac{1}{2}((L_{X^c}\widetilde{g})(Y_1^c, Z_1^c) + (L_{X^c}\widetilde{g})(Y_2^v, Z_2^v)) \\ &= \frac{1}{2}((L_X g)(Y_1, Z_1) + (L_X g)(Y_2, Z_2)) \\ &= (S+\lambda)(g(Y_1, Z_1) + g(Y_2, Z_2)) \\ &= (\widetilde{S}+\lambda)\widetilde{g}(\widetilde{Y},\widetilde{Z}),\end{aligned}$$

for all left invariant vector fields $\widetilde{Y}, \widetilde{Z}$ on TG. i.e. $(TG, \widetilde{g}, X^c)$ is a Yamabe soliton. □

Theorem 3.4 *Let (G, g) be a bi-invariant Lie group and X be a left invariant vector field on G. Then the following statements hold:*

(1) If $(TG, \widetilde{g}, X^v)$ is a Yamabe soliton, then X is Killing.

(2) If $(TG, \widetilde{g}, X^c)$ is a Yamabe soliton, then (G, g, X) is Yamabe soliton.

(3) If $(TG, \widetilde{g}, X^c + X^v)$ is a Yamabe soliton, then (G, g, X) is a Yamabe soliton.

Proof (1). Using the definition of the Yamabe soliton we have

$$\frac{1}{2}(L_{X^v}\widetilde{g})(\widetilde{Y},\widetilde{Z}) = (\widetilde{S}+\lambda)\widetilde{g}(\widetilde{Y},\widetilde{Z}). \tag{3.8}$$

If we consider $\widetilde{Y} = Y^c$ and $\widetilde{Z} = Z^v$, where Y, Z are left invariant vector fields on G, then we have

$$\frac{1}{2}(L_{X^v}\widetilde{g})(Y^c, Z^v) = (S+\lambda)\widetilde{g}(Y^c, Z^v).$$

Therefore, we conclude that $-\frac{1}{2}g([X,Y], Z) = 0$ for all left invariant vector fields Y, Z on G. Hence, X is a Killing vector field.

(2). Setting $\widetilde{Y} = Y^c$ and $\widetilde{Z} = Z^c$ in (3.8), where Y, Z are left invariant vector fields on G, and using (2.2) and (3.1), we get

$$\frac{1}{2}(L_X g)(Y, Z) = (S+\lambda)g(Y, Z).$$

Hence, (G, X, g) is a Yamabe soliton.

(3). By the definition of the Yamabe soliton we have

$$\frac{1}{2}(L_{X^c+X^v}\widetilde{g})(\widetilde{Y},\widetilde{Z}) = (\widetilde{S}+\lambda)\widetilde{g}(\widetilde{Y},\widetilde{Z}).$$

If we consider $\widetilde{Y} = Y^c$ and $\widetilde{Z} = Z^c$, where Y, Z are left invariant vector fields on G, then we obtain

$$\frac{1}{2}(L_{X^c}\widetilde{g})(Y^c, Z^c) + \frac{1}{2}(L_{X^v}\widetilde{g})(Y^c, Z^c) = (S+\lambda)g(Y, Z).$$

Using the identities (3.1) and (3.5), we deduce that $\frac{1}{2}(L_X g)(Y, Z) = (S+\lambda)g(Y, Z)$ for all left invariant vector fields Y, Z on G. Hence, (G, g, X) is a Yamabe soliton. □

4. Einstein-like and soliton structures on bi-invariant Lie groups

Definition 4.1 *[16] A Riemannian manifold (M,g) is said to be Einstein-like of $\mathcal{A}$-type (resp. of $\mathcal{B}$-type) if its Ricci tensor is cyclic-parallel; that is:*

$$(\nabla_X Ric)(Y,Z)+(\nabla_Y Ric)(Z,X)+(\nabla_Z Ric)(X,Y)=0, \quad \forall X,Y,Z\in\chi(M), \tag{4.1}$$

(resp. its Ricci tensor is a Codazzi tensor; that is:

$$(\nabla_X Ric)(Y,Z)=(\nabla_Y Ric)(X,Z), \quad \forall X,Y,Z\in\chi(M).) \tag{4.2}$$

Remark 4.2 *If the Ricci tensor of a Riemannian manifold (M,g) satisfies (4.1) and (4.2), then we get $(\nabla_X Ric)(Y,Z)=0$, for any $X,Y,Z\in\chi(M)$. In this case, (M,g) is a Ricci-parallel manifold. Also, (4.1) is equivalent to $(\nabla_X Ric)(X,X)=0$, for any $X\in\chi(M)$.*

Theorem 4.3 *Every bi-invariant Lie group (G,g) is an Einstein-like Lie group of $\mathcal{A}$-type.*

Proof We have

$$(\nabla_X Ric)(X,X)=X(Ric(X,X))-Ric(\nabla_X X,X)-Ric(Y,\nabla_X X),$$

for any left invariant vector fields X on G. Since the Riemannian curvature is left invariant, the Ricci tensor is constant. Also, we have $\nabla_X X=\frac{1}{2}[X,X]=0$. From the above equation we deduce $(\nabla_X Ric)(X,X)=0$ for any left invariant vector fields X on G. Therefore, from Definition 4.1 and Remark 4.2, we conclude that (G,g) is Einstein-like of $\mathcal{A}$-type. □

Theorem 4.4 *Let (G,g) be an Einstein-like bi-invariant Lie group of $\mathcal{B}$-type. Then $(TG,\widetilde{g})$ is a Ricci-parallel manifold.*

Proof Let $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ be left invariant vector fields on TG. Then there exist left invariant vector fields X_1, X_2, Y_1, Y_2, Z_1, and Z_2 on G such that $\widetilde{X}=X_1^c+X_2^v$, $\widetilde{Y}=Y_1^c+Y_2^v$, $\widetilde{Z}=Z_1^c+Z_2^v$. Therefore, by calculating the covariant derivative of the Ricci tensor, we have the following:

$$(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Ric})(\widetilde{Y},\widetilde{Z})=\widetilde{X}(\widetilde{Ric}(\widetilde{Y},\widetilde{Z}))-\widetilde{Ric}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},\widetilde{Z})-\widetilde{Ric}(\widetilde{Y},\widetilde{\nabla}_{\widetilde{X}}\widetilde{Z}). \tag{4.3}$$

Obviously, the first term of the right side of the above equation is zero. Now we study the second term. With attention to Corollary 2.3, we have:

$$\begin{aligned}
&\widetilde{Ric}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},\widetilde{Z})=\widetilde{Ric}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},Z_1^c)+\widetilde{Ric}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},Z_2^v)\\
&=\widetilde{Ric}(\widetilde{\nabla}_{X_1^c}Y_1^c,Z_1^c)+\widetilde{Ric}(\widetilde{\nabla}_{X_1^c}Y_2^v,Z_1^c)+\widetilde{Ric}(\widetilde{\nabla}_{X_2^v}Y_2^v,Z_1^c)+\widetilde{Ric}(\widetilde{\nabla}_{X_2^v}Y_1^c,Z_1^c)\\
&+\widetilde{Ric}(\widetilde{\nabla}_{X_1^c}Y_1^c,Z_2^v)+\widetilde{Ric}(\widetilde{\nabla}_{X_1^c}Y_2^v,Z_2^v)+\widetilde{Ric}(\widetilde{\nabla}_{X_2^v}Y_2^v,Z_2^v)+\widetilde{Ric}(\widetilde{\nabla}_{X_2^v}Y_1^c,Z_2^v)\\
&=\frac{1}{2}\widetilde{Ric}([X_1,Y_1]^c,Z_1^c)+\widetilde{Ric}([X_1,Y_2]^v,Z_1^c)+\frac{1}{2}\widetilde{Ric}(([X_1,Y_1]^c,Z_2^v)+\widetilde{Ric}([X_1,Y_2]^v,Z_2^v)\\
&=\frac{1}{2}Ric([X_1,Y_1],Z_1)=Ric(\nabla_{X_1}Y_1,Z_1).
\end{aligned}$$

In a similar way we have:

$$\widetilde{Ric}(\widetilde{Y}, \widetilde{\nabla}_{\widetilde{X}} \widetilde{Z}) = Ric(Y_1, \nabla_{X_1} Z_1). \tag{4.4}$$

Setting these equations in (4.3) we get:

$$(\widetilde{\nabla}_{\widetilde{X}} \widetilde{Ric})(\widetilde{Y}, \widetilde{Z}) = (\nabla_{X_1} Ric)(Y_1, Z_1). \tag{4.5}$$

Since (G, g) has a Codazzi–Ricci tensor, we have

$$(\nabla_{X_1} Ric)(Y_1, Z_1) = (\nabla_{Y_1} Ric)(X_1, Z_1). \tag{4.6}$$

Using (4.4), (4.5), and (4.6) we conclude that $\widetilde{Ric}$ is a Codazzi tensor. Using (4.4) and (4.6), Theorem 4.3 implies that $\widetilde{Ric}$ is cyclic-parallel. Therefore, $(TG, \widetilde{g})$ is a Ricci-parallel manifold. □

Definition 4.5 *Let (M, g) be a Riemannian manifold. We say that (M, g) is a gradient Ricci soliton if there is smooth function f on M such that*

$$Ric(Y, Z) + (\mathrm{Hessf})(Y, Z) = \lambda g(Y, Z), \quad \forall Y, Z \in \chi(M), \tag{4.7}$$

where λ is a real number and Hessf *is the Hessian of f given by*

$$\mathrm{Hessf}(X, Y) = \nabla df(X, Y) = X(Y(f)) - (\nabla_X Y)(f). \tag{4.8}$$

Also, we say that (M, g, X) is a Ricci almost soliton if the following identity holds:

$$Ric(Y, Z) + \frac{1}{2} L_X g(Y, Z) = \lambda g(Y, Z), \quad \forall Y, Z \in \chi(M), \tag{4.9}$$

where λ is a smooth function on M and X is a vector field on M. When λ is a constant function, then (M, g) is called a Ricci soliton.

Theorem 4.6 *Let (G, g) be a left invariant Lie group. Considering $(G, \bar{g} = e^{2u} g)$ for some smooth function u, we have:*

(i) If (G, g) is a gradient Ricci soliton and grad$f$ *is a left invariant vector field, then $(G, \bar{g})$ is gradient Ricci soliton if and only if*

$$\begin{aligned}(\bar{\lambda} e^{2u} - \lambda + \Delta u + (n-2) \| \nabla u \|^2) g(Y, Z) &= -(Y(u)Z + Z(u)Y - g(Y, Z)\mathrm{grad}u)(f) \\ &\quad + (n-2)[Y(u)Z(u) - \nabla^2 u(Y, Z)]. \end{aligned} \tag{4.10}$$

(ii) If (G, g, X) is a Ricci almost soliton, then $(G, \bar{g}, X)$ is Ricci almost soliton if and only if

$$\begin{aligned}\frac{1}{2}(e^{2u} - 1)(L_X g)(Y, Z) &= \{\bar{\lambda} e^{2u} + \Delta u + (n-2) \| \nabla u \|^2 - X(u) e^{2u} - \lambda\} g(Y, Z) \\ &\quad + (2-n)[Y(u)Z(u) - \nabla^2 u(Y, Z)]. \end{aligned} \tag{4.11}$$

Proof (i) At first, considering $\bar{g} = e^{2u}g$, we get

$$\overline{\nabla}_Y Z = \nabla_Y Z + Y(u)Z + Z(u)Y - g(Y,Z)\text{grad}u. \tag{4.12}$$

If we denote the Ricci tensors of g and $\bar{g}$ by Ric and $\overline{Ric}$, respectively, then using the above equation and the definition of Ricci tensor we get the following:

$$\overline{Ric}(Y,Z) = Ric(Y,Z) + (n-2)[Y(u)Z(u) - \nabla^2 u(Y,Z)] - [\Delta u + (n-2)\| \nabla u \|^2]g(Y,Z), \tag{4.13}$$

where Δ is the Laplacian of (G,g). Also, we have

$$\overline{\text{Hess}}f(Y,Z) = Y\bar{g}(\text{grad}f, Z) - \bar{g}(\text{grad}f, \overline{\nabla}_Y Z).$$

Since $\text{grad}f$ and $\bar{g}$ are left invariant, then we have that $\bar{g}(\text{grad}f, Z)$ is constant and so $Y\bar{g}(\text{grad}f, Z) = 0$. Thus, the above equation reduces to the following:

$$\overline{\text{Hess}}f(Y,Z) = -\bar{g}(\text{grad}f, \overline{\nabla}_Y Z) = -(\overline{\nabla}_Y Z)(f).$$

Setting (4.12) in the above equation we obtain

$$\overline{\text{Hess}}f(Y,Z) = \text{Hess}f(Y,Z) - (Y(u)Z + Z(u)Y - g(Y,Z)\text{grad}u)(f). \tag{4.14}$$

Since (G,g) is a gradiant Ricci soliton (with real number λ), using (4.7), (4.13), and (4.14) we deduce that $(G,\bar{g})$ is a gradient Ricci soliton (with real number $\bar{\lambda}$) if and only if (4.10) holds.

(ii) Considering $\bar{g} = e^{2u}g$, we get

$$(L_X\bar{g})(Y,Z) = 2X(u)e^{2u}g(Y,Z) + e^{2u}(L_X g)(Y,Z). \tag{4.15}$$

(4.13) and (4.15) give us

$$\begin{aligned}\overline{Ric}(Y,Z) + (L_X\bar{g})(Y,Z) = {} & Ric(Y,Z) + (n-2)[Y(u)Z(u) - \nabla^2 u(Y,Z)] - [\Delta u + (n-2)\| \nabla u \|^2]g(Y,Z)\\ & + X(u)e^{2u}g(Y,Z) + \frac{1}{2}e^{2u}(L_X g)(Y,Z).\end{aligned} \tag{4.16}$$

If (G,g,X) is a Ricci almost soliton (with a smooth function λ), then from (4.9) and (4.16) we deduce that $(G,\bar{g},X)$ is a Ricci almost soliton (with a smooth function $\bar{\lambda}$) if and only (4.11) holds. □

Theorem 4.7 *Let (G,g) be a Lie group with a bi-invariant metric such that its Ricci tensor is cyclic-parallel. If we consider $(G,\bar{g} = e^{2u}g)$ for some smooth function u, then the Ricci tensor of $(G,\bar{g})$ is cyclic-parallel if and only if*

$$\begin{aligned}&(n-2)X(X(u)X(u) - \nabla^2 u(X,Y)) - X(\Delta u + (n-2) \parallel \nabla u \|^2)g(X,X) - 4X(u)\{Ric(X,X)\\ &+ (n-2)(X(u)X(u) - \nabla^2 u(X,X)) - (\Delta u + (n-2) \parallel \nabla u \|^2)g(X,X)\} + 2g(X,X)\{Ric(\text{grad}u, X)\\ &+ (n-2)((\text{grad}u)(u)X(u) - \nabla^2 u(\text{grad}u, X)) - (\Delta u + (n-2) \parallel \nabla u \|^2)g(\text{grad}u, X)\} = 0.\end{aligned}$$

Proof Using

$$\overline{\nabla}_X X = 2X(u)X - g(X,X)\mathrm{gradu},$$

we get

$$\overline{Ric}(\overline{\nabla}_X X, X) = 2X(u)\overline{Ric}(X,X) - g(X,X)\overline{Ric}(\mathrm{grad}u, X), \tag{4.17}$$

where X is a left invariant vector field on G. Since $Ric(X,X)$ is constant, then considering the above equation and using the formula of $(\overline{\nabla}_X \overline{Ric})(X,X)$ we conclude the assertion. □

Theorem 4.8 *Let (G,g) be a bi-invariant Lie group, X be a left invariant vector field on G, and X^c be its complete lift of X on TG. If $(TG,\widetilde{g},X^c)$ is a Ricci almost soliton (Ricci soliton), then X is a conformal vector field (homothetic vector field).*

Proof Let Y, Z be left invariant vector fields on G. We consider $\widetilde{Y} = Y^v$ and $\widetilde{Z} = Z^v$. By the definition of a Ricci almost soliton there exists a smooth function λ such that:

$$2\lambda\widetilde{g}(\widetilde{Y},\widetilde{Z}) = 2\widetilde{Ric}(\widetilde{Y},\widetilde{Z}) + L_{X^c}\widetilde{g}(\widetilde{Y},\widetilde{Z}) = L_{X^c}\widetilde{g}(Y^v,Z^v) = L_X g(Y,Z),$$

which gives us $L_X g(Y,Z) = 2\lambda g(Y,Z)$. Thus, λ is constant on each fiber of TG, and X is then a conformal vector field of G. When $(TG,\widetilde{g},X^c)$ is a Ricci soliton Lie group, then λ is a real number and so X is a homothetic vector field. □

Definition 4.9 *Let (M,g) be a Riemannian manifold and X be a vector field on M. We say that (M,g,X) is a ρ-Einstein soliton if the following identity holds:*

$$Ric(Y,Z) + \frac{1}{2}L_X g(Y,Z) = (\rho S + \lambda)g(Y,Z), \tag{4.18}$$

for all vector fields Y, Z tangent to M, where ρ , λ are real numbers and ρ is a nonzero constant.

Theorem 4.10 *Let (G,g) be a bi-invariant Lie group and X be a left invariant vector field on G. Then the following assertions hold:*

(1) If $(TG,\widetilde{g},X^v)$ is a Ricci soliton (ρ-Einstein soliton), then G is an Einstein manifold.

(2) If $(TG,\widetilde{g},,X^c + X^v)$ is a Ricci soliton (ρ-Einstein soliton), then G is an Einstein manifold.

Proof (1). Let $\widetilde{Y}$, $\widetilde{Z}$ be left invariant vector fields on TG. By the definition of a Ricci soliton we have

$$\widetilde{Ric}(\widetilde{Y},\widetilde{Z}) + \frac{1}{2}L_{X^v}\widetilde{g}(\widetilde{Y},\widetilde{Z}) = \lambda\widetilde{g}(\widetilde{Y},\widetilde{Z}).$$

Replacing $\widetilde{Y} = Y^c + Y^v$ and $\widetilde{Z} = Z^c + Z^v$ in the above equation, we obtain the following:

$$\widetilde{Ric}(Y^c + Y^v, Z^c + Z^v) + \frac{1}{2}(L_{X^v}\widetilde{g})(Y^c + Y^v, Z^c + Z^v) = \lambda\widetilde{g}(\widetilde{Y},\widetilde{Z}),$$

which gives

$$Ric(Y,Z) + \frac{1}{2}\{-g([X,Y],Z) - g(Y,[X,Z])\} = 2\lambda g(Y,Z),$$

and consequently

$$Ric(Y,Z) + \frac{1}{2}(L_X g)(Y,Z) = 2\lambda g(Y,Z).$$

Therefore, (G,X,g) is a Ricci soliton. Now, in the Ricci soliton equation of TG, if we consider $\widetilde{Y} = Y^c$ and $\widetilde{Z} = Z^v$, then we conclude that $g([X,Y],Z) = 0$, so X is Killing. Since (G,X,g) is a Ricci soliton and X is Killing, we deduce that G is an Einstein manifold.

(2). In the Ricci soliton equation of TG, if we consider $\widetilde{Y} = Y^c$ and $\widetilde{Z} = Z^c$, then we conclude that G is a Ricci soliton. Moreover, in the Ricci soliton equation of TG, if we consider $\widetilde{Y} = Y^v$ and $\widetilde{Z} = Z^c$, we deduce that X is Killing. Since (G,X,g) is a Ricci soliton and X is Killing, G is an Einstein manifold. □

Theorem 4.11 *Let (G,g) be a bi-invariant Lie group and X be a left invariant vector field on G. If $(TG, X^c, \widetilde{g})$ is a Ricci almost soliton (Ricci soliton), then X is a conformal (homothetic) vector field.*

Proof In the Ricci soliton equation of TG, if we consider $\widetilde{Y} = Y^v$ and $\widetilde{Z} = Z^v$, then the proof will be complete. □

Theorem 4.12 *Let G be a bi-invariant Lie group and X be a left invariant vector field on G. If $(TG, X^c, \widetilde{g})$ is a ρ-Einstein soliton, then G is a ρ-Einstein soliton and X is a homothetic vector field.*

Proof In the ρ-Einstein soliton equation of TG, if we consider $\widetilde{Y} = Y^c$ and $\widetilde{Z} = Z^c$, then G is a ρ-Einstein soliton. Moreover, in the ρ-Einstein soliton equation of TG, if we consider $\widetilde{Y} = Y^v$ and $\widetilde{Z} = Z^v$, then X is a homothetic vector field. □

5. Tensors related to Riemannian curvature on TG

In Riemannian geometry there are some important tensors such as Schouten, Cotton, Weyl, and Bach tensors (see [3], [14], [4], and [7] for more details). In the general theory of relativity, the curvature tensor describing the gravitational field consists of two parts: the matter part and the free gravitational part. Also, the projective curvature tensor is the measure of the failure of a Riemannian manifold to be of constant curvature. Furthermore, in differential geometry, the Weyl curvature tensor and the Cotton and Bach tensors are essential tools for the study of the curvature, conformal flatness, and conformal relativity of a space time or a pseudo-Riemannian manifold. The Cotton tensor on a pseudo-Riemannian manifold restricts the relation between the Ricci tensor and the energy-momentum tensor of matter in the Einstein equations and plays an important role in the Hamilton formalism of general relativity.

The aim of this section is the study of Schouten, Cotton, Weyl, and Bach tensors of the tangent Lie group TG whenever G is a bi-invariant Lie group. Moreover, in this section we show that if (G,g) is a bi-invariant Ricci flat Lie group and $(TG,\widetilde{g})$ has a harmonic Weyl tensor, then $(TG,\widetilde{g})$ and (G,g) are Bach flat Lie groups. At the end of this section we compute the components of the Bach tensor of TG.

5.1. Schouten and Cotton tensors of TG

Let (M, g) be an n-dimensional Riemannian manifold, $(n > 2)$ and let the Ricci tensor and the scalar curvature be denoted by Ric and S, respectively. We define the Schouten tensor P and Cotton tensor C as follows:

$$P = \frac{1}{n-2}(Ric - \frac{1}{2(n-1)}Sg), \tag{5.1}$$

$$C(X, Y, Z) = \nabla_Z P(X, Y) - \nabla_Y P(X, Z). \tag{5.2}$$

Lemma 5.1 *Let (G, g) be a bi-invariant Lie group. Then the Schouten tensor $\widetilde{P}$ of the tangent Lie group $(TG, \widetilde{g})$ is as follows:*

$$\widetilde{P}(X^c, Y^c) = \frac{1}{2n-2}(Ric(X, Y) - \frac{S}{4n-2}g(X, Y)),$$

$$\widetilde{P}(X^c, Y^v) = \widetilde{P}(X^v, Y^c) = 0,$$

$$\widetilde{P}(X^v, Y^v) = -\frac{S}{(2n-2)(4n-2))}g(X, Y).$$

Proof Let S and $\widetilde{S}$ be the scalar curvatures of G and TG, respectively. We know that in a bi-invariant Lie group, $S = \widetilde{S}$. Therefore, we have:

$$\begin{aligned}\widetilde{P}(X^c, Y^c) &= \frac{1}{2n-2}(\widetilde{Ric}(X^c, Y^c) - \frac{S}{4n-2}\widetilde{g}(X^c, Y^c)) \\ &= \frac{1}{2n-2}(Ric(X, Y) - \frac{S}{4n-2}g(X, Y)).\end{aligned}$$

Similarly, we can conclude the others relations. □

Lemma 5.2 *Let (G, g) be a bi-invariant n-dimensional Lie group $(n > 2)$, and let $C(X, Y, Z)$ and $\widetilde{C}(\widetilde{X}, \widetilde{Y}, \widetilde{Z})$ be Cotton tensors of (G, g) and $(TG, \widetilde{g})$, respectively. Then*

$$\begin{aligned}\widetilde{C}(\widetilde{X}, \widetilde{Y}, \widetilde{Z}) = {}& \frac{1}{4n-4}\frac{S}{4n-2}\{g(X_2, [Z_2, Y_1]) + g([Z_2, X_1], Y_2) - g(X_2, [Y_2, Z_1]) - g([Y_2, X_1], Z_2)\} \\ &+ \frac{n-2}{2n-2}C(X_1, Y_1, Z_1),\end{aligned}$$

where $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ are left invariant vector fields on TG, and $\widetilde{X} = X_1^c + X_2^v$, $\widetilde{Y} = Y_1^c + Y_2^v$ and $\widetilde{Z} = Z_1^c + Z_2^v$ for left invariant vector fields X_i, Y_i, Z_i on G, $i = 1, \cdots n$.

Proof The definitions of Schouten and Cotton tensors imply

$$\begin{aligned}\nabla_Z P(X, Y) &= \frac{1}{n-2}((\nabla_Z Ric)(X, Y) - \frac{S}{2n-2}(\nabla_Z g)(X, Y)) \\ &= \frac{1}{n-2}(\nabla_Z Ric)(X, Y),\end{aligned}$$

$$\begin{aligned}C(X,Y,Z) &= \nabla_Z P(X,Y) - \nabla_Y P(X,Z)\\ &= \frac{1}{n-2}((\nabla_Z Ric)(X,Y) - (\nabla_Y Ric)(X,Z)),\end{aligned}$$

$$(\nabla_Z Ric)(X,Y) - (\nabla_Y Ric)(X,Z) = (n-2)C(X,Y,Z).$$

Let $\widetilde{X}$, $\widetilde{Y}$, $\widetilde{Z}$ be left invariant vector fields on TG. Therefore, there exist the left invariant vector fields X_1, X_2, Y_1, Y_2, Z_1, Z_2 on G such that $\widetilde{X} = X_1^c + X_2^v$, $\widetilde{Y} = Y_1^c + Y_2^v$, $\widetilde{Z} = Z_1^c + Z_2^v$. Using Corollary 2.3 and Lemma 5.1, the components of the Cotton tensor satisfy the following:

$$\begin{aligned}\widetilde{\nabla}_{\widetilde{Z}}\widetilde{P}(X_1^c, Y_1^c) &= \widetilde{\nabla}_{Z_1^c+Z_2^v}\widetilde{P}(X_1^c, Y_1^c) = \widetilde{\nabla}_{Z_1^c}\widetilde{P}(X_1^c, Y_1^c) + \widetilde{\nabla}_{Z_2^v}\widetilde{P}(X_1^c, Y_1^c)\\ &= (Z_1^c(\widetilde{P}(X_1^c, Y_1^c)) - \widetilde{P}(\widetilde{\nabla}_{Z_1^c}X_1^c, Y_1^c) - \widetilde{P}(X_1^c, \widetilde{\nabla}_{Z_1^c}Y_1^c))\\ &+ (Z_2^v(\widetilde{P}(X_1^c, Y_1^c)) - \widetilde{P}(\widetilde{\nabla}_{Z_2^v}X_1^c, Y_1^c) - \widetilde{P}(X_1^c, \widetilde{\nabla}_{Z_2^v}Y_1^c))\\ &= -\widetilde{P}((\nabla_{Z_1}X_1)^c, Y_1^c) - \widetilde{P}(X_1^c, (\nabla_{Z_1}Y_1)^c)\\ &= -\frac{1}{2n-2}(Ric(\nabla_{Z_1}X_1, Y_1) - \frac{S}{4n-2}g(\nabla_{Z_1}X_1, Y_1))\\ &- \frac{1}{2n-2}(Ric(X_1, \nabla_{Z_1}Y_1) - \frac{S}{4n-2}g(X_1, \nabla_{Z_1}Y_1))\\ &= \frac{1}{2n-2}(-Ric(\nabla_{Z_1}X_1, Y_1) - Ric(X_1, \nabla_{Z_1}Y_1))\\ &+ \frac{S}{(2n-2)(4n-2)}(g(\nabla_{Z_1}X_1, Y_1) + g(X_1, \nabla_{Z_1}Y_1))\\ &= \frac{1}{2n-2}(\nabla_{Z_1}Ric)(X_1, Y_1).\end{aligned}$$

Again, Corollary 2.3 and Lemma 5.1 imply that

$$\begin{aligned}\widetilde{\nabla}_{\widetilde{Z}}\widetilde{P}(X_2^v, Y_2^v) &= \widetilde{\nabla}_{Z_1^c+Z_2^v}\widetilde{P}(X_2^v, Y_2^v) = \widetilde{\nabla}_{Z_1^c}\widetilde{P}(X_2^v, Y_2^v) + \widetilde{\nabla}_{Z_2^v}\widetilde{P}(X_2^v, Y_2^v)\\ &= (Z_1^c(\widetilde{P}(X_2^v, Y_2^v)) - \widetilde{P}(\widetilde{\nabla}_{Z_1^c}X_2^v, Y_2^v) - \widetilde{P}(X_2^v, \widetilde{\nabla}_{Z_1^c}Y_2^v))\\ &+ (Z_2^v(\widetilde{P}(X_2^v, Y_2^v)) - \widetilde{P}(\widetilde{\nabla}_{Z_2^v}X_2^v, Y_2^v) - \widetilde{P}(X_2^v, \widetilde{\nabla}_{Z_2^v}Y_2^v))\\ &= -\widetilde{P}(([Z_1, X_2]^v, Y_2^v) - \widetilde{P}(X_2^v, ([Z_1, Y_2]^v) - \widetilde{P}(0, Y_2^v) - \widetilde{P}(X_2^v, 0)\\ &= \frac{S}{(4n-2)(2n-2)}(g([Z_1, X_2], Y_2) + g(X_2, [Z_1, Y_2])) = 0.\end{aligned}$$

In a similar way, we get:

$$\begin{aligned}\widetilde{\nabla}_{\widetilde{Z}}\widetilde{P}(X_2^v, Y_1^c) &= \frac{S}{2(2n-2)(4n-2)}g(X_2, [Z_2, Y_1]),\\ \widetilde{\nabla}_{\widetilde{Z}}\widetilde{P}(X_1^c, Y_2^v) &= \frac{S}{2(2n-2)(4n-2)}g([Z_2, X_1], Y_2).\end{aligned}$$

Thus, we can obtain the following:

$$\begin{aligned}
\widetilde{\nabla}_{\widetilde{Z}}\widetilde{P}(\widetilde{X},\widetilde{Y}) &= \widetilde{\nabla}_{\widetilde{Z}}\widetilde{P}(X_1^c+X_2^v, Y_1^c+Y_2^v)\\
&= \widetilde{\nabla}_{\widetilde{Z}}\widetilde{P}(X_1^c,Y_1^c)+\widetilde{\nabla}_{\widetilde{Z}}\widetilde{P}(X_2^v,Y_2^v)+\widetilde{\nabla}_{\widetilde{Z}}\widetilde{P}(X_1^c,Y_2^v)+\widetilde{\nabla}_{\widetilde{Z}}\widetilde{P}(X_2^v,Y_1^c)\\
&= \frac{1}{2n-2}(\nabla_{Z_1}Ric)(X_1,Y_1)+\frac{S}{2(2n-2)(4n-2)}\{g(X_2,[Z_2,Y_1])+g([Z_2,X_1],Y_2)\}.
\end{aligned}$$

Therefore, we have:

$$\begin{aligned}
\widetilde{C}(\widetilde{X},\widetilde{Y},\widetilde{Z}) &= \widetilde{\nabla}_{\widetilde{Z}}\widetilde{P}(\widetilde{X},\widetilde{Y})-\widetilde{\nabla}_{\widetilde{Y}}\widetilde{P}(\widetilde{X},\widetilde{Z})\\
&= \frac{1}{2n-2}((\nabla_{Z_1}Ric)(X_1,Y_1)-(\nabla_{Y_1}Ric)(X_1,Z_1))\\
&+\frac{S}{(4n-4)(4n-2)}\{g(X_2,[Z_2,Y_1])+g([Z_2,X_1],Y_2)-g(X_2,[Y_2,Z_1])-g([Y_2,X_1],Z_2)\}\\
&= \frac{n-2}{2n-2}C(X_1,Y_1,Z_1)\\
&+\frac{S}{(4n-4)(4n-2)}\{g(X_2,[Z_2,Y_1])+g([Z_2,X_1],Y_2)-g(X_2,[Y_2,Z_1])-g([Y_2,X_1],Z_2)\}.
\end{aligned}$$

□

Definition 5.3 *A Riemannian manifold (M^n,g) has harmonic Weyl tensor if the Cotton tensor vanishes.*

Theorem 5.4 *Let (G,g) be a bi-invariant n-dimensional Lie group $(n>2)$. If $(TG,\widetilde{g})$ has a harmonic Weyl tensor then (G,g) has a harmonic Weyl tensor.*

Proof Let X,Y,Z be left invariant vector fields on G. Lemma 5.2 gives us

$$\widetilde{C}(X^c,Y^c,Z^c)=\frac{n-2}{2n-2}C(X,Y,Z)+0. \tag{5.3}$$

By hypothesis, (5.3) implies that $C(X,Y,Z)=0$. Hence, G has a harmonic Weyl tensor. □

5.2. Weyl tensor of TG

The components of (0,4)-versions of Riemann tensor and Weyl tensor W are related by the following formula:

$$\begin{aligned}
W_{abcd} &= R_{abcd}+\frac{S}{(n-1)(n-2)}(g_{ac}.g_{bd}-g_{ad}.g_{bc})\\
&-\frac{1}{n-2}(Ric_{ac}.g_{bd}-Ric_{ad}.g_{bc}+Ric_{bd}.g_{ac}-Ric_{bc}.g_{ad}),\qquad a,b,c,d=1,\cdots n.
\end{aligned}$$

We recall that an n-dimensional Riemannian manifold $(n>3)$ is conformally flat if its Weyl tensor is zero.

Theorem 5.5 *Let (G,g) be an n-dimensional bi-invariant Lie group $(n>3)$. $(TG,\widetilde{g})$ is conformally flat if and only if (G,g) is flat.*

Proof Uing Corollary 2.3, we get

$$\begin{aligned}\widetilde{W}(X^c,Y^c,Z^c,T^c) &= (R(X,Y)Z,T) + \frac{S}{(2n-1)(2n-2)}(g(X,Z)g(Y,T) - g(X,T)g(Y,Z))\\ &- \frac{1}{2n-2}(Ric(X,Z)g(Y,T) - Ric(X,T)g(Y,Z)\\ &+ Ric(Y,T)g(X,Z) - Ric(Y,Z)g(X,T)),\end{aligned} \tag{5.4}$$

$$\widetilde{W}(X^c,Y^v,Z^v,T^c) = -\frac{S}{(2n-1)(2n-2)}g(X,T)g(Y,Z) + \frac{1}{2n-2}Ric(X,T)g(Y,Z), \tag{5.5}$$

$$\widetilde{W}(X^c,Y^c,Z^v,T^c) = \widetilde{W}(X^v,Y^v,Z^v,T^c) = 0, \tag{5.6}$$

$$\widetilde{W}(X^v,Y^v,Z^c,T^c) = \widetilde{W}(X^v,Y^c,Z^c,T^c) = 0, \tag{5.7}$$

$$\begin{aligned}\widetilde{W}(X^v,Y^c,Z^c,T^v) &= -\frac{S}{(2n-1)(2n-2)}g(X,T)g(Y,Z)\\ &+ \frac{1}{2n-2}Ric(Y,Z)g(X,T),\end{aligned} \tag{5.8}$$

$$\widetilde{W}(X^v,Y^v,Z^v,T^v) = \frac{S}{(2n-1)(2n-2)}(g(X,Z)g(Y,T) - g(X,T)g(Y,T)), \tag{5.9}$$

$$\widetilde{W}(X^v,Y^v,Z^c,T^v) = \widetilde{W}(X^c,Y^c,Z^c,T^v) = \widetilde{W}(X^c,Y^v,Z^v,T^v) = 0, \tag{5.10}$$

$$\widetilde{W}(X^c,Y^c,Z^v,T^v) = 0. \tag{5.11}$$

Obviously, if $R=0$, then $\widetilde{W}=0$. Conversely, if $\widetilde{W}=0$, then from (5.9) we conclude that $S=0$. Using (5.8) we deduce that $Ric=0$. Therefore, applying (5.4), we get $R=0$. □

Definition 5.6 *Let (M,g) be an n-dimensional Riemannian manifold $(n>1)$, and let the Ricci tensor and Riemann curvature tensor be denoted by Ric and R, respectively. We define the projective curvature tensor Pr as follows:*

$$Pr(X,Y,Z) = R(X,Y)Z - \frac{1}{n-1}(Ric(Y,Z)X - Ric(X,Z)Y). \tag{5.12}$$

We say that (M,g) is projectively flat if the projective curvature tensor Pr vanishes identically.

Theorem 5.7 *Let (G,g) be an n-dimensional bi-invariant Lie group $(n>1)$. $(TG,\widetilde{g})$ is projectively flat if and only if (G,g) is flat.*

Proof Let $\widetilde{Pr}$ and Pr be the projective curvature tensors of the manifolds $(TG, \widetilde{g})$ and (G, g), respectively. Now direct calculation shows that all components of projective curvature tensor $\widetilde{Pr}$ are zero except:

$$\widetilde{g}(\widetilde{Pr}(X^c, Y^c, Z^c), W^c) = g(R(X,Y)Z, W) - \frac{1}{2n-1}(Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W)),$$

$$\widetilde{g}(\widetilde{Pr}(X^v, Y^c, Z^c), W^v) = -\frac{1}{2n-1}(Ric(Y,Z)g(X,W)).$$

Now, from the hypothesis and the above equations, we derive that $\widetilde{Pr} = 0$ if and only if $R = 0$. □

Definition 5.8 *Let (M, g) be an n-dimensional Riemannain manifold $(n > 1)$. We define the concircular curvature tensor Con as follows:*

$$Con(X, Y, Z) = R(X,Y)Z - \frac{S}{n(n-1)}(g(Y,Z)X - g(X,Z)Y). \tag{5.13}$$

We say that (M, g) is concircularly flat if the concircular curvature tensor Con vanishes identically.

Theorem 5.9 *Let (G, g) be an n-dimensional bi-invariant Lie group $(n > 1)$. $(TG, \widetilde{g})$ is concircularly flat if and only if (G, g) is flat.*

Proof Let $\widetilde{Con}$ and Con be the noncircular curvature tensors of Lie groups $(TG, \widetilde{g})$ and (G, g), respectively. Now direct computations imply that all components of concircular curvature tensor $\widetilde{Con}$ are zero except the following:

$$\widetilde{g}(\widetilde{Con}(X^c, Y^c, Z^c), W^c) = g(R(X,Y)Z, W) - \frac{S}{2n(2n-1)}(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)),$$

$$\widetilde{g}(\widetilde{Con}(X^c, Y^v, Z^v), W^c) = -\frac{S}{2n(2n-1)}(g(Y,Z)g(X,W)),$$

$$\widetilde{g}(\widetilde{Con}(X^v, Y^v, Z^v), W^v) = -\frac{S}{2n(2n-1)}(g(Y,Z)g(X,W) - g(X,Z)g(Y,W)),$$

$$\widetilde{g}(\widetilde{Con}(X^v, Y^c, Z^c), W^v) = -\frac{S}{2n(2n-1)}g(Y,Z)g(X,W).$$

Now, by the above equations and the hypothesis (similar to Theorem (5.5)), we see that the concircular flatness of TG is equivalent to the flatness of G. □

Definition 5.10 *Let (M, g) be an n-dimensional Riemannian manifold $(n > 1)$. We define the m-projective curvature tensor $M(X, Y, Z)$ as follows:*

$$g(M(X,Y,Z), W) = g(R(X,Y)Z, W) + \frac{1}{2(n-1)}(Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W) \tag{5.14}$$

$$+ g(Y,Z)Ric(X,W) - g(X,Z)Ric(Y,W)). \tag{5.15}$$

Theorem 5.11 *Let (G,g) be an n-dimensional bi-invariant Lie group $(n>1)$. $(TG,\widetilde{g})$ is m-projectively flat if and only if G is flat.*

Proof Let $\widetilde{M}$ and M be the m-projective curvature tensors of Lie groups $(TG,\widetilde{g})$ and (G,g), respectively. Now direct calculations show that all components of the m-projective curvature tensor $\widetilde{M}$ are zero except:

$$\begin{aligned}\widetilde{g}(\widetilde{M}(X^c,Y^c,Z^c),W^c) = g(R(X,Y)Z,W) - \frac{1}{2(2n-1)}(&Ric(Y,Z)g(X,W) - Ric(X,Z)g(Y,W)\\ &+ g(Y,Z)Ric(X,W) - g(X,Z)Ric(Y,W)),\end{aligned}$$

$$\widetilde{g}(\widetilde{M}(X^c,Y^v,Z^v),W^c) = -\frac{1}{2(2n-1)}g(Y,Z)Ric(X,W),$$

$$\widetilde{g}(\widetilde{M}(X^v,Y^c,Z^c),W^v) = -\frac{1}{2(2n-1)}(Ric(Y,Z)g(X,W) + g(Y,Z)Ric(X,W)).$$

The hypothesis and the above equations imply that $R=0$ if and only if $\widetilde{M}=0$. □

5.3. Bach tensor of TG

In 1921 Bach introduced a tensor to study conformal relativity in the context of conformally Einstein spaces. This tensor is known as the Bach tensor and it is a symmetric $(0,2)$-tensor B on an n-dimensional pseudo-Riemannian manifold (M,g), defined by

$$B(X,Y) = \frac{1}{n-2}\{\sum_{i=1}^{n}(\nabla_{\partial_i}C)(X,Y,\partial_i) + \sum_{i,j=1}^{n} Ric(\partial_i,\partial_j)W(X,\partial_i,Y,\partial_j)\},$$

where $\{\partial_i, i=1,\cdots,n\}$ is a local frame on (M,g), Ric is the Ricci tensor of type $(0,2)$, C is the Cotton tensor, and W denotes the Weyl tensor of type $(0,4)$. The metric g is said to be Bach flat when $B=0$. The aim of this section is investigation of the components of the Bach tensor for a tangent Lie group TG.

Theorem 5.12 *Let (G,g) be a Ricci flat bi-invariant Lie group. If $(TG,\widetilde{g})$ has a harmonic Weyl tensor, then $(TG,\widetilde{g})$ and (G,g) are Bach flat Lie groups.*

Proof In the definition of the Bach tensor $\widetilde{B}$ of TG, the first sum is zero, because TG has a harmonic Weyl tensor. Also, the Ricci flatness of G implies that TG is Ricci flat. Therefore, the second sum of $\widetilde{B}$ is zero, and so TG is Bach flat. Moreover, in the definition of the Bach tensor B of G, the second sum of $\widetilde{B}$ is zero, and by Theorem 5.4 this implies that G has a harmonic Weyl tensor. Therefore, the first sum of B is zero, and so G is Bach flat. □

Theorem 5.13 *Let (G,g) be an n-dimensional bi-invariant Lie group $(n>4)$. The components of the Bach tensor of the tangent Lie group $(TG,\widetilde{g})$ are as follows:*

$$\widetilde{B}(X^v,Y^c) = 0,$$

$$\widetilde{B}(X^v,Y^v)=\frac{1}{2n-2}\sum_{i=1}^{2n}-A\{g([X,X_i],[Y,X_i])+g(X,[[Y,X_i],X_i])\}$$
$$-\sum_{i,j=1}^{2n}Ric(X_i,X_j)\{-\frac{S}{(2n-1)(2n-2)}g(X,Y)g(X_i,X_j)$$
$$+\frac{1}{2n-2}Ric(X_i,X_j)g(X,Y)\},$$

$$\widetilde{B}(X^c,Y^c)=\frac{1}{2n-2}\sum_{i=1}^{2n}(\{-2Ag([X_i,X],[X_i,Y])+Ag([[X_i,Y],X],X_i)\}$$
$$+\{\frac{n-2}{2n-2}X_i^c(C(X,Y,X_i))-\frac{1}{2}\frac{n-2}{2n-2}\{C([X_i,X],Y,X_i)+C(X,[X_i,Y],X_i)\}\})$$
$$+\sum_{i,j=1}^{2n}Ric(X_i,X_j)\{R(X,Y,Z,T)+\frac{S}{(2n-1)(2n-2)}(g(X,Z)g(Y,T)-g(X,T)g(Y,Z))$$
$$-\frac{1}{2n-2}(Ric(X,Z)g(Y,T)-Ric(X,T)g(Y,Z)+Ric(Y,T)g(X,Z)-Ric(Y,Z)g(X,T))\},$$

for all left invariant vector fields X, Y *on* G, *where* $A=\frac{S}{(4n-4)(4n-2)}$

Proof Let X, Y be left invariant vector fields on G. From Lemma 5.2, we deduce that $\widetilde{C}(X^v,Y^v,X_i^v)=0$ and $\widetilde{C}(X^v,Y^c,X_i^v)$ is constant. Therefore, $X_i^v(\widetilde{C}(X^v,Y^c,X_i^v))=0$. Now, by the covariant derivative of the Cotton tensor, we have

$$(\widetilde{\nabla}_{X_i^v}\widetilde{C})(X^v,Y^c,X_i^v)=X_i^v(\widetilde{C}(X^v,Y^c,X_i^v))-\widetilde{C}(\widetilde{\nabla}_{X_i^v}X^v,Y^c,X_i^v)$$
$$-\widetilde{C}(X^v,\widetilde{\nabla}_{X_i^v}Y^c,X_i^v)-\widetilde{C}(X^v,Y^c,\widetilde{\nabla}_{X_i^v}X_i^v). \tag{5.16}$$

In the above equation all terms are zero, and so $(\widetilde{\nabla}_{X_i^v}\widetilde{C})(X^v,Y^c,X_i^v)=0$. Also, by Lemma 5.2, we have $\widetilde{C}(X^v,Y^c,X_i^c)=0$. We deduce, using Corollary 2.3, that all the summands of the right-hand side of the following equation are zero:

$$(\widetilde{\nabla}_{X_i^c}\widetilde{C})(X^v,Y^c,X_i^c)=X_i^c(\widetilde{C}(X^v,Y^c,X_i^c))-\widetilde{C}(\widetilde{\nabla}_{X_i^c}X^v,Y^c,X_i^c)$$
$$-\widetilde{C}(X^v,\widetilde{\nabla}_{X_i^c}Y^c,X_i^c)-\widetilde{C}(X^v,Y^c,\widetilde{\nabla}_{X_i^c}X_i^c), \tag{5.17}$$

and so $(\widetilde{\nabla}_{X_i^c}\widetilde{C})(X^v,Y^c,X_i^c)=0$. Now, by the definition of the Bach tensor for the tangent Lie group TG, we obtain the following:

$$\widetilde{B}(X^v,Y^c)=\frac{1}{2n-2}\{\sum_{i=1}^{2n}\{(\widetilde{\nabla}_{X_i^v}\widetilde{C})(X^v,Y^c,X_i^v)+(\widetilde{\nabla}_{X_i^c}\widetilde{C})(X^v,Y^c,X_i^c)\}$$
$$+\sum_{i,j=1}^{2n}\widetilde{Ric}(X_i^c,X_j^c)\widetilde{W}(X^v,X_i^c,Y^c,X_j^c)\}. \tag{5.18}$$

Using (5.16), (5.17), and (5.7) in the above equation, we get $\widetilde{B}(X^v, Y^c) = 0$.
The two others identities of the theorem can be proved in the same manner using Lemma 5.2 and Corollary 2.3. □

6. Operators related to Riemannian curvatures

Some geometrical properties and applications of the Szabo operator and Jacobi operator are in the study of curvature theory and they play a central role in the study of Osserman manifolds. Furthermore, the spectral properties of these operators have been studied extensively. The study of the eigenvalues of these operators is particularly distinguished in Riemannian geometry. In this section, we study Szabo and Jacobi operators on the tangent bundle of a bi-invariant Lie group.

6.1. Szabo operator of TG

Let (M, g, ∇) be an n-dimensional Riemannian manifold with Levi-Civita connection and let R be the associated curvature operator of ∇. We define the Riemannian Szabo operator $S^\nabla(X)$ with respect to $X \in T_pM$ as follows:

$$\begin{aligned} S^\nabla(X)Y &= (\nabla_X R)(\bar{Y}, \bar{X})\bar{X} \\ &= \nabla_X(R(\bar{Y}, \bar{X})\bar{X}) - R(\nabla_X \bar{Y}, X)X - R(Y, \nabla_X \bar{X})X - R(Y, X)\nabla_X \bar{X}, \end{aligned} \tag{6.1}$$

where $\bar{X}$ and $\bar{Y}$ are any vector fields on G extending X and Y, respectively. We say that M is Szabo flat if $S^\nabla(X) = 0$ for every vector field X on M.

Theorem 6.1 *If (G, g) is a Riemannian bi-invariant Lie group, then (G, g) is Szabo flat.*

Proof Let X, Y be left invariant vector fields on G. Since $\nabla_X X = \frac{1}{2}[X, X] = 0$, then the two last terms of (6.1) are zero. Also, we have

$$\nabla_X(R(Y, X)X) = -\frac{1}{4}\nabla_X[[Y, X], X] = -\frac{1}{8}[[[X, Y], X], X].$$

Similar calculations give us the following:

$$R(\nabla_X Y, X)X = \frac{1}{8}[[[X, Y], X], X].$$

Setting the two above equations in (6.1), we get $S^\nabla = 0$. □

Theorem 6.2 *Let (G, g) be a Riemannian bi-invariant Lie group. Then $(TG, \widetilde{g})$ is Szabo flat.*

Proof Let $\widetilde{X} = X_1^c + X_2^v$ and $\widetilde{Y} = Y_1^c + Y_2^v$ be left invariant vector fields on TG. Using (6.1) we get

$$S^{\widetilde{\nabla}}_{\widetilde{X}}(\widetilde{Y}) = \widetilde{\nabla}_{\widetilde{X}}\widetilde{R}(\widetilde{Y}, \widetilde{X})\widetilde{X} - \widetilde{R}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y}, \widetilde{X})\widetilde{X} - \widetilde{R}(\widetilde{Y}, \widetilde{\nabla}_{\widetilde{X}}\widetilde{X})\widetilde{X} - \widetilde{R}(\widetilde{Y}, \widetilde{X})\widetilde{\nabla}_{\widetilde{X}}\widetilde{X}. \tag{6.2}$$

Using Corollary 2.3 we obtain

$$\widetilde{R}(\widetilde{Y}, \widetilde{\nabla}_{\widetilde{X}}\widetilde{X})\widetilde{X} = \widetilde{R}(Y_1^c, \widetilde{\nabla}_{X_1^c}X_1^c)X_1^c = \frac{1}{2}\widetilde{R}(Y_1^c, [X_1, X_1]^c)X_1^c = 0. \tag{6.3}$$

Similarly, we derive that

$$\widetilde{R}(\widetilde{Y},\widetilde{X})\widetilde{\nabla}_{\widetilde{X}}\widetilde{X}=0. \tag{6.4}$$

Applying Corollary 2.3, we get

$$\widetilde{\nabla}_{\widetilde{X}}(\widetilde{R}(\widetilde{Y},\widetilde{X})\widetilde{X})=\widetilde{\nabla}_{X_1^c}(\widetilde{R}(Y_1,X_1)X_1)^c=-\frac{1}{8}[[[X_1,Y_1],X_1],X_1]^c. \tag{6.5}$$

Again Corollary 2.3 implies

$$\widetilde{R}(\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y},\widetilde{X})\widetilde{X}=\widetilde{R}(\widetilde{\nabla}_{X_1^c}Y_1^c,X_1^c)X_1^c=-\frac{1}{8}[[[X_1,Y_1],X_1],X_1]^c. \tag{6.6}$$

Setting (6.3)–(6.6) in (6.2), we deduce $S^{\widetilde{\nabla}}_{\widetilde{X}}(\widetilde{Y})=0$ for any left invariant vector fields $\widetilde{X}$ and $\widetilde{Y}$ on TG. □

6.2. Jacobi operator of TG

Let (M,g) be a Riemannian manifold, ∇ be the Levi-Civita connection, R be the Riemann curvature tensor, and X be a vector field on M. We define the Jacobi operator with respect to X as follows:

$$J_X(Y)=R(Y,X,X),\ \forall Y\in\chi(M). \tag{6.7}$$

Definition 6.3 *Let (M,g) be a Riemannian manifold with the Levi-Civita connection. We say that M is nilpotent Osserman at $p\in M$, if spec$\{J_X\}=\{0\}$ for all $X\in T_pM$.*

Theorem 6.4 *Let (G,g) be a Riemannian bi-invariant Lie group. If G is nilpotent Osserman at identity, then $\widetilde{J}_{X^v}=0$ and $\widetilde{J}_{X^c}$ is a nilpotent operator.*

Proof Let $\widetilde{X}$, $\widetilde{Y}$ be left invariant vector fields on TG. There exist left invariant vector fields X_1, X_2, Y_1, Y_2 on G such that $\widetilde{X}=X_1^c+X_2^v$ and $\widetilde{Y}=Y_1^c+Y_2^v$. Direct calculation gives us

$$\widetilde{J}_{\widetilde{X}}(\widetilde{Y})=\widetilde{R}(\widetilde{Y},\widetilde{X})\widetilde{X}=\widetilde{R}(Y_1^c+Y_2^v,\widetilde{X})\widetilde{X}=\widetilde{R}(Y_1^c,X_1^c)X_1^c=(R(Y_1,X_1)X_1))^c=(J_{X_1}(Y_1))^c. \tag{6.8}$$

In particular, when $X_1=0$ and $X_2=X$ in (6.8), we obtain $\widetilde{J}_{X^v}(\widetilde{Y})=0$, which proves our first assertion. On the other hand, we obtain by induction

$$\widetilde{J}^k_{\widetilde{X}}(\widetilde{Y})=(J^k_{X_1}(Y_1))^c, \tag{6.9}$$

where $k>0$. In particular, when $X_1=X$ and $X_2=0$ in (6.9), we obtain $\widetilde{J}^k_{X^c}(\widetilde{Y})=(J^k_X(Y_1))^c$ for all integer k, which proves our second assertion. □

Theorem 6.5 *Let (G,g) be a Riemannian bi-invariant Lie group. If G is nilpotent Osserman at identity, then $(TG,\widetilde{g})$ is nilpotent Osserman at identity and equivalently the nilpotency of the Jacobi operator of G can be transfered to the nilpotency of the Jacobi operator of TG.*

Proof Under the same hypotheses and notations as in the proof of Theorem 6.4, Theorem 6.5 follows easily from (6.9). □

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