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Coefficient estimates for a new subclasses of λ-pseudo biunivalent functions with respect to symmetrical points associated with the Horadam Polynomials

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Abstract: In the present article, we introduce two new subclasses of λ-pseudo biunivalent functions with respect to symmetrical points in the open unit disk $\mathbb{U}$ defined by means of the Horadam polynomials. For functions belonging to these subclasses , estimates on the Taylor -Maclaurin coefficients ja2j and ja3j are obtained . Fekete–Szegö inequalities of functions belonging to these subclasses are also founded. Furthermore, we point out several new special cases of our results.

Key words: Analytic function, univalent and biunivalent functions, Fekete–Szegö problem, λ-pseudo biunivalent functions with respect to symmetrical points, Horadam polynomials, coefficient bounds, subordination

1. Introduction and preliminaries

Let $\mathcal{A}$ denote the class of all functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the open unit open disk $\mathbb{U} = \{z : z \in \mathbb{C}, |z| < 1\}$, and denoted by $\mathcal{A}$. Let $\mathcal{S}$ be class of all functions in $\mathcal{A}$ which are univalent and normalized by the conditions

$$f(0) = 0 = f^{'}(0) - 1$$

in $\mathbb{U}$. For two functions f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, written as $f(z) \prec g(z), \quad (z \in \mathbb{U})$, provided that there exists an analytic function (that is, Schwarz function) $w(z)$ defined on $\mathbb{U}$ with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \text{ for all } z \in \mathbb{U},$$

such that $f(z) = g(w(z))$ for all $z \in \mathbb{U}$.

Besides, it is known that

$$f(z) \prec g(z)\ (z \in \mathbb{U}) \ \Leftrightarrow \ f(0) = g(0) \ \text{ and } \ f(\mathbb{U}) \subset g(\mathbb{U}).$$

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It is well known that every univalent function f has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U}),$$

and

$$f^{-1}(f(w)) = w \quad (|w| < r_0(f); r_0(f) \geq \frac{1}{4}),$$

where

$$f^{-1}(w) = w + a_2 w^2 + (2a_2^2 - 3a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be biunivalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$, and denoted by Σ.

In 1967, the class Σ of biunivalent functions was first investigated by Lewin [12] and it was derived that $|a_2| < 1.51$. Brannan and Taha [6] also considered certain subclasses of biunivalent functions, and obtained estimates for the initial coefficients. In 2010, Srivastava et al. [17] revived the investigation of various classes of bi univalent functions. Moreover, many other authors (see [1–4, 7]) have introduced and investigated subclasses of biunivalent functions.

By $S^*(\varphi)$ and $K(\varphi)$ we denote the following classes of functions

$$S^*(\varphi) = \left\{ f : \ f \in \mathcal{A}, \ \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \ z \in \mathbb{U},$$

and

$$K(\varphi) = \left\{ f : \ f \in A, \ 1 + \frac{zf'(z)}{f(z)} \prec \varphi(z) \right\}, \ z \in \mathbb{U},$$

where $S^*(\varphi)$ and $K(\varphi)$ are the class of starlike and convex functions, respectively, were defined and studied by Ma and Minda [14]. It is clear that if $f(z) \in K$, then $zf'(z) \in S^*$.

Sakaguchi [16] introduced the class S_s^* of functions starlike with respect to symmetric points, which consists of functions $f(z) \in S$ satisfying the condition

$$\Re \left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in \mathbb{U}.$$

Moreover, Wang et al. [18] introduced the class K_s of functions convex with respect to symmetric points, which consists of functions $f(z) \in S$ satisfying the condition

$$\Re \left\{ \frac{(zf'(z))'}{(f(z) - f(-z))'} \right\} > 0, \quad z \in \mathbb{U}.$$

It is easily seen that if $f(z) \in K_s$, then $zf'(z) \in S_s^*$. For such a function φ, Ravichandran [15] introduced the following subclasses: A function $f \in A$ is in the class $S_s^*(\varphi)$ if

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \varphi(z), \quad z \in \mathbb{U},$$

and in the class $K_s(\varphi)$ if

$$\frac{2(zf^{'}(z))^{'}}{f^{'}(z)+f^{'}(-z)} \prec \varphi(z) \quad z \in \mathbb{U}.$$

Recently, Babalola [5] defined the class L_λ of λ-pseudo-starlike functions as follows: Let $f \in A$ and $\lambda \geq 1$ is real. Then $f(z)$ belongs to the class L_λ of λ-pseudo-starlike functions in the unit disc $\mathbb{U}$ if and only if

$$\Re\left\{\frac{z(f^{'}(z))^{\lambda}}{f(z)}\right\} \geq 0, \quad z \in \mathbb{U}.$$

The Horadam polynomials $h_n(x)$ are given by the following recurrence relation (see [10])

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x), \quad (n \in \mathbb{N} \geq 2), \tag{1.3}$$

with $h_1 = a,\ h_2 = bx,$ and $h_3 = pbx^2 + aq$ where (a, b, p, q are some real constants).
The characteristic equation of recurrence relation (1.3) is

$$t^2 - pxt - q = 0. \tag{1.4}$$

This equation has two real roots;

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2},$$

and

$$\beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

Note that, some particular cases of Horadam polynomials sequence are listed as follows:

- If $a = b = p = q = 1$, the Fibonacci polynomials sequence is obtained

$$F_n(x) = xF_{n-1}(x) + F_{n-2}(x),\ F_1(x) = 1,\ F_2(x) = x.$$

- If $a = 2, b = p = q = 1$, the Lucas polynomials sequence is obtained

$$L_{n-1}(x) = xL_{n-2}(x) + L_{n-3}(x),\ L_0(x) = 2,\ L_1(x) = x.$$

- If $a = q = 1, b = p = 2$, the Pell polynomials sequence is obtained

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x),\ p_1(x) = 1,\ P_2(x) = 2x.$$

- If $a = b = p = 2, q = 1$, the Pell-Lucas polynomials sequence is obtained

$$Q_{n-1}(x) = 2xQ_{n-2}(x) + Q_{n-3}(x),\ Q_0(x) = 2,\ Q_1(x) = 2x.$$

- If $a = b = 1, p = 2, q = -1$, the Chebyshev polynomials of first kind sequence is obtained

$$T_{n-1}(x) = 2xT_{n-2}(x) + T_{n-3}(x),\ T_0(x) = 1,\ T_1(x) = x.$$

- If $a = 1, b = p = 2, q = -1$, the Chebyshev polynomials of second kind sequence is obtained

$$U_{n-1}(x) = 2xU_{n-2}(x) + U_{n-3}(x),\ U_0(x) = 1,\ U_1(x) = 2x.$$

- If $x = 1$, the Horadam numbers sequence is obtained

$$h_{n-1}(1) = ph_{n-2}(1) + qh_{n-3}(1),\ h_0(1) = a,\ h_1(1) = b.$$

For more information associated with these polynomials see [8], ([9, 11, 13]).

Remark 1.1 *[9] Let $\Omega(x,z)$ be the generating function of the Horadam polynomials $h_n(x)$. Then*

$$\Omega(x,z) = \frac{a + (b-ap)xt}{1 - pxt - qt^2} = \sum_{n=1}^{\infty} h_n(x)z^{n-1}. \tag{1.5}$$

In this paper, we introduce two new subclasses of λ-pseudo biunivalent functions with respect to symmetrical points by using the Horadam polynomials $h_n(x)$ and the generating function $\Omega(x,z)$ which are given by the recurrence relation (1.3) and (1.5), respectively. Furthermore, we find the initial coefficients and the Fekete–Szegö inequality for functions belonging to the classes $L\Sigma(\lambda,\alpha,x)$ and $M\Sigma(\lambda,\alpha,x)$. Also, several special cases to our results were obtained.

2. Coefficient bounds for the function class $L\Sigma(\lambda,\alpha,x)$

Definition 2.1 *A function $f \in \Sigma$ given by (1.1) is said to be in the class $L\Sigma(\lambda,\alpha,x)$, if the following conditions are satisfied:*

$$(1-\alpha)\frac{2z[f'(z)]^{\lambda}}{f(z)-f(-z)} + \alpha\frac{2[(zf'(z))']^{\lambda}}{[f(z)-f(-z)]'} \prec \Omega(x,z) + 1 - \alpha \tag{2.1}$$

and

$$(1-\alpha)\frac{2w[g'(w)]^{\lambda}}{g(w)-g(-w)} + \alpha\frac{2[(wg'(w))']^{\lambda}}{[g(w)-g(-w)]'} \prec \Omega(x,w) + 1 - \alpha \tag{2.2}$$

where the real constants a, b, and q are as in (1.3) and $g(w) = f^{-1}(z)$ is given by (1.2).

We first state and prove the following result.

Theorem 2.2 *Let the function $f \in \Sigma$ given by (1.1) be in the class $L\Sigma(\lambda,\alpha,x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[((2\lambda^2+\lambda-1)+2\alpha(3\lambda^2-1))b - 4p\lambda^2(1+\alpha)^2]bx^2 - 4qa\lambda^2(1+\alpha)^2|}} \tag{2.3}$$

$$|a_3| \leq \frac{|bx|}{(3\lambda-1)(1+2\alpha)} + \frac{(bx)^2}{4\lambda^2(1+\alpha)^2}, \tag{2.4}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{(3\lambda-1)(1+2\alpha)} & ,\ |\eta - 1| \leq \frac{1}{2(3\lambda-1)(1+2\alpha)}\,|A| \\ \frac{|bx|^3|1-\eta|}{|[(2\lambda^2+\lambda-1)+2\alpha(3\lambda^2-1)][bx]^2 - 4\lambda^2(1+\alpha)^2(pbx^2+qa)|} & ,\ |\eta - 1| \geq \frac{1}{2(3\lambda-1)(1+2\alpha)}\,|A| \end{cases} \tag{2.5}$$

where $A = (2\lambda^2+\lambda-1) + 2\alpha(3\lambda^2-1) - \frac{4\lambda^2(1+\alpha)^2(pbx^2+qa)}{b^2x^2}$.

Proof Let $f \in \Sigma$ be given by the Taylor-Maclaurin expansion (1.1). Then, for some analytic functions Ψ and Φ such that $\Psi(0) = \Phi(0) = 0$, $|\psi(z)| < 1$ and $|\Phi(w)| < 1$, $z, w \in \mathbb{U}$ and using Definition 2.1, we can write

$$(1-\alpha)\frac{2z[f'(z)]^{\lambda}}{f(z)-f(-z)} + \alpha\frac{2[(zf'(z))']^{\lambda}}{[f(z)-f(-z)]'} = \omega(x, \Phi(z)) + 1 - \alpha$$

and

$$(1-\alpha)\frac{2w[g'(w)]^{\lambda}}{g(w)-g(-w)} + \alpha\frac{2[(wg'(w))']^{\lambda}}{[g(w)-g(-w)]'} = \omega(x, \psi(w)) + 1 - \alpha$$

or, equivalently,

$$(1-\alpha)\frac{2z[f'(z)]^{\lambda}}{f(z)-f(-z)} + \alpha\frac{2[(zf'(z))']^{\lambda}}{[f(z)-f(-z)]'} = 1 + h_1(x) - a + h_2(x)\Phi(z) + h_3(x)[\Phi(z)]^3 + \cdots \tag{2.6}$$

and

$$(1-\alpha)\frac{2w[g'(w)]^{\lambda}}{g(w)-g(-w)} + \alpha\frac{2[(wg'(w))']^{\lambda}}{[g(w)-g(-w)]'} = 1 + h_1(x) - a + h_2(x)\psi(w) + h_3(x)[\psi(w)]^3 + \cdots . \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$(1-\alpha)\frac{2z[f'(z)]^{\lambda}}{f(z)-f(-z)} + \alpha\frac{2[(zf'(z))']^{\lambda}}{[f(z)-f(-z)]'} = 1 + h_2(x)p_1 z + [h_2(x)p_2 + h_3(x)p_1^2]z^2 + \cdots \tag{2.8}$$

and

$$(1-\alpha)\frac{2w[g'(w)]^{\lambda}}{g(w)-g(-w)} + \alpha\frac{2[(wg'(w))']^{\lambda}}{[g(w)-g(-w)]'} = 1 + h_2(x)p_1 w + [h_2(x)q_2 + h_3(x)q_1^2]w^2 + \cdots . \tag{2.9}$$

Notice that if

$$|\Phi(z)| = |p_1 z + p_2 z^2 + p_3 z^3 + \cdots| < 1 \quad (z \in \mathbb{U})$$

and

$$|\psi(w)| = |q_1 w + q_2 w^2 + q_3 w^3 + \cdots| < 1 \quad (w \in \mathbb{U}),$$

then

$$|p_i| \leq 1 \quad \textit{and} \quad |q_i| \leq 1 \quad (i \in \mathbb{N}).$$

Thus, upon comparing the corresponding coefficients in (2.8) and (2.9), we have

$$2\lambda(1+\alpha)a_2 = h_2(x)p_1 \tag{2.10}$$

$$2\lambda(\lambda-1)(1+3\alpha)a_2^2 + (3\lambda-1)(1+2\alpha)a_3 = h_2(x)p_2 + h_3(x)p_1^2 \tag{2.11}$$

$$-2\lambda(1+\alpha)a_2 = h_2(x)q_1 \tag{2.12}$$

$$\left[2(\lambda^2+2\lambda-1) + 2\alpha(3\lambda^3+3\lambda-2)\right]a_2^2 - (3\lambda-1)(1+2\alpha)a_3 = h_2(x)q_2 + h_3(x)q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we find that

$$p_1 = -q_1 \tag{2.14}$$

and

$$8\lambda^2(1+\alpha)^2 a_2^2 = h_2^2(x)(p_1^2 + q_1^2). \tag{2.15}$$

Moreover, by using (2.13) and (2.11), we obtain

$$[2(2\lambda^2+\lambda-1)+4\alpha(3\lambda^2-1)]a_2^2 = h_2(x)(p_2+q_2) + h_3(x)(p_1^2+q_1^2). \tag{2.16}$$

By using (2.14) in (2.16), we get

$$\left[2(2\lambda^2+\lambda-1)+4\alpha(3\lambda^2-1) - \frac{8\lambda^2(1+\alpha)^2 h_3(x)}{[h_2(x)]^2}\right] a_2^2 = h_2(x)(p_2+q_2). \tag{2.17}$$

From (1.3) and (2.17), we have the desired inequality (2.3).

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.11) and using (2.14) and (2.15) we get

$$a_3 = \frac{h_2(x)(p_2-q_2)}{2(3\lambda-1)(1+2\alpha)} + \frac{h_2(x)[p_1^2+q_1^2]}{8\lambda^2(1+\alpha)^2}. \tag{2.18}$$

Hence, using (2.14) and applying (1.3), we get desired inequality (2.4).

Now, by using (2.16) and (2.18) for some $\eta \in \mathbb{R}$, we get

$$a_3 - \eta a_2^2 = \frac{[h_2(x)]^3(1-\eta)(p_2+q_2)}{[2(2\lambda^2+\lambda-1)+4\alpha(3\lambda^2-1)][h_2(x)]^2 - 8\lambda^2(1+\alpha)^2 h_3(x)} + \frac{h_2(x)(p_2-q_2)}{2(3\lambda-1)(1+2\alpha)}$$

$$= h_2(x)\left[\left(\Theta(\eta,x) + \frac{1}{2(3\lambda-1)(1+2\alpha)}\right)p_2 + \left(\Theta(\eta,x) - \frac{1}{2(3\lambda-1)(1+2\alpha)}\right)q_2\right],$$

where

$$\Theta(\eta,x) = \frac{[h_2(x)]^2(1-\eta)}{[2(\lambda^2+\lambda-1)+4\alpha(3\lambda^2-1)][h_2(x)]^2 - 8\lambda^2(1+\alpha)^2 h_3(x)}.$$

Thus, we conclude that

$$|a_3 - \eta a_2^2| \le \begin{cases} \frac{h_2(x)}{(3\lambda-1)(1+2\alpha)} & , \ |\Theta(\eta,x)| \le \frac{1}{2(3\lambda-1)(1+2\alpha)} \\ 2|h_2(x)||\Theta(\eta,x)| & , \ |\Theta(\eta,x)| \ge \frac{1}{2(3\lambda-1)(1+2\alpha)} \end{cases}.$$

This proves Theorem 2.2. □

For $\alpha = 0$ the class $L\Sigma(\lambda,\alpha,x)$ reduced to the class of $\lambda-$pseudo bistarlike functions with respect to symmetrical points. For functions belong to this class we have the following corollary:

Corollary 2.3 *Let the function $f \in \Sigma$ given by (1.1) be in the class $L\Sigma(\lambda,x)$. Then*

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(2\lambda^2+\lambda-1)b - 4p\lambda^2]bx^2 - 4qa\lambda^2|}} \tag{2.19}$$

$$|a_3| \leq \frac{|bx|}{3\lambda - 1} + \frac{(bx)^2}{4\lambda^2}, \tag{2.20}$$

and for some $\eta \in \mathbb{R}$*,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{3\lambda-1} & , \ |\eta - 1| \leq \frac{1}{2(3\lambda-1)}\,|A_0| \\ \frac{|bx|^3|1-\eta|}{|(2\lambda^2+\lambda-1)[bx]^2-4\lambda^2(pbx^2+qa)|} & , \ |\eta - 1| \geq \frac{1}{2(3\lambda-1)}\,|A_0| \end{cases} \tag{2.21}$$

where $A_0 = (2\lambda^2 + \lambda - 1) - \frac{4\lambda^2(pbx^2+qa)}{b^2x^2}$.

For $\alpha = 1$ the class $L\Sigma(\lambda, 1, x)$ reduced to the class of $\lambda-$pseudo biconvex functions with respect to symmetrical points. For functions belong to this class we have the following corollary:

Corollary 2.4 *Let the function* $f \in \Sigma$ *given by (1.1) be in the class* $L\Sigma(\lambda, 1, x)$*. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[((2\lambda^2 + \lambda - 1) + 2(3\lambda^2 - 1))b - 16p\lambda^2]bx^2 - 16qa\lambda^2|}} \tag{2.22}$$

$$|a_3| \leq \frac{|bx|}{3(3\lambda - 1)} + \frac{(bx)^2}{16\lambda^2}, \tag{2.23}$$

and for some $\eta \in \mathbb{R}$*,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{3(3\lambda-1)} & , \ |\eta - 1| \leq \frac{1}{6(3\lambda-1)}\,|A_1| \\ \frac{|bx|^3|1-\eta|}{|[(2\lambda^2+\lambda-1)+2(3\lambda^2-1)][bx]^2-16\lambda^2(pbx^2+qa)|} & , \ |\eta - 1| \geq \frac{1}{6(3\lambda-1)}\,|A_1| \end{cases} \tag{2.24}$$

where $A_1 = (2\lambda^2 + \lambda - 1) + 2(3\lambda^2 - 1) - \frac{16\lambda^2(pbx^2+qa)}{b^2x^2}$.

For $\lambda = 1$ the class $L\Sigma(1, \alpha, x)$ reduced to the class of biunivalent functions with respect to symmetrical points. For functions belonging to this class we have the following corollary:

Corollary 2.5 *Let the function* $f \in \Sigma$ *given by (1.1) be in the class* $L\Sigma(1, \alpha, x)$*. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[2(1 + 2\alpha)b - 4p(1 + \alpha)^2]bx^2 - 4qa(1 + \alpha)^2|}} \tag{2.25}$$

$$|a_3| \leq \frac{|bx|}{2(1 + 2\alpha)} + \frac{(bx)^2}{4(1 + \alpha)^2}, \tag{2.26}$$

and for some $\eta \in \mathbb{R}$*,*

$$|a_3 - \eta a_2^2| \leq \begin{cases} \frac{|bx|}{2(1+2\alpha)} & , \ |\eta - 1| \leq \frac{1}{4(1+2\alpha)}\,|B| \\ \frac{|bx|^3|1-\eta|}{|2(1+\alpha)[bx]^2-4(1+\alpha)^2(pbx^2+qa)|} & , \ |\eta - 1| \geq \frac{1}{4(1+2\alpha)}\,|B| \end{cases} \tag{2.27}$$

where $B = 2(1 + 2\alpha) - \frac{4(1+\alpha)^2(pbx^2+qa)}{b^2x^2}$.

3. Coefficient bounds for the function class $M\Sigma(\lambda,\alpha,x)$

Definition 3.1 *A function $f \in \Sigma$ given by (1.1) is said to be in the class $M\Sigma(\lambda,\alpha,x)$, if the following conditions are satisfied:*

$$\left(\frac{2z[f^{'}(z)]^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2[(zf^{'}(z))^{'}]^{\lambda}}{[f(z)-f(-z)]^{'}}\right)^{1-\alpha} \prec \Omega(x,z)+1-\alpha \tag{3.1}$$

and

$$\left(\frac{2w[g^{'}(w)]^{\lambda}}{g(w)-g(-w)}\right)^{\alpha}\left(\frac{2[(wg^{'}(w))^{'}]^{\lambda}}{[g(w)-g(-w)]^{'}}\right)^{1-\alpha} \prec \Omega(x,w)+1-\alpha \tag{3.2}$$

where the real constants a, b, and q are as in (1.3) and $g(w)=f^{-1}(z)$ is given by (1.2).

Theorem 3.2 *Let the function $f \in \Sigma$ given by (1.1) be in the class $M\Sigma(\lambda,\alpha,x)$. Then*

$$|a_2| \leq \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(2\lambda^2(\alpha-2)^2+(\lambda+2\alpha-3))b-4p\lambda^2(\alpha-2)^2]bx^2-4qa\lambda^2(\alpha-2)^2)|}} \tag{3.3}$$

$$|a_3| \leq \frac{|bx|}{(3\lambda-1)(|3-2\alpha|)}+\frac{(bx)^2}{4\lambda^2(\alpha-2)^2}, \tag{3.4}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3-\eta a_2^2| \leq \begin{cases} \frac{|bx|}{(3\lambda-1)(3-2\alpha)} & ,\ |\eta-1| \leq \frac{1}{2(3\lambda-1)(3-2\alpha)}\,|C| \\ \frac{|bx|^3|1-\eta|}{|[2\lambda^2(\alpha-2)^2+(\lambda+2\alpha-3)][bx]^2-4\lambda^2(\alpha-2)^2(pbx^2+qa)|} & ,\ |\eta-1| \geq \frac{1}{2(3\lambda-1)(3-2\alpha)}\,|C| \end{cases} \tag{3.5}$$

where $C=2\lambda^2(\alpha-2)^2+(\lambda+2\alpha-3)-\frac{4\lambda^2(\alpha-2)^2(pbx^2+qa)}{b^2x^2}$.

Proof Let $f \in \Sigma$ be given by the Taylor–Maclaurin expansion (1.1). Then, for some analytic functions Ψ and Φ such that $\Psi(0)=\Phi(0)=0$, $|\psi(z)|<1$ and $|\Phi(w)|<1$, $z,w \in \mathbb{U}$ and using Definition 3.1, we can write

$$\left(\frac{2z[f^{'}(z)]^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2[(zf^{'}(z))^{'}]^{\lambda}}{[f(z)-f(-z)]^{'}}\right)^{1-\alpha} = \omega(x,\Phi(z))+1-\alpha$$

and

$$\left(\frac{2w[g^{'}(w)]^{\lambda}}{g(w)-g(-w)}\right)^{\alpha}\left(\frac{2[(wg^{'}(w))^{'}]^{\lambda}}{[g(w)-g(-w)]^{'}}\right)^{1-\alpha} = \omega(x,\psi(w))+1-\alpha$$

or, equivalently,

$$\left(\frac{2z[f^{'}(z)]^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2[(zf^{'}(z))^{'}]^{\lambda}}{[f(z)-f(-z)]^{'}}\right)^{1-\alpha} = 1+h_1(x)-a+h_2(x)\Phi(z)+h_3(x)[\Phi(z)]^3+\cdots \tag{3.6}$$

and

$$\left(\frac{2w[g^{'}(w)]^{\lambda}}{g(w)-g(-w)}\right)^{\alpha}\left(\frac{2[(wg^{'}(w))^{'}]^{\lambda}}{[g(w)-g(-w)]^{'}}\right)^{1-\alpha}=1+h_1(x)-a+h_2(x)\psi(w)+h_3(x)[\psi(w)]^3+\cdots. \tag{3.7}$$

From (3.6) and (3.7), we obtain

$$\left(\frac{2z[f^{'}(z)]^{\lambda}}{f(z)-f(-z)}\right)^{\alpha}\left(\frac{2[(zf^{'}(z))^{'}]^{\lambda}}{[f(z)-f(-z)]^{'}}\right)^{1-\alpha}=1+h_2(x)p_1z+[h_2(x)p_2+h_3(x)p_1^2]z^2+\cdots \tag{3.8}$$

and

$$\left(\frac{2w[g^{'}(w)]^{\lambda}}{g(w)-g(-w)}\right)^{\alpha}\left(\frac{2[(wg^{'}(w))^{'}]^{\lambda}}{[g(w)-g(-w)]^{'}}\right)^{1-\alpha}=1+h_2(x)p_1w+[h_2(x)q_2+h_3(x)q_1^2]w^2+\cdots. \tag{3.9}$$

Notice that if

$$|\Phi(z)|=|p_1z+p_2z^2+p_3z^3+\cdots|<1 \quad (z\in\mathbb{U})$$

and

$$|\psi(w)|=|q_1w+q_2w^2+q_3w^3+\cdots|<1 \quad (w\in\mathbb{U}),$$

then

$$|p_i|\leq 1 \quad \textit{and} \quad |q_i|\leq 1 \quad (i\in\mathbb{N}).$$

Thus, upon comparing the corresponding coefficients in (3.8) and (3.9), we have

$$-2\lambda(\alpha-2)a_2=h_2(x)p_1 \tag{3.10}$$

$$[2\lambda^2(\alpha-2)^2+2\lambda(3\alpha-4)]a_2^2+(3\lambda-1)(3-2\alpha)a_3=h_2(x)p_2+h_3(x)p_1^2 \tag{3.11}$$

$$2\lambda(\alpha-2)a_2=h_2(x)q_1 \tag{3.12}$$

$$[2\lambda^2(\alpha-2)^2+2\lambda(5-3\alpha)+2(2\alpha-3)]a_2^2+(3\lambda-1)(2\alpha-3)a_3=h_2(x)q_2+h_3(x)q_1^2. \tag{3.13}$$

From (3.10) and (3.12), we find that

$$p_1=-q_1 \tag{3.14}$$

and

$$8\lambda^2(\alpha-2)^2a_2^2=h_2^2(x)(p_1^2+q_1^2). \tag{3.15}$$

Moreover, by using (3.13) and (3.11), we obtain

$$[4\lambda^2(\alpha-2)^2+2(\lambda+2\alpha-3)]a_2^2=h_2(x)(p_2+q_2)+h_3(x)(p_1^2+q_1^2). \tag{3.16}$$

By using (3.14) in (3.16), we get

$$\left[4\lambda^2(\alpha-2)^2+2(\lambda+2\alpha-3)-\frac{8\lambda^2(\alpha-2)^2h_3(x)}{[h_2(x)]^2}\right]a_2^2=h_2(x)(p_2+q_2). \tag{3.17}$$

From (1.3), and (3.17), we have the desired inequality (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.13) from (3.11) and using (3.14) and (3.15) we get

$$a_3 = \frac{h_2(x)(p_2 - q_2)}{2(3\lambda - 1)(|3 - 2\alpha|)} + \frac{h_2(x)[p_1^2 + q_1^2]}{8\lambda^2(\alpha - 2)^2}. \tag{3.18}$$

Hence, using (3.14) and applying (1.3), we get desired inequality (3.4).
Now, by using (3.16) and (3.18) for some $\eta \in \mathbb{R}$, we get

$$a_3 - \eta a_2^2 = \frac{[h_2(x)]^3(1-\eta)(p_2+q_2)}{[4\lambda^2(\alpha-2)^2 + 2(\lambda+2\alpha-3)][h_2(x)]^2 - 8\lambda^2(\alpha-2)^2 h_3(x)} + \frac{h_2(x)(p_2 - q_2)}{2(3\lambda-1)(3-2\alpha)}$$

$$= h_2(x)\left[\left(\Theta(\eta,x) + \frac{1}{2(3\lambda-1)(3-2\alpha)}\right)p_2 + \left(\Theta(\eta,x) - \frac{1}{2(3\lambda-1)(3-2\alpha)}\right)q_2\right],$$

where

$$\Theta(\eta,x) = \frac{[h_2(x)]^2(1-\eta)}{[4\lambda^2(\alpha-2)^2 + 2(\lambda+2\alpha-3)][h_2(x)]^2 - 8\lambda^2(\alpha-2)^2 h_3(x)}.$$

Thus, we conclude that

$$|a_3 - \eta a_2^2| \le \begin{cases} \frac{h_2(x)}{(3\lambda-1)(3-2\alpha)} & , \ |\Theta(\eta,x)| \le \frac{1}{2(3\lambda-1)(3-2\alpha)} \\ 2|h_2(x)||\Theta(\eta,x)| & , \ |\Theta(\eta,x)| \ge \frac{1}{2(3\lambda-1)(3-2\alpha)} \end{cases}.$$

This proves Theorem 3.2. □

For $\lambda = 1$ the class $M\Sigma(\lambda,\alpha,x)$ reduced to the class of biunivalent functions with respect to symmetrical points. For functions belong to this class we have the following corollary:

Corollary 3.3 *Let the function $f \in \Sigma$ given by (1.1) be in the class $M\Sigma(1,\alpha,x)$. Then*

$$|a_2| \le \frac{|bx|\sqrt{|bx|}}{\sqrt{|[(2(\alpha-2)^2 + 2(\alpha-1))b - 4p(\alpha-2)^2]bx^2 - 4qa(\alpha-2)^2)|}} \tag{3.19}$$

$$|a_3| \le \frac{|bx|}{2(|3-2\alpha|)} + \frac{(bx)^2}{4(\alpha-2)^2}, \tag{3.20}$$

and for some $\eta \in \mathbb{R}$,

$$|a_3 - \eta a_2^2| \le \begin{cases} \frac{|bx|}{2(3-2\alpha)} & , \ |\eta - 1| \le \frac{1}{4(3-2\alpha)}|C_1| \\ \frac{|bx|^3|1-\eta|}{|[2(\alpha-2)^2+2(\alpha-1)][bx]^2 - 4(\alpha-2)^2(pbx^2+qa)|} & , \ |\eta - 1| \ge \frac{1}{4(3-2\alpha)}|C_1| \end{cases} \tag{3.21}$$

where $C_1 = 2(\alpha-2)^2 + 2(\alpha-1) - \frac{4(\alpha-2)^2(pbx^2+qa)}{b^2x^2}$.

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