A metric invariant of Möbius transformations

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Abstract: The complex unit disk $\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}$ is endowed with Möbius addition $\oplus_M$ defined by

$$w \oplus_M z = \frac{w + z}{1 + \overline{w}z}.$$

We prove that the metric d_T defined on $\mathbb{D}$ by $d_T(w, z) = \tan^{-1}|-w \oplus_M z|$ is an invariant of Möbius transformations carrying $\mathbb{D}$ onto itself. We also prove that $(\mathbb{D}, d_T)$ and $(\mathbb{D}, d_P)$, where d_P denotes the Poincaré metric, have the same isometry group and then classify the isometries of $(\mathbb{D}, d_T)$.

Key words: Möbius transformation, Poincaré metric, transformation invariant, isometry group, gyrogroup

1. Introduction

Recall that the Poincaré disk model (also called the conformal disk model) consists of the (open) complex unit disk,

$$\mathbb{D} = \{z \in \mathbb{C}\colon |z| < 1\}, \tag{1.1}$$

naturally associated with the Poincaré metric. From a complex-analysis point of view, the Poincaré metric is the most natural metric on the complex unit disk (cf. [13, p. 53]). The right transformations of the Poincaré disk model are the Möbius transformations that carry $\mathbb{D}$ onto itself (also called conformal self-maps of $\mathbb{D}$). Several characterizations of Möbius transformations are obtained; see, for instance, [3–7, 9–12, 14–16, 21].

From an algebraic point of view, the complex unit disk has a nonassociative group-like structure when it is equipped with *Möbius addition* $\oplus_M$ defined by the equation

$$w \oplus_M z = \frac{w + z}{1 + \overline{w}z} \tag{1.2}$$

for all $w, z \in \mathbb{D}$ [20]. More precisely, $(\mathbb{D}, \oplus_M)$ satisfies the following properties:

   I. (IDENTITY) The zero number 0 satisfies $0 \oplus_M z = z = z \oplus_M 0$ for all $z \in \mathbb{D}$.

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II. (INVERSE) For each $z \in \mathbb{D}$, the negative of z belongs to $\mathbb{D}$ and satisfies

$$-z \oplus_M z = 0 = z \oplus_M -z.$$

III. (THE GYROASSOCIATIVE LAW) For all $a, b \in \mathbb{D}$, define a map $\mathrm{gyr}[a, b]$ by

$$\mathrm{gyr}[a,b]z = \frac{1 + a\bar{b}}{1 + \bar{a}b}z \tag{1.3}$$

for all $z \in \mathbb{D}$. Then $\mathrm{gyr}[a, b]$ is an algebraic automorphism of $\mathbb{D}$ with respect to Möbius addition for all $a, b \in \mathbb{D}$. Furthermore,

$$a \oplus_M (b \oplus_M c) = (a \oplus_M b) \oplus_M \mathrm{gyr}[a,b]c$$
$$(a \oplus_M b) \oplus_M c = a \oplus_M (b \oplus_M \mathrm{gyr}[b,a]c) \tag{1.4}$$

for all $a, b, c \in \mathbb{D}$.

IV. (THE LOOP PROPERTY) For all $a, b \in \mathbb{D}$,

$$\mathrm{gyr}[a \oplus_M b, b] = \mathrm{gyr}[a, b] \quad \text{and} \quad \mathrm{gyr}[a, b \oplus_M a] = \mathrm{gyr}[a, b]. \tag{1.5}$$

V. (THE GYROCOMMUTATIVE LAW) For all $a, b \in \mathbb{D}$,

$$a \oplus_M b = \mathrm{gyr}[a,b](b \oplus_M a). \tag{1.6}$$

We remark that Möbius addition is not associative so that $(\mathbb{D}, \oplus_M)$ does not form a group. However, it shares common properties with groups. The space $(\mathbb{D}, \oplus_M)$ is known as a gyrogroup, the term coined by Ungar [19]. The map $\mathrm{gyr}[a, b]$ in item III is called a gyroautomorphism due to the fact that it represents a rotation of the complex unit disk and preserves Möbius addition. Equation (1.4) resembles the associative law in groups, called the gyroassociative law, and Equation (1.6) resembles the commutative law in abelian groups, called the gyrocommutative law. From now on, the space $(\mathbb{D}, \oplus_M)$ is referred to as the (complex) Möbius gyrogroup. The next theorem collects some algebraic identities of the Möbius gyrogroup extended from group-theoretic identities.

Theorem 1.1 *(See [17, 19].) The following properties are true in* $(\mathbb{D}, \oplus_M)$:

1. $-a \oplus_M (a \oplus_M b) = b$;   (LEFT CANCELLATION LAW)

2. $-(a \oplus_M b) = \mathrm{gyr}[a, b](-b \oplus_M -a)$;

3. $(-a \oplus_M b) \oplus_M \mathrm{gyr}[-a, b](-b \oplus_M c) = -a \oplus_M c$;

4. $\mathrm{gyr}[-a, -b] = \mathrm{gyr}[a, b]$;   (EVEN PROPERTY)

5. $\mathrm{gyr}[b, a] = \mathrm{gyr}^{-1}[a, b]$, *where* $\mathrm{gyr}^{-1}[a, b]$ *denotes the inverse of* $\mathrm{gyr}[a, b]$ *with respect to composition of functions;*   (INVERSIVE SYMMETRY)

6. $L_a : z \mapsto a \oplus_M z$ *is a bijective self-map of* $\mathbb{D}$ *and* $L_a^{-1} = L_{-a}$.

Theorem 1.1 will prove useful in studying the geometric structure of the complex unit disk in Sections 2 and 3. The map L_a described in Theorem 1.1(6) is in fact a Möbius transformation of $\mathbb{D}$, which is called a hyperbolic translation or a left gyrotranslation as it resembles a Euclidean translation.

2. The metric induced by Möbius addition

In this section, we prove that the complex unit disk possesses another metric, denoted by d_T (here, "T" stands for "$\tan^{-1}$"). We then establish that the Möbius transformations of $\mathbb{D}$ preserve d_T. Hence, d_T becomes a numerical invariant of Möbius transformations of $\mathbb{D}$. We also describe a few fundamental classes of isometries of $(\mathbb{D}, d_T)$ as well as some geometry of $(\mathbb{D}, d_T)$. Let us begin with the fact that the Cauchy–Schwarz inequality in $\mathbb{R}^2$ may be expressed via complex multiplication and conjugation.

Lemma 2.1 *For all $w, z \in \mathbb{D}$,*

$$-2|w||z| \leq w\overline{z} + \overline{w}z \leq 2|w||z|. \tag{2.1}$$

Proof Let $w = a + bi$ and $z = c + di$, where $a, b, c, d \in \mathbb{R}$. Then $\dfrac{w\overline{z} + \overline{w}z}{2} = ac + bd = \langle \mathbf{w}, \mathbf{z} \rangle$, where $\mathbf{w} = (a, b)$ and $\mathbf{z} = (c, d)$. According to the Cauchy–Schwarz inequality in $\mathbb{R}^2$, we obtain $|\langle \mathbf{w}, \mathbf{z} \rangle| \leq \|\mathbf{w}\|\|\mathbf{z}\| = |w||z|$, which implies $-|w||z| \leq \langle \mathbf{w}, \mathbf{z} \rangle \leq |w||z|$ and so (2.1) holds. □

Theorem 2.2 *The inequality*

$$\frac{|w| - |z|}{1 + |w||z|} \leq |w \oplus_M z| \leq \frac{|w| + |z|}{1 - |w||z|} \tag{2.2}$$

holds in the complex unit disk.

Proof Let $w, z \in \mathbb{D}$. Using the triangle inequality and the reverse triangle inequality, we obtain

$$(|w| - |z|)^2 \leq |w + z|^2 \leq (|w| + |z|)^2.$$

Note that $|1 + \overline{w}z|^2 = (1 + \overline{w}z)\overline{1 + \overline{w}z} = 1 + w\overline{z} + \overline{w}z + |w|^2|z|^2$, which implies $(1 - |w||z|)^2 \leq |1 + \overline{w}z|^2 \leq (1 + |w||z|)^2$ by Lemma 2.1. It follows that

$$|w \oplus_M z|^2 = \frac{|w + z|^2}{|1 + \overline{w}z|^2} \leq \frac{(|w| + |z|)^2}{(1 - |w||z|)^2}$$

and so $|w \oplus_M z| \leq \dfrac{|w| + |z|}{1 - |w||z|}$. Similarly, we have

$$|w \oplus_M z|^2 = \frac{|w + z|^2}{|1 + \overline{w}z|^2} \geq \frac{(|w| - |z|)^2}{(1 + |w||z|)^2}$$

and hence $|w \oplus_M z| \geq \dfrac{|w| - |z|}{1 + |w||z|}$. □

Define a function $|\cdot|_T$ by

$$|z|_T = \tan^{-1}|z| \tag{2.3}$$

for all $z \in \mathbb{D}$. The next theorem summarizes elementary properties of $|\cdot|_T$.

Theorem 2.3 *$|\cdot|_T$ satisfies the following properties:*

  1. $|z|_T \geq 0$ and $|z|_T = 0$ if and only if $z = 0$

2. $|-z|_T = |z|_T$

3. $|w|_T - |z|_T \le |w \oplus_M z|_T \le |w|_T + |z|_T$

4. $|\mathrm{gyr}[a,b]z|_T = |z|_T$

for all $a, b, w, z \in \mathbb{D}$.

Proof Item (1) follows from the fact that $\tan^{-1}$ is increasing and injective. Item (2) is clear. To prove item (3), set $r = |w|_T = \tan^{-1}|w|$ and $s = |z|_T = \tan^{-1}|z|$. By Theorem 2.2,

$$\frac{|w| - |z|}{1 + |w||z|} \le |w \oplus_M z| \le \frac{|w| + |z|}{1 - |w||z|}$$

and so $\tan(r-s) \le |w \oplus_M z| \le \tan(r+s)$. Since $\tan^{-1}$ is increasing, it follows that

$$r - s \le \tan^{-1}|w \oplus_M z| \le r + s,$$

as claimed. Item (4) follows from the fact that $\mathrm{gyr}[a,b]$ preserves the complex modulus. $\square$

In view of Theorem 2.3, one can define a new metric on the complex unit disk using $|\cdot|_T$. In fact, define d_T by

$$d_T(w,z) = |-w \oplus_M z|_T \tag{2.4}$$

for all $w, z \in \mathbb{D}$. We remark that d_T is a bounded metric on $\mathbb{D}$, whereas the Poincaré metric is unbounded.

Theorem 2.4 *d_T is a bounded metric on the complex unit disk.*

Proof By Theorem 2.3(1), $d_T(w,z) \ge 0$. Furthermore, $d_T(w,z) = 0$ if and only if $-w \oplus_M z = 0$ if and only if $w = z$. Let $x, y, z \in \mathbb{D}$. Using some properties of Möbius addition mentioned in Theorem 1.1, we obtain

$$|-y \oplus_M x|_T = |-(-y \oplus_M x)|_T = |\mathrm{gyr}[-y,x](-x \oplus_M y)|_T = |-x \oplus_M y|_T.$$

Hence, $d_T(y,x) = d_T(x,y)$. Furthermore, direct computation shows that

$$\begin{aligned}
d_T(x,z) &= |-x \oplus_M z|_T \\
&= |(-x \oplus_M y) \oplus_M \mathrm{gyr}[-x,y](-y \oplus_M z)|_T \\
&\le |-x \oplus_M y|_T + |\mathrm{gyr}[-x,y](-y \oplus_M z)|_T \\
&= |-x \oplus_M y|_T + |-y \oplus_M z|_T \\
&= d_T(x,y) + d_T(y,z).
\end{aligned}$$

This proves that d_T defines a metric on $\mathbb{D}$.

Note that $d_T(0,z) = |z|_T = \tan^{-1}|z| < \tan^{-1}1 = \dfrac{\pi}{4}$ for all $z \in \mathbb{D}$. Hence,

$$d_T(w,z) \le d_T(w,0) + d_T(0,z) < \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2}$$

for all $w, z \in \mathbb{D}$. $\square$

There are two important classes of isometries of $\mathbb{D}$ with respect to d_T: (i) modulus-preserving automorphisms of $(\mathbb{D}, \oplus_M)$ and (ii) hyperbolic translations; see Theorems 2.5 and 2.6. This allows us to prove that $(\mathbb{D}, d_T)$ is a homogeneous space and to construct point-reflections of $(\mathbb{D}, d_T)$.

Theorem 2.5 *If $\tau \in \mathrm{Aut}(\mathbb{D}, \oplus_M)$ and τ preserves the complex modulus, then τ is an isometry of $(\mathbb{D}, d_T)$. In particular, every rotation of the complex unit disk is an isometry of $(\mathbb{D}, d_T)$.*

Proof Let $\tau \in \mathrm{Aut}(\mathbb{D}, \oplus_M)$ and suppose that τ preserves $|\cdot|$. Then

$$d_T(\tau(w), \tau(z)) = |-\tau(w) \oplus_M \tau(z)|_T = |\tau(-w \oplus_M z)|_T = |-w \oplus_M z|_T = d_T(w, z)$$

for all $w, z \in \mathbb{D}$.

Let ρ be a rotation of $\mathbb{D}$; that is, $\rho(z) = \omega z$ for all $z \in \mathbb{D}$, where ω is a unimodular complex number. Clearly, ρ preserves $|\cdot|$ and is bijective. Next, we prove that ρ preserves Möbius addition. Let $w, z \in \mathbb{D}$. Direct computation shows that $\rho(w) \oplus_M \rho(z) = \dfrac{\omega(w + z)}{1 + |\omega|^2 \overline{w} z} = \omega \dfrac{w + z}{1 + \overline{w} z} = \rho(w \oplus_M z)$. It follows that ρ is an isometry of $(\mathbb{D}, d_T)$. $\qquad\square$

Theorem 2.6 *The hyperbolic translation $L_w \colon z \mapsto w \oplus_M z$ is an isometry of $(\mathbb{D}, d_T)$ for all $w \in \mathbb{D}$.*

Proof Let $w \in \mathbb{D}$. The hyperbolic translation L_w is bijective since $L_w \circ L_{-w}$ and $L_{-w} \circ L_w$ are the identity transformation of $\mathbb{D}$ by the left cancellation law. Using some properties of Möbius addition mentioned in Theorem 1.1, we obtain

$$\begin{aligned}
|-(w \oplus_M a) \oplus_M (w \oplus_M b)|_T &= |\mathrm{gyr}[w, a](-a \oplus_M -w) \oplus_M (w \oplus_M b)|_T \\
&= |(-a \oplus_M -w) \oplus_M \mathrm{gyr}[a, w](w \oplus_M b)|_T \\
&= |(-a \oplus_M -w) \oplus_M \mathrm{gyr}[-a, -w](w \oplus_M b)|_T \\
&= |-a \oplus_M b|_T.
\end{aligned}$$

Hence, $d_T(L_w(a), L_w(b)) = d_T(a, b)$. $\qquad\square$

Theorem 2.7 (Homogeneity) *For each pair of points w and z in $\mathbb{D}$, there is an isometry ψ of $(\mathbb{D}, d_T)$ such that $\psi(w) = z$. In particular, $(\mathbb{D}, d_T)$ is homogeneous.*

Proof Let $w, z \in \mathbb{D}$. Define $\psi = L_z \circ L_{-w}$. Then ψ is an isometry of $\mathbb{D}$, being the composite of isometries of $\mathbb{D}$. Clearly, $\psi(w) = z \oplus_M (-w \oplus_M w) = z$. $\qquad\square$

With the aid of Möbius addition, a *point-reflection* symmetry of $\mathbb{D}$ is easy to construct, as shown in the following theorem.

Theorem 2.8 (Symmetry) *For each $z \in \mathbb{D}$, there is a point-reflection symmetry σ_z of $(\mathbb{D}, d_T)$; that is, σ_z is an isometry of $(\mathbb{D}, d_T)$ such that σ_z^2 is the identity transformation of $\mathbb{D}$ and z is the unique fixed point of σ_z.*

Proof Let ι be the negative map of $\mathbb{D}$; that is, $\iota(z) = -z$ for all $z \in \mathbb{D}$. By Theorem 2.5, ι is an isometry of $\mathbb{D}$. Define $\sigma_z = L_z \circ \iota \circ L_{-z}$. Then σ_z is a nonidentity isometry of $\mathbb{D}$, being the composite of isometries of $\mathbb{D}$. Since $L_{-z} = L_z^{-1}$, it follows that $\sigma_z^2 = (L_z \circ \iota \circ L_z^{-1}) \circ (L_z \circ \iota \circ L_z^{-1}) = L_z \circ \iota^2 \circ L_z^{-1} = L_z \circ L_z^{-1} = I$, where I is the identity transformation of $\mathbb{D}$. By construction, z is a fixed point of σ_z. Suppose that w is a fixed point of σ_z; that is, $\sigma_z(w) = w$. It follows that $z \oplus_M \iota(-z \oplus_M w) = w$ and hence $\iota(-z \oplus_M w) = -z \oplus_M w$. Since 0 is the unique fixed point of ι, we obtain that $-z \oplus_M w = 0$ and so $w = z$. $\qquad\square$

3. Connections between the Poincaré metric and d_T-metric

In this section, we indicate some fruitful connections between the Poincaré metric, the metric d_T, and Möbius transformations.

3.1. The metric structure of the complex unit disk

Recall that the *Poincaré metric* on $\mathbb{D}$ is given by

$$d_P(w, z) = 2\tanh^{-1}\left|\frac{z - w}{1 - \overline{w}z}\right| = 2\tanh^{-1}|-w \oplus_M z| \tag{3.1}$$

and the *pseudo-hyperbolic distance* δ_M (also called the *Möbius gyrometric*) is given by

$$\delta_M(w, z) = \left|\frac{z - w}{1 - \overline{w}z}\right| = |-w \oplus_M z| \tag{3.2}$$

for all $w, z \in \mathbb{D}$. The pseudo-hyperbolic distance δ_M is indeed a metric on $\mathbb{D}$ [2, Theorem 2.4]. Although the metric d_T is quite different from the Poincaré metric, they are *topologically* equivalent. To see this, note that $d_T(w, z) \leq d_P(w, z)$ for all $w, z \in \mathbb{D}$ since $f(x) = 2\tanh^{-1} x - \tan^{-1} x$ defines a strictly increasing function on the open interval $(0, 1)$. Hence, the topology induced by d_P is finer than the topology induced by d_T. If $w \in \mathbb{D}$ and if $\epsilon > 0$, then set $\delta = \tan^{-1}\left(\tanh\frac{\epsilon}{2}\right)$ so that $B_{d_T}(w, \delta) \subseteq B_{d_P}(w, \epsilon)$. This proves that the topology induced by d_T is finer than the topology induced by d_P and so they coincide. The topology of $\mathbb{D}$ induced by d_P is in fact the usual Euclidean metric topology; see, for instance, Proposition 3 on p. 48 of [13].

It is well known that the metric space $(\mathbb{D}, d_P)$ is complete; see, for instance, Proposition 4 on p. 49 of [13]. Hence, every Cauchy sequence in $(\mathbb{D}, d_P)$ converges. This leads to a natural question of determining whether $(\mathbb{D}, d_T)$ is complete since $(\mathbb{D}, d_P)$ and $(\mathbb{D}, d_T)$ are topologically equivalent. We begin with the fact that $(\mathbb{D}, d_T)$ is not compact.

Let $B_P(z, \epsilon)$ be the open ball of $(\mathbb{D}, d_P)$ of radius ϵ centered at z and let $B_T(z, \epsilon)$ be the open ball of $(\mathbb{D}, d_T)$ of radius ϵ centered at z. For all $\epsilon > 0$, note that

$$d_P(z, w) < \epsilon \qquad \text{if and only if} \qquad d_T(z, w) < \tan^{-1}\left(\tanh\frac{\epsilon}{2}\right). \tag{3.3}$$

This shows that $B_P(z, \epsilon) = B_T(z, \epsilon')$, where $\epsilon' = \tan^{-1}\left(\tanh\frac{\epsilon}{2}\right)$, for all $\epsilon > 0$. This also implies that $B_T(z, \epsilon) = B_P(z, \epsilon')$, where $\epsilon' = 2\tanh^{-1}(\tan\epsilon)$, for all ϵ with $0 < \epsilon < \frac{\pi}{4}$ because the domain of $\tanh^{-1}$ is $(-1, 1)$.

Theorem 3.1 $(\mathbb{D}, d_T)$ *is not compact.*

Proof Define $V = \left\{ B_T \left(0, \frac{\pi}{4} - \frac{1}{n} \right) : n \in \mathbb{N} \text{ and } n \geq 2 \right\}$. Then V is an open cover of $(\mathbb{D}, d_T)$ without finite subcollections covering $\mathbb{D}$. In fact, suppose that

$$U = \left\{ B_T \left(0, \frac{\pi}{4} - \frac{1}{n_1} \right), B_T \left(0, \frac{\pi}{4} - \frac{1}{n_2} \right), \ldots, B_T \left(0, \frac{\pi}{4} - \frac{1}{n_r} \right) \right\}$$

is a finite subcollection of V. Set $m = \max \{n_1, n_2, \ldots, n_r\}$. Then

$$\bigcup_{i=1}^{r} B_T \left(0, \frac{\pi}{4} - \frac{1}{n_i} \right) = B_T \left(0, \frac{\pi}{4} - \frac{1}{m} \right) \subsetneq \mathbb{D}.$$

Therefore, $(\mathbb{D}, d_T)$ is not compact. □

A similar argument used in the proof of Theorem 3.1 shows that $(\mathbb{D}, d_P)$ is not compact. In fact, $V = \{B_P(0, n) : n \in \mathbb{N}\}$ is an open cover of $(\mathbb{D}, d_P)$ without finite subcollections covering $\mathbb{D}$ since

$$\bigcup_{i=1}^{r} B_P(0, n_i) = B_P(0, m) \subsetneq \mathbb{D},$$

where $m = \max \{n_1, n_2, \ldots, n_r\}$. Since $(\mathbb{D}, d_P)$ is complete, it follows that $(\mathbb{D}, d_P)$ is not totally bounded as a metric space is compact if and only if it is complete and totally bounded. This in turn implies that $(\mathbb{D}, d_T)$ is not totally bounded, as shown in the next theorem.

Theorem 3.2 $(\mathbb{D}, d_T)$ *is not totally bounded.*

Proof Suppose to the contrary that $(\mathbb{D}, d_T)$ is totally bounded. Let $\epsilon > 0$. For $\epsilon' = \tan^{-1} \left(\tanh \frac{\epsilon}{2} \right)$, there exist open balls $B_T(z_1, \epsilon'), B_T(z_2, \epsilon'), \ldots, B_T(z_k, \epsilon')$ of $(\mathbb{D}, d_T)$ such that $\bigcup_{i=1}^{k} B_T(z_i, \epsilon') = \mathbb{D}$. As proved above, $B_T(z_i, \epsilon') = B_P(z_i, \epsilon)$ for all $i = 1, 2, \ldots, k$. Hence, $\bigcup_{i=1}^{k} B_P(z_i, \epsilon) = \mathbb{D}$ and so $(\mathbb{D}, d_P)$ is totally bounded, a contradiction. □

In view of Theorems 3.1 and 3.2, the problem of determining whether $(\mathbb{D}, d_T)$ is complete is not immediate. However, this is indeed the case, as shown in the following theorem.

Theorem 3.3 $(\mathbb{D}, d_T)$ *is a complete metric space.*

Proof Let (x_n) be a Cauchy sequence in $(\mathbb{D}, d_T)$. We first prove that (x_n) is also a Cauchy sequence in $(\mathbb{D}, d_P)$. Let $\epsilon > 0$. For $\epsilon' = \tan^{-1} \left(\tanh \frac{\epsilon}{2} \right) > 0$, there is a positive integer N such that $d_T(x_m, x_k) < \epsilon'$ for all $m, k \geq N$. By (3.3), $d_P(x_m, x_k) < \epsilon$ for all $m, k \geq N$. Hence, (x_n) is a Cauchy sequence with respect to d_P. As $(\mathbb{D}, d_P)$ is complete, (x_n) converges in $(\mathbb{D}, d_P)$, namely that $x_n \to x$ and $x \in \mathbb{D}$.

We claim that (x_n) converges to the same point x in $(\mathbb{D}, d_T)$. Let $\epsilon > 0$. Set $m = \min\{\epsilon, \pi/8\}$ and $M = \min\{\tan m, 1/2\}$. Then $M \in (0,1)$ and we can let $\epsilon' = 2\tanh^{-1} M$. Then $\epsilon' > 0$ and so there is a positive integer N such that $d_P(x_k, x) < \epsilon'$ for all $k \geq N$. By (3.3),

$$d_T(x_k, x) < \tan^{-1}\left(\tanh\frac{\epsilon'}{2}\right) = \tan^{-1} M \leq \tan^{-1}(\tan m) = m \leq \epsilon$$

for all $k \geq N$. This proves the claim. □

3.2. The isometry groups of $(\mathbb{D}, d_P)$, $(\mathbb{D}, \delta_M)$ and $(\mathbb{D}, d_T)$

The next theorem shows that $(\mathbb{D}, d_T)$, $(\mathbb{D}, \delta_M)$, and $(\mathbb{D}, d_P)$ have the same isometry group.

Theorem 3.4 *Let ψ be a self-map of $\mathbb{D}$. Then the following statements are equivalent:*

1. *ψ preserves d_T;*

2. *ψ preserves δ_M;*

3. *ψ preserves d_P.*

Therefore, $\mathrm{Iso}(\mathbb{D}, d_T) = \mathrm{Iso}(\mathbb{D}, \delta_M) = \mathrm{Iso}(\mathbb{D}, d_P)$, where $\mathrm{Iso}(\mathbb{D}, d_T), \mathrm{Iso}(\mathbb{D}, \delta_M)$, and $\mathrm{Iso}(\mathbb{D}, d_P)$ are the isometry groups of $(\mathbb{D}, d_T), (\mathbb{D}, \delta_M)$, and $(\mathbb{D}, d_P)$, respectively.

Proof　The theorem follows from the fact that

$$d_T(\psi(w), \psi(z)) = d_T(w, z) \quad \Leftrightarrow \quad |-\psi(w) \oplus_M \psi(z)| = |-w \oplus_M z|$$
$$\Leftrightarrow \quad d_P(\psi(w), \psi(z)) = d_P(w, z).$$

for all $w, z \in \mathbb{D}$. □

Let $\mathcal{R}$ be the group of rotations of $\mathbb{D}$; that is, $\mathcal{R}$ consists precisely of transformations of the form $z \mapsto \omega z$, $z \in \mathbb{D}$, where ω is a unimodular complex number. Also, define

$$\mathcal{N} = \{\eta\colon \eta \text{ is a bijective self-map of } \mathbb{D} \text{ and } |\eta(z)| = |z| \text{ for all } z \in \mathbb{D}\}. \tag{3.4}$$

It is clear that $\mathcal{N}$ forms a group under composition and that $\mathcal{R}$ is a subgroup of $\mathcal{N}$. Furthermore, $\mathcal{R}$ is proper in $\mathcal{N}$ as shown in (3.6). Let $\mathcal{M}(\mathbb{D})$ be the group of Möbius transformations carrying $\mathbb{D}$ onto itself; that is, $\mathcal{M}(\mathbb{D})$ consists precisely of holomorphic functions from $\mathbb{D}$ to itself that are injective and surjective. By Theorem 6.2.3 of [8], τ is a Möbius transformation of $\mathbb{D}$ if and only if $\tau = \rho \circ \varphi_a$, where ρ is a rotation in $\mathcal{R}$, $a \in \mathbb{D}$, and φ_a is given by

$$\varphi_a(z) = \frac{z-a}{1-\bar{a}z}, \qquad z \in \mathbb{D}.$$

Clearly, $\varphi_a = L_{-a}$. By Equation (55) of [17], $\rho \circ L_{-a} = L_{\rho(-a)} \circ \rho$. Moreover, if $w \in \mathbb{D}$ and $\rho \in \mathcal{R}$, then $L_w \circ \rho = \rho \circ L_{\rho^{-1}(w)} = \rho \circ \varphi_{-\rho^{-1}(w)}$. This proves that

$$\mathcal{M}(\mathbb{D}) = \{L_w \circ \rho\colon w \in \mathbb{D} \text{ and } \rho \in \mathcal{R}\}. \tag{3.5}$$

Lemma 3.5 *If ψ is an isometry of $(\mathbb{D}, d_T)$ that leaves 0 fixed, then ψ preserves the complex modulus.*

Proof Let $z \in \mathbb{D}$. Then

$$|\psi(z)|_T = |-\psi(0) \oplus_M \psi(z)|_T = d_T(\psi(0), \psi(z)) = d_T(0, z) = |z|_T.$$

Since $\tan^{-1}$ is injective, it follows that $|\psi(z)| = |z|$. □

We remark that there is a transformation of $\mathbb{D}$ that preserves the complex modulus but is not additive:

$$f(z) = \begin{cases} z & \text{if } |z| \neq \dfrac{1}{2} \\ -z & \text{if } |z| = \dfrac{1}{2} \end{cases} \tag{3.6}$$

for all $z \in \mathbb{D}$. Moreover, f is not a rotation of $\mathbb{D}$. The following theorem shows that the Möbius transformations of $\mathbb{D}$ are indeed isometries of $(\mathbb{D}, d_T)$. However, there exists an isometry of $(\mathbb{D}, d_T)$ that is not Möbius; see Example 3.9.

Theorem 3.6 *The following inclusions hold:*

$$\mathcal{M}(\mathbb{D}) \subseteq \mathrm{Iso}(\mathbb{D}, d_T) \subseteq \{L_w \circ \eta \colon w \in \mathbb{D} \text{ and } \eta \in \mathcal{N}\}.$$

Proof By Theorems 2.5 and 2.6, $\mathcal{M}(\mathbb{D}) \subseteq \mathrm{Iso}(\mathbb{D}, d_T)$. Let $\psi \in \mathrm{Iso}(\mathbb{D}, d_T)$. Then ψ is a bijective self-map of $\mathbb{D}$. By Theorem 11 of [18], $\psi = L_{\psi(0)} \circ \eta$, where η is a bijective self-map of $\mathbb{D}$ fixing 0. Since $\eta = L_{\psi(0)}^{-1} \circ \psi = L_{-\psi(0)} \circ \psi$, it follows that η is an isometry of $(\mathbb{D}, d_T)$. By Lemma 3.5, η preserves the complex modulus and so $\eta \in \mathcal{N}$. Hence, $\psi \in \{L_w \circ \eta \colon w \in \mathbb{D} \text{ and } \eta \in \mathcal{N}\}$. This proves the second inclusion. □

Corollary 3.7 *If a self-map σ of $\mathbb{D}$ does not preserve the metric d_T, then σ is not Möbius.*

Example 3.8 *The function f defined by (3.6) preserves the complex modulus but does not define an isometry of $(\mathbb{D}, d_T)$. In fact, if $w = \dfrac{1}{4}$ and $z = \dfrac{1}{2}$, then $|-f(w) \oplus_M f(z)| = \dfrac{6}{7}$, whereas $|-w \oplus_M z| = \dfrac{2}{7}$. This shows that $\mathrm{Iso}(\mathbb{D}, d_T)$ is proper in $\{L_w \circ \eta \colon w \in \mathbb{D} \text{ and } \eta \in \mathcal{N}\}$.*

From Example 3.8, we have seen that a modulus-preserving bijection need not be an isometry of $\mathbb{D}$ with respect to d_T. However, a modulus-preserving bijection that respects Möbius addition must be an isometry by Theorem 2.5.

Example 3.9 *The complex conjugation $\kappa \colon z \mapsto \overline{z}$, $z \in \mathbb{D}$, is an isometry of $(\mathbb{D}, d_T)$ but not a Möbius transformation of $\mathbb{D}$. In fact, κ is an automorphism of $(\mathbb{D}, \oplus_M)$ that preserves the complex modulus and hence becomes an isometry of $\mathbb{D}$ with respect to d_T and d_P. However, κ is not Möbius for it is not holomorphic. This shows that $\mathcal{M}(\mathbb{D})$ is proper in $\mathrm{Iso}(\mathbb{D}, d_T)$.*

Example 3.9 motivates the following question:

Under what conditions is a d_T-preserving self-map of $\mathbb{D}$ a Möbius transformation?

According to the Schwarz–Pick lemma (see, for instance, Theorem 4 on p. 15 of [13]), an obvious answer of the previous question is the condition of "being holomorphic". We remark that this condition is very strong.

Theorem 3.10 *If ψ is a d_T-preserving self-map of $\mathbb{D}$ that is holomorphic, then ψ is a Möbius transformation of $\mathbb{D}$.*

Proof Let $w, z \in \mathbb{D}$ with $w \neq z$. By assumption, $d_T(\psi(w), \psi(z)) = d_T(w, z)$, which implies

$$\left| \frac{\psi(z) - \psi(w)}{1 - \overline{\psi(w)}\psi(z)} \right| = \left| \frac{z - w}{1 - \overline{w}z} \right|$$

since $\tan^{-1}$ is injective. By the Schwarz–Pick lemma, ψ is Möbius. □

In order to describe the exact isometry group of $(\mathbb{D}, d_T)$, we need a Euclidean version of Möbius addition. Under the identification

$$z = u + vi \quad \leftrightarrow \quad \mathbf{z} = (u, v) \quad \text{for all } u, v \in \mathbb{R},$$

Möbius addition becomes a binary operation on the open unit ball of $\mathbb{R}^2$ given by

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{(1 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{v}\|^2)\mathbf{u} + (1 - \|\mathbf{u}\|^2)\mathbf{v}}{1 + 2\langle \mathbf{u}, \mathbf{v} \rangle + \|\mathbf{u}\|^2\|\mathbf{v}\|^2}, \tag{3.7}$$

where $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ are the Euclidean inner product and norm of $\mathbb{R}^2$, respectively; see Section 4 of [20]. In view of (3.7), we obtain a complete description of the isometry group of $(\mathbb{D}, d_T)$ using Abe's result [1] as follows. Define

$$\mathrm{O}(\mathbb{D}) = \{\Phi|_{\mathbb{D}} \colon \Phi \text{ is an orthogonal transformation of } \mathbb{R}^2\}, \tag{3.8}$$

where $\Phi|_{\mathbb{D}}$ denotes the restriction of Φ to $\mathbb{D}$.

Theorem 3.11 *The isometry group of $(\mathbb{D}, d_T)$ is given by*

$$\mathrm{Iso}(\mathbb{D}, d_T) = \{L_w \circ \varphi \colon w \in \mathbb{D} \text{ and } \varphi \in \mathrm{O}(\mathbb{D})\}. \tag{3.9}$$

Proof Note that every transformation in $\mathrm{O}(\mathbb{D})$ is a bijective self-map of $\mathbb{D}$ since $\mathbb{D}$ is invariant under the orthogonal transformations of $\mathbb{R}^2$. Using (3.7), we obtain that if $\varphi \in \mathrm{O}(\mathbb{D})$, then φ is an isometry of $(\mathbb{D}, d_T)$ since φ preserves both Möbius addition and the Euclidean inner product. This combined with Theorem 2.6 implies that

$$\{L_w \circ \varphi \colon w \in \mathbb{D} \text{ and } \varphi \in \mathrm{O}(\mathbb{D})\} \subseteq \mathrm{Iso}(\mathbb{D}, d_T).$$

To prove the reverse inclusion, let $\tau \in \mathrm{Iso}(\mathbb{D}, d_T)$. By Theorem 11 of [18], $\tau = L_{\tau(0)} \circ \varphi$, where φ is a bijective self-map of $\mathbb{D}$ fixing 0. Thus, $\varphi = L_{\tau(0)}^{-1} \circ \tau$ preserves the metric d_T as well as the Möbius gyrometric δ_M. By Theorem 3.2 of [1], $\varphi = \Phi|_{\mathbb{D}}$, where Φ is an orthogonal transformation of $\mathbb{R}^2$. Hence, $\varphi \in \mathrm{O}(\mathbb{D})$ and the proof completes. □

It is a standard result in linear algebra that any orthogonal transformation of $\mathbb{R}^2$ is either a rotation about the origin (which is Möbius) or a reflection about a line passing through the origin (which is not Möbius). Theorem 3.11 shows that any isometry of $(\mathbb{D}, d_T)$ is of the form $\psi = L_w \circ \varphi$, where $w \in \mathbb{D}$ and φ is the restriction of an orthogonal transformation of $\mathbb{R}^2$ to $\mathbb{D}$. Furthermore, ψ is Möbius if and only if φ is a rotation about the origin.

Acknowledgments

References

[1] Abe T. Gyrometric preserving maps on Einstein gyrogroups, Möbius gyrogroups and Proper Velocity gyrogroups. Nonlinear Functional Analysis and Applications 2014; 19: 1-17.

[2] Beardon AF, Minda D. The hyperbolic metric and geometric function theory. In: Quasiconformal mappings and their applications. Narosa, New Delhi; 2007. pp. 9-56.

[3] Bulut S, Yilmaz ÖN. A new characterization of Möbius transformations by the use of Apollonius points of $(2n-1)$-gons. Acta Mathematica Sinica, English Series 2005; 21 (3): 667-672.

[4] Demirel O. A characterization of Möbius transformations by use of hyperbolic regular star polygons. Proceedings of the Romanian Academy, Series A 2013; 14: 13-19.

[5] Demirel O. A characterization of Möbius transformations by use of hyperbolic triangles. Journal of Mathematical Analysis and Applications 2013; 398 (2): 457-461. doi: 10.1016/j.jmaa.2012.05.044

[6] Demirel O, Seyrantepe ES. A characterization of Möbius transformations by use of hyperbolic regular polygons. Journal of Mathematical Analysis and Applications 2011; 374 (2): 566-572. doi: 10.1016/j.jmaa.2010.08.049

[7] Dyakonov KM. A characterization of Möbius transformations. Comptes Rendus Mathématique, Académie des Sciences, Paris 2014; 352 (7-8): 593-595. doi: 10.1016/j.crma.2014.05.009

[8] Greene RE, Krantz SG. Function Theory of One Complex Variable, 3rd edition. Providence, RI, USA: AMS, 2006.

[9] Haruki H, Rassias TM. A new invariant characteristic property of Möbius transformations from the standpoint of conformal mapping. Journal of Mathematical Analysis and Applications 1994; 181 (2): 320-327. doi: 10.1006/jmaa.1994.1024

[10] Haruki H, Rassias TM. A new characteristic of Möbius transformations by use of Apollonius points of triangles. Journal of Mathematical Analysis and Applications 1996; 197: 14-22. doi: 10.1006/jmaa.1996.0002

[11] Haruki H, Rassias TM. A new characteristic of Möbius transformations by use of Apollonius quadrilaterals. Proceedings of the American Mathematical Society 1998; 126 (10): 2857-2861.

[12] Haruki H, Rassias TM. A new characterization of Möbius transformations by use of Apollonius hexagons. Proceedings of the American Mathematical Society 2000; 128 (7): 2105-2109.

[13] Krantz SG. Complex Analysis: The Geometric Viewpoint, 2nd edition. Washington, DC, USA: MAA, 2004.

[14] Niamsup P. A characterization of Möbius transformations. International Journal of Mathematics and Mathematical Sciences 2000; 24 (10): 663-666. doi: 10.1155/S0161171200010255

[15] Niamsup P. A note on the characteristics of Möbius transformations. Journal of Mathematical Analysis and Applications 2000; 248: 203-215. doi: 10.1006/jmaa.2000.6888

[16] Niamsup P. A note on the characteristics of Möbius transformations II. Journal of Mathematical Analysis and Applications 2001; 261: 151-158. doi: 10.1006/jmaa.2001.7485

[17] Suksumran T. The algebra of gyrogroups: Cayley's theorem, Lagrange's theorem, and isomorphism theorems. In: Rassias TM and Pardalos PM (editors). Essays in Mathematics and its Applications: In Honor of Vladimir Arnold. Switzerland: Springer, 2016, pp. 369-437.

[18] Suksumran T, Wiboonton K. Isomorphism theorems for gyrogroups and L-subgyrogroups. Journal of Geometry and Symmetry in Physics 2015; 37: 67-83. doi: 10.7546/jgsp-37-2015-67-83

[19] Ungar AA. Analytic Hyperbolic Geometry and Albert Einstein's Special Theory of Relativity. Hackensack, NJ, USA: World Scientific, 2008.

[20] Ungar AA. From Möbius to gyrogroups. The American Mathematical Monthly 2008; 115 (2): 138-144.

[21] Yang S. A characterization of Möbius transformations. Proceedings of the Japan Academy, Series A, Mathematical Sciences 2008; 84 (2): 35-38.