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Multiplicative Lie algebras

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Abstract: A multiplicative Lie algebra is a group together with a "bracket function" that satisfies the basic properties of the commutator function. This paper investigates the construction of such functions.

Key words: Lie algebra, perfect group, Lie product

1. Introduction

In his paper [1], Graham Ellis defined the concept of a multiplicative Lie algebra. According to his definition we have the following.

Definition 1.1 *A multiplicative Lie algebra consists of a group G together with a bracket function $\{,\}: G\times G \to G$ (called a Lie product) satisfying the following identities for all $x, y, z \in G$:*

1. $\{x, x\} = 1$,
2. $\{x, yz\} = \{x, y\}\ {}^{y}\{x, z\}$,
3. $\{xy, z\} =^{x} \{y, z\}\{x, z\}$,
4. $\{\{y, x\}, {}^{x}z\}\{\{x, z\}, {}^{z}y\}\{\{z, y\}, {}^{y}x\} = 1$,
5. ${}^{z}\{x, y\} = \{{}^{z}x, {}^{z}y\}$.

In this definition, ${}^{y}x$ is short for yxy^{-1}, $[x, y]$ is the commutator $xyx^{-1}y^{-1}$, and (iv) is a Jacobi–Witt–Hall type identity. The study of such properties began in the papers by MacDonald and Neumann ([2], [3]), who were interested in the interrelationships between various commutator laws. Graham Ellis was interested in showing that any universal commutator identity was a consequence of the identities in the above definition.

The papers by MacDonald and Neumann claimed to give a set of commutator identities from which all universal commutator identities can be deduced. However, they assumed an identity of the form $\{\{x, y\}, z\} = \{xyx^{-1}y^{-1}, z\}$ and they defined conjugation in terms of the commutator they had defined.

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2. Preliminaries

From [1] the results of the following theorem are easy consequences of Definition 1.1. We include the proofs for the sake of completeness.

Theorem 2.1 *Let G be a group. Then, for all $x, y, z, a, b \in G$, we have the following:*

1. $\{1, x\} = \{x, 1\} = 1$,
2. $\{x, y\} = \{y, x\}^{-1}$,
3. ${}^{\{x,y\}}\{a, b\} = {}^{[x,y]}\{a, b\}$ *(in particular we have that $\{x, y\}$ and $[x, y]$ must commute),*
4. $\{x^{-1}, y\} = {}^{x^{-1}}\{x, y\}^{-1}$ *and* $\{x, y^{-1}\} = {}^{y^{-1}}\{x, y\}^{-1}$,
5. $[\{x, y\}, z] = \{[x, y], z\}$.

Proof

1. Now $1 = \{1, x\} = \{1 \cdot 1, x\} = {}^{1}\{1, x\}\{1, x\} = \{1, x\}^2$. It follows that $1 = \{1, x\}$. Similarly, $\{x, 1\} = 1$.

2. Now

$$\begin{aligned} 1 = \{xy, xy\} &= {}^{x}\{y, xy\}\{x, xy\} \\ &= {}^{x}(\{y, x\}\ {}^{x}\{y, y\})\{x, x\}\ {}^{x}\{x, y\} \\ &= {}^{x}\{y, x\}\ {}^{x}\{x, y\} = {}^{x}(\{y, x\}\{x, y\}). \end{aligned}$$

   It follows that $\{y, x\}\{x, y\} = 1$, giving the result.

3. For this proof we need to compute $\{xa, yb\}$ in two different ways. First we get

$$\begin{aligned} \{xa, yb\} &= {}^{x}\{a, yb\}\{x, yb\} \\ &= {}^{x}(\{a, y\}\ {}^{y}\{a, b\})\{x, y\}\ {}^{y}\{x, b\} \\ &= {}^{x}\{a, y\}\ {}^{xy}\{a, b\}\{x, y\}\ {}^{y}\{x, b\}. \end{aligned}$$

   Secondly we get

$$\begin{aligned} \{xa, yb\} &= \{xa, y\}\ {}^{y}\{xa, b\} \\ &= {}^{x}\{a, y\}\{x, y\}\ {}^{y}({}^{x}\{a, b\}\{x, b\}) \\ &= {}^{x}\{a, y\}\{x, y\}\ {}^{yx}\{a, b\}^{y}\{x, b\}. \end{aligned}$$

   Canceling like terms gives

$${}^{xy}\{a, b\}\{x, y\} = \{x, y\}\ {}^{yx}\{a, b\}.$$

   Now, replacing a by $a^{x^{-1}y^{-1}}$ and b by $b^{x^{-1}y^{-1}}$ gives

$${}^{[x,y]}\{a, b\}\{x, y\} = \{x, y\}\{a, b\}.$$

Thus,

$$^{[x,y]}\{a,b\} = ^{\{x,y\}}\{a,b\},$$

as required.

4. Now

$$1 = \{x^{-1}x, y\} = ^{x^{-1}}\{x,y\}\{x^{-1},y\}.$$

It follows that $\{x^{-1},y\} = ^{x^{-1}}\{x,y\}^{-1}$. Similarly, $\{x,y^{-1}\} = ^{y^{-1}}\{x,y\}^{-1}$.

5. Now

$$\begin{aligned}
[\{x,y\},z] &= \{x,y\}\ ^{z}\{x,y\}^{-1} \text{ and by } 4\\
&= \{x,y\}\ ^{zx}\{x^{-1},y\}\\
&= \{x,y\}\ ^{[z,x]xz}\{x^{-1},y\} \text{ and by } 3\\
&= \{x,y\}\ ^{\{z,x\}xz}\{x^{-1},y\}\\
&= \{x,y\}\{x,z\}^{-1}\ ^{xz}\{x^{-1},y\}\{x,z\} \text{ again by } 4\\
&= \{x,y\}\ ^{x}\{x^{-1},z\}\ ^{xz}\{x^{-1},y\}\{x,z\}\\
&= \{x,y\}\ ^{x}(\{x^{-1},z\}\ ^{z}\{x^{-1},y\})\{x,z\}\\
&= \{x,y\}\ ^{x}\{x^{-1},zy\}\{x,z\} \text{again by } 4\\
&= ^{x}\{x^{-1},y\}^{-1}\ ^{x}\{x^{-1},yz\}\{x,z\}\\
&= ^{x}\{x^{-1},y\}^{-1}\ ^{x}\{x^{-1},y(y^{-1}zy)\}\{x,z\}\\
&= ^{x}\{x^{-1},y\}\ ^{x}(\{x^{-1},y\}^{y}\{x^{-1},y^{-1}zy\})\{x,z\}\\
&= ^{xy}\{x^{-1},y^{-1}zy\}\{x,z\}\\
&= ^{x}\{yx^{-1}y^{-1},z\}\{x,z\}\\
&= \{xyx^{-1}y^{-1},z\} = \{[x,y],z\}, \text{ as required.}
\end{aligned}$$

This completes the proof. □

Now let us look at some examples.

Example 2.2 *Let G be a group. We can make G into a multiplicative Lie algebra by defining either for all $x,y \in G, \{x,y\} = 1$ or for all $x,y \in G, \{x,y\} = [x,y] = xyx^{-1}y^{-1}$. If these are the only possible Lie products that can be defined on G, we say the trivial consequence holds for G.*

Example 2.3 *Any Lie algebra over $\mathbb{Z}$ is a multiplicative Lie algebra with $\{x,y\}$ defined to be the ordinary Lie bracket.*

Example 2.4 *(Ellis [1]) Let E be a group and let $P = \frac{E}{Z(E)}$. Define an action of P on E by for $u \in E, x \in P$ (letting $x = \overline{x}Z(E), \overline{x} \in E), ^{x}u = \overline{x}u\overline{x}^{-1}$. Let G be the semidirect product of E by P using the above action. Then, $\{(u_1,x_1),(u_2,x_2)\} = ([u_1\overline{x_1}, u_2\overline{x_2}])$ defines a Lie product on G, which is in general different from the usual commutator defined on G.*

Example 2.5 *In general, suppose that* G *is a group,* $H \leq G$*, and* $f : G \to H$ *is a homomorphism so that for all* $x \in G, x^{-1}f(x) \in C_G(H)$ *(note that if* $G = H \times K$*, then* π_H*, the projection function onto* H*, is such a homomorphism). Then defining for all* $x, y \in G$ $\{x, y\} = [f(x), f(y)]$ *gives a Lie product on* G*. Furthermore, if* $H \leq G$ *and* $G = HC_G(H)$*, we can define a Lie product on* G *by defining for* $x = h_1k_1$ *and* $y = h_2k_2$ *with* $h_1, h_2 \in H, k_1, k_2 \in K, \{x, y\} = [h_1, h_2]$*.*

Example 2.6 *Let* $G = \langle a \rangle \times \langle b \rangle$ *and suppose that* $x \in G$ *is such that* $|x|$ *divides both* $|a|$ *and* $|b|$ *(here we are assuming that anything will divide infinity). Now we can define a Lie product on* G *by* $\{a^{i_1}b^{j_1}, a^{i_2}b^{j_2}\} = x^{i_1j_2-i_2j_1}$*.*

Here are a few remarks.

Remark 2.7 *If* $\mathbb{Q}$ *is the additive group of rational numbers, then if* $\{,\}$ *is a Lie product defined on* $\mathbb{Q}$*, we must have for all* $x, y \in \mathbb{Q}, \{x, y\} = 0$*.*

Remark 2.8 *Let* F *be any free group. Then, if* $\{,\}$ *is a Lie product defined on* F*, we must have either for all* $x, y \in F, \{x, y\} = 1$ *or for all* $x, y \in F, \{x, y\} = xyx^{-1}y^{-1} = [x, y]$*. That is, the trivial consequence must hold for free groups.*

The results of the last two remarks could be determined directly from the definition of a Lie product, but as they will follow from some general results given later, their proofs are omitted for now. The last remark is actually found in [1] and in [3]. These two remarks serve to motivate the following question.

Question For which groups must the trivial consequence hold?

3. Some results

Note that any subgroup of a group that can be defined in terms of commutators will have an analog defined by a given Lie product. We will indicate (in general) these subgroups by using script in the usual notations. Thus, if G is a group and $\{,\}$ is a Lie product of G, we define $\mathcal{G}' := \langle \{\{x, y\}|\ x, y \in G\} \rangle$ and $\mathcal{Z}(G) := \{y \in G|\ \{x, y\} = 1 \text{ for all } x \in G\}$. Note that both $\mathcal{G}'$ and $\mathcal{Z}(G)$ are normal subgroups of G. Also, if H and K are subsets of G, we define $\{H, K\} := \langle \{\{H, K\}|\ h \in H, k \in K\} \rangle$. In particular, $\mathcal{G}' = \{G, G\}$.

The next result is a slight extension of the above remarks:

Lemma 3.1 *Let* G *be a group with Lie product* $\{,\}$*. Then we must have:*

1. $\{C_G(\mathcal{G}'), G'\} = 1$*,*
2. *for all* $x, y \in G$ $\{x, y\}^{-1}[x, y] \in C_G(\mathcal{G}')$*,*
3. $G' \leq \mathcal{G}'C_G(\mathcal{G}')$*,*
4. *For all* $a, b, c, d \in G, \{\{a, b\}^{-1}[a, b], \{c, d\}^{-1}[c, d]\} = \{\{a, b\}, \{c, d\}\}[\{a, b\}, \{c, d\}]^{-1}$*,*
5. *If* $\{C_G(\mathcal{G}), C_G(\mathcal{G})\} = 1$*, then for all* $a, b, c, d \in G$*, we get* $\{\{a, b\}, \{c, d\}\} = [\{a, b\}, \{c, d\}]$*.*

Proof

1. From Theorem 2.1 parts 5 and 2 we know that $\{x,[a,b]\}=[x,\{a,b\}]$ for all $x,a,b\in G$. Now if $x\in C_G(\mathcal{G}')$ we get $\{x,[a,b]\}=1$. The result follows.

2. This follows directly from Theorem 2.1 3.

3. This follows from 2.

4. Now

$$\begin{aligned}
\{\{a,b\}^{-1}[a,b],\{c,d\}^{-1}[c,d]\} &= {}^{\{a,b\}^{-1}}\{[a,b],\{c,d\}^{-1}[c,d]\}\{\{a,b\}^{-1},\{c,d\}^{-1}[c,d]\} \\
&= \{\{a,b\}^{-1},\{c,d\}^{-1}[c,d]\} \text{ by 1 and 2} \\
&= \{\{a,b\}^{-1},\{c,d\}^{-1}\}\ {}^{\{c,d\}^{-1}}\{\{a,b\}^{-1},[c,d]\} \text{ by Theorem 2.1 4} \\
&= {}^{\{a,b\}^{-1}}\{\{a,b\},\{c,d\}^{-1}\}^{-1}\ {}^{\{c,d\}^{-1}\{a,b\}^{-1}}\{\{a,b\},[c,d]\}^{-1} \text{ by Theorem 2.1 2} \\
&= {}^{\{a,b\}^{-1}\{c,d\}^{-1}}\{\{a,b\},\{c,d\}\}\ {}^{\{c,d\}^{-1}\{a,b\}^{-1}}\{\{a,b\},[c,d]\}^{-1}.
\end{aligned}$$

Now since $\{a,b\}^{-1}[a,b]$ and $\{c,d\}^{-1}[c,d]$ are in $C_G(\mathcal{G}')$ we obtain

$$\begin{aligned}
\{\{a,b\}^{-1}[a,b],\{c,d\}^{-1}[c,d]\} &= {}^{[\{a,b\},\{c,d\}]}\{\{a,b\},\{c,d\}\}\{\{a,b\},[c,d]\}^{-1} \\
&= \{\{a,b\},\{c,d\}\}\{\{a,b\},[c,d]\}^{-1} \\
&= \{\{a,b\},\{c,d\}\}[\{a,b\},\{c,d\}]^{-1} \text{ by using Theorem 2.1 5.}
\end{aligned}$$

Note that we have used the fact from Theorem 2.1 3 that $\{x,y\}$ and $[x,y]$ must commute.

5. This follows from 4.

□

We can use this result to prove the remark about free groups (Remark 2.8).

Theorem 3.2 *Let F be a free group. Then, if $\{,\}$ is a Lie product defined on F, we must have either for all $x,y\in F,\{x,y\}=1$ or for all $x,y\in F,\{x,y\}=xyx^{-1}y^{-1}=[x,y]$.*

Proof The important fact that we need about (nonabelian) free groups is that the centralizer of a nontrivial normal subgroup must be trivial. Note that Theorem 2.1 3 implies that for all $a,b,x,y\in F,\{x,y\}^{-1}[x,y]\in C_G(\{a,b\})$. Hence, we must have for all $x,y\in F,\{x,y\}^{-1}[x,y]\in C_F(\mathcal{F}')$.

It follows that either $\mathcal{F}'=1$ and thus for all $x,y\in F,\{x,y\}=1$ or $C_F(\mathcal{F}')=1$ and thus for all $x,y\in F,\{x,y\}=[x,y]$. □

Theorem 3.3 *Let G be a group having the following property:*

for all $1\neq H\lhd G$ so that $G'\leq HC_G(H)$ we must have $C_G(H)=1$. (*)

Then the trivial consequence must hold for G.

Proof From Lemma 3.1 3 we have $G' \leq \mathcal{G}'C_G(\mathcal{G}')$. The property (*) now implies that either $\mathcal{G}' = 1$ and for all $x, y \in G, \{x, y\} = 1$ or $C_G(\mathcal{G}') = 1$, which implies, as above, for all $x, y \in G, \{x, y\} = [x, y]$, as required. □

The next result is a slight extension of the following results.

Theorem 3.4 *Let G be a group and suppose that $\{,\}$ is a Lie product defined on G. Define $\ell : G\times G \to C_G(\mathcal{G}')$ by for all $x, y \in G, \ell(x, y) = \{x, y\}^{-1}[x, y]$. Then ℓ satisfies properties 1, 2, 3, and 5 of the definition of a Lie product, Definition 1.1. Furthermore, $\ell_1 = \ell|_{C_G(\mathcal{G}')}$ is a Lie product on $C_G(\mathcal{G}')$.*

Proof We show that ℓ and ℓ_1 satisfy the appropriate conditions.
For all $x, y, z \in G$:

$$\begin{aligned}
(i)\ \ell(x,x) &= \{x,x\}^{-1}[x,x] = 1,\\
(ii)\ \ell(x,yz) &= \{x,yz\}^{-1}[x,yz] = (\{x,y\}\ {}^{y}\{x,z\})^{-1}([x,y]\ {}^{y}[x,z])\\
&= {}^{y}\{x,z\}^{-1}(\{x,y\}^{-1}[x,y])\ {}^{y}[x,z]\\
&= (\{x,y\}^{-1}[x,y])\ {}^{y}(\{x,z\}^{-1}[x,z])\\
&= \ell(x,y)\ {}^{y}\ell(x,z),\\
(iii)\ \ell(xy,z) &= {}^{x}\ell(y,z)\ell(x,z) \text{ is similar to (ii)},\\
(v)\ {}^{z}\ell(x,y) &= {}^{z}(\{x,y\}^{-1}[x,y]) = \{{}^{z}x,{}^{z}y\}^{-1}\ [{}^{z}x,{}^{z}y]\\
&= \ell({}^{z}x,{}^{z}y).
\end{aligned} \tag{3.1}$$

In the next part of the proof we are assuming that $x, y, z \in C_G(\mathcal{G}')$.

(iv) Note that

$$\begin{aligned}
\ell(\ell(y,x),{}^{x}z) &= \ell(\{y,x\}^{-1}[y,x],{}^{x}z)\\
&= {}^{\{y,x\}^{-1}}\ell([y,x],{}^{x}z)\ell(\{y,x\}^{-1},{}^{x}z)\\
&= \ell([y,x],{}^{x}z)\ell(\{y,x\}^{-1},{}^{x}z)\\
&= \{[y,x],{}^{x}z\}^{-1}[[y,x],{}^{x}z]\{\{y,x\}^{-1},{}^{x}z\}[\{y,x\}^{-1},{}^{x}z]\\
&= \text{ by Proposition 2.14}\\
&= [\{y,x\},{}^{x}z]^{-1}[[y,x],{}^{x}z]\ {}^{\{y,x\}^{-1}}\{\{y,x\},{}^{x}z\}\\
&= \text{ by Proposition 2.14, 5}\\
&= [[y,x],{}^{x}z]\{\{y,x\},{}^{x}z\}.
\end{aligned}$$

Thus, $\ell(\ell(y,x),{}^{x}z)\ell(\ell(x,z),{}^{z}y)\ell(\ell(z,y),{}^{y}x) =$

$$\begin{aligned}
&= [[y,x],{}^{x}z]\{\{y,x\},{}^{x}z\}[[x,z],{}^{z}y]\{\{x,z\},{}^{z}y\}[[z,y],{}^{x}z]\{\{y,x\},{}^{x}z\}\\
&= [[y,x],{}^{x}z][[x,z],{}^{z}y][[z,y],{}^{z}y]\{\{y,x\},{}^{x}z\}\{\{x,z\},{}^{z}y\}\{\{z,y\},{}^{y}x\} = 1.
\end{aligned}$$

Note that we are using the fact that if $x \in C_G(\mathcal{G}')$, then $[[a,b],x] \in C_G(\mathcal{G}')$.

□

Corollary 3.5 *Let G be a group and suppose that $\{,\}$ is a Lie product defined on G. Assume that $C_G(\mathcal{G}')$ is an abelian group for which the trivial consequence must hold. (That is, we must have $\{C_G(\mathcal{G}'), C_G(\mathcal{G}')\} = 1$.) Then, for all $x, y \in C_G(\mathcal{G}')\mathcal{G}'$, we must have $\{x, y\} = [x, y]$. Furthermore, if G is a perfect group, then for all $x, y \in G$, we must have $\{x, y\} = [x, y]$.*

Proof Since $\ell(x, y) = 1$ for all $x, y \in C(\mathcal{G}')$, we must have for all $a, b, c, d \in G$ that $\{\{a, b\}^{-1}[a, b], \{c, d\}^{-1}[c, d]\} = 1$. Some simple calculations similar to the above calculations (see Theorem 3.1 3) give that for all $a, b, c, d \in G, \{\{a, b\}, \{c, d\}\} = [\{a, b\}, \{c, d\}]$.

It then follows that for all $x, y \in \mathcal{G}', \{x, y\} = [x, y]$. Now some easy calculations give that for all $x, y \in C_G(\mathcal{G}')\mathcal{G}', \{x, y\} = [x, y]$. The last comment follows from the fact that $G' \leq C_G(\mathcal{G}')\mathcal{G}'$ (Theorem 3.1 2). □

Lemma 3.6 *Suppose that G is a perfect group and $C_G(\mathcal{G}')$ is abelian. Then $C_G(\mathcal{G}') = Z(G)$.*

Proof Since G is perfect, we have from Theorem 3.1 3 that $G = C_G(\mathcal{G}')\mathcal{G}'$. It follows that $[G, C_G(\mathcal{G}')] = [C_G(\mathcal{G}'), C_G(\mathcal{G}')] = 1$. The result follows. □

This result was an inspiration for the following theorem.

Theorem 3.7 *Let G be a group and suppose that there is a function $g : \frac{G}{G'} \times \frac{G}{G'} \to Z(G)$ that satisfies the following conditions for all $x, y, z \in \frac{G}{G'}$:*

1. $g(xy, z) = g(x, z)g(y, z)$,
2. $g(x, yz) = g(x, y)g(x, z)$,
3. $g(x, x) = 1$,
4. $g(g(y, x)G', {}^x y)g(g(x, z)G', {}^z y)g(g(z, y)G', {}^y x) = 1$.

Then, for all $x, y \in G$, we can define $\{x, y\} = [x, y]g(xG', yG')$. This defines a Lie product on G. Furthermore, if G is a group having a Lie product such that $C_G(\mathcal{G}') = Z(G)$, then the Lie product on G must have arisen in this way.

Proof First, we will show that if g has the desired properties, then $\{,\}$ does satisfy the properties to be a Lie product.

1. $\{x, x\} = [x, x]g(xG', xG') = 1$,
2. $$\begin{aligned}\{xy, z\} = [xy, z]g(xyG', zG') &= {}^x[y, z][x, z]g(xG', zG')g(yG', zG') \\ &= {}^x([y, z]g(yG', zG'))[x, z]g(xG', zG') = {}^x\{y, z\}\{x, z\},\end{aligned}$$
3. $\{x, yz\} = \{x, y\}\ {}^y\{x, z\}$ and the proof is similar to the proof of 2.

4. Now notice that

$$\{\{y,x\},^{x}z\} = \{[y,x]g(yG',xG',^{x}z\} \tag{3.2}$$
$$= \{g(yG',xG')[y,x],^{x}z\} \tag{3.3}$$
$$= {}^{g(yG',xG')}\{[y,x],^{x}z\}\{g(yG',xG'),^{x}z\} \tag{3.4}$$
$$= [[y,x],^{x}z]g([y,x]G',zG')\{g(g(yG',xG')G',^{x}zG'\} \tag{3.5}$$
$$= [[y,x],^{x}z]g(g(yG',xG')G',^{x}zG'), \tag{3.6}$$

where we have used the fact that $g(1,x)=1$.

Using similar reasoning we can conclude that

$$\begin{aligned}\{\{y,x\},^{x}z\}\{\{x,z\},^{x}y\}\{\{z,y\},^{y}x\} &= [[y,x],^{x}z]g(g(yG',xG'),^{x}zG')\\ &\quad [[x,z],^{z}y]g(g(xG',zG'),^{z}yG')\\ &\quad [[z,y],^{y}x]g(g(zG',yG'),^{y}xG')\\ &= [[y,x],^{x}z][[x,z],^{z}y][[z,y],^{y}x]\\ &\quad g(g(yG',xG')G',^{x}zG')g(g(xG',zG')G',^{z}G')\\ &\quad g(g(zG',yG')G',^{y}xG') = 1, \text{ as required.}\end{aligned}$$

5. $\{^{z}x,^{z}y\} = [^{z}x,^{z}y]g(^{z}xG',^{z}yG') =^{z} ([x,y]g(xG',yG')) =^{x} \{x,y\}$. Thus, we have shown that $\{,\}$ has all the properties to be a Lie product.

Now suppose that G is a group having a Lie product defined on it and so $C_G(\mathcal{G}')=Z(G)$. Define $\ell : G\times G \to Z(G)$ by $\ell(x,y)=\{x,y\}^{-1}[x,y]$ as in Theorem 3.4. Now by Theorem 3.4 we know that for all $x,y,z\in G$ we have $\ell(x,x)=1, \ell(xy,z)=\ell(x,z)\ell(y,z), \ell(x,yz)=\ell(x,y)\ell(y,z)$. It follows that $\ell(x,1)=\ell(1,x)=1$ and $\ell(x^{-1},y)=\ell(x,y^{-1})=\ell(x,y)^{-1}$. Hence, $\ell(x,[y,z])=1$. Thus, if $x\in G$ and $w\in G'$, then $\ell(x,w)=\ell(w,x)=1$.

We define $g : \frac{G}{G'}\times\frac{G}{G'} \to Z(G)$ by saying for all $x,y\in G$, that $g(xG',yG')=\ell(x,y)^{-1}$. Note that if $xG'=x_1G'$ and $yG'=y_1G'$, then $x_1^{-1}x, y_1^{-1}y\in G'$. It follows that $\ell(x_1,y_1)=\ell(x,y_1)=\ell(x,y)$ and hence G is well defined. Now using the properties of ℓ and the fact that g maps into the center, it is easy to check that g satisfies the appropriate conditions above.

Hence, for all $x,y\in G, g(xG',yG')^{-1}=\ell(x,y)=\{x,y\}^{-1}[x,y]$. It follows that $\{x,y\}=[x,y]g(xG',yG')$ is defined as in the theorem, as required. □

Remark 3.8 *If G is a dihedral group of order 2^n, then the function g of Theorem 3.7 can be viewed as a homomorphism from the alternating tensor square of $\frac{G}{G'}\cong\mathbb{Z}_2\times\mathbb{Z}_2$ into $Z(G)\cong\mathbb{Z}_2$. As the alternating tensor square of $\mathbb{Z}_2\times\mathbb{Z}_2$ is a module over $\mathbb{Z}_2$ of dimension 3, we can construct functions g satisfying the conditions of Theorem 3.7. It follows that the trivial consequence can never hold for any dihedral 2-group.*

Remark 3.9 *If G is an abelian group, then for all subgroups H of G we must have $C_G(H)=Z(G)=G$. Hence, by Theorem 3.7 all possible Lie products must arise as in the theorem. Notice that all such functions g must factor through the alternating tensor square of G. It follows that if G is an abelian group with trivial alternating square (such as a rank 1 group), then the trivial consequence must hold for G.*

The following corollary is a slight extension of Corollary 3.5

Corollary 3.10 *Let $1 \neq G$ be a perfect group with a Lie product $\{,\}$ so that $C_G(\mathcal{G}')$ is abelian. Then, for all $x, y \in G$, we have $\{x, y\} = [x, y]$.*

Proof We know from Lemma 3.1 3 that $G' \leq C_G(\mathcal{G}')\mathcal{G}'$. Thus, $G = C_G(\mathcal{G}')\mathcal{G}'$. Since G is perfect and $C_G(\mathcal{G}')$ is abelian, we must have $G = \mathcal{G}'$. Hence, $C_G(\mathcal{G}') = Z(G)$. Now the result follows from Theorem 3.7. □

The next result allows us to determine the possible Lie products for perfect groups.

Theorem 3.11 *Suppose that G is a perfect group with Lie product $\{,\}$. Then there is a subgroup H of G that is perfect so that $G = HC_G(H)$ (that is, G is a central product), and for all $x, y \in G$ with $x = h_1k_1, y = h_2k_2, h_1, h_2 \in H, k_1, k_2 \in C_G(H)$, we have $\{x, y\} = [h_1, h_2]$.*

Proof Let $H = \mathcal{G}', K = C_G(\mathcal{G}')$. As above, since G is perfect, $G = HK$. Note that $H', K' \lhd G$, and since $\frac{G}{H'K'}$ is abelian, we get $G = H'K'$. Now for $x, y \in G$, we can write $x = h_1k_1, y = h_2k_2$ with $h_1, h_2 \in H'$ and $k_1, k_2 \in K'$. This gives:

$$\begin{aligned} \{x,y\} = \{h_1k_1, h_2k_2\} &= {}^{h_1}\{k_1, h_2k_2\}\{h_1, h_2k_2\} \\ \text{by Lemma 3.1} &= \{h_1, h_2k_2\} \\ &= \{h_1, h_2\}\, {}^{h_2}\{k_1, k_2\} \\ \text{again by Lemma 3.1} &= \{h_1, h_2\}. \end{aligned}$$

Notice that we have from Theorem 2.1 5 that $\{[x, y], h_2\} = [\{x, y\}, h_2] \in H'$ and so it follows that in the above equations $\{h_1, h_2\} \in H'$. Thus, for all $x, y \in G$, we have $\{x, y\} \in H'$. Hence, $H = H'$ and H is perfect. Also note that $\{,\}$ defines a Lie product on the group H and with respect to this Lie product that $\mathcal{H}' = \{H, H\} = H$. It follows that $C_H(\mathcal{H}') = Z(H)$. Now by Corollary 3.10 we must have for all $x, y \in H$, that $\{x, y\} = [x, y]$, as required. □

We give one more result and a few corollaries. This next result gives further information about the structure of $C_G(\mathcal{G}')$ for a group with a Lie product. In this result we again let $\mathcal{Z}(\mathcal{G}) = \{g \in G |\ \{g, x\} = 1, \text{ for all } x \in G\}$.

Theorem 3.12 *Let G be a group with Lie product $\{,\}$. Then $\frac{C_G(\mathcal{G}')}{\mathcal{Z}(\mathcal{G})}$ is isomorphic to a subgroup of $\prod_{y \in G}(C_G(\mathcal{G}') \cap C_G(G'))$. In particular, $\frac{C_G(\mathcal{G}')}{\mathcal{Z}(\mathcal{G})}$ must be nilpotent of class ≤ 2.*

Proof Note that for all $a, b, y \in G, x \in C_G(\mathcal{G}')$, we have by Theorem 2.1 5 that $[\{y, x\}, [a, b]] = \{[y, x], [a, b]\} = [[y, x], \{a, b\}] = 1$, as $[y, x] \in C_G(\mathcal{G}')$. It follows that $\{y, x\} \in C_G(G')$. Furthermore, $\{y, x\} \in C_G(\mathcal{G}')$ since both $[y, x]$ and $\{x, y\}^{-1}[x, y] \in C_G(\mathcal{G}')$, by Lemma 3.1 2. Now we can define for all $y \in G, T_y : C_G(\mathcal{G}') \to C_G(\mathcal{G}') \cap C_G(G')$ by $T_y(x) = \{y, x\}$. Note that for all $y \in G$ we have $T_y(x_1x_2) = \{y, x_1x_2\} = \{y, x_1\}\, {}^{x_1}\{y, x_2\} = \{y, x_1\}\{y, x_2\}$ and we have that each T_y is a homomorphism. Notice that

$$\bigcap_{y \in G} \ker(T_y) = \{x \in C_G(\mathcal{G}') |\ \{y, x\} = 1, \text{ for all } y \in G\} = \mathcal{Z}(\mathcal{G}).$$

Note that it is clear that $\mathcal{Z}(\mathcal{G}) \leq C_G(\mathcal{G}')$. The result now follows as $C_G(G')$ is nilpotent of class ≤ 2. □

Corollary 3.13 *Let G be a group having a Lie product $\{,\}$ so that $C_G(\mathcal{G}') \cap C_G(G') = 1$. Then for all $x \in G, y \in \mathcal{G}'$, we must have $\{x, y\} = [x, y]$.*

Proof By Theorem 3.12 we have $C_G(\mathcal{G}') = \mathcal{Z}(\mathcal{G})$. It follows that for all $a, b, c \in G$ we have $\{\{a,b\}^{-1}[a,b], c\} = 1$. Thus, we obtain

$$\begin{aligned} 1 &= \{\{a,b\}^{-1}[a,b], c\} \\ &= {}^{\{a,b\}^{-1}}\{[a,b], c\}\{\{a,b\}^{-1}, c\} \\ \text{by lemma 2.1 4} &= {}^{\{a,b\}^{-1}}\{[a,b], c\}\ {}^{\{a,b\}^{-1}}\{\{a,b\}, c\}^{-1} \\ &= {}^{\{a,b\}^{-1}}(\{[a,b], c\}\{\{a,b\}, c\}^{-1}). \end{aligned}$$

It follows that $\{[a,b], c\} = \{\{a,b\}, c\}$. Now using Lemma 2.1 5 we get

$$[\{a,b\}, c] = \{[a,b], c\}.$$

The result now follows from a simple calculation. □

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