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Tauberian conditions under which convergence follows from summability by the discrete power series method

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Abstract: In this paper, we obtain Tauberian conditions to recover convergence of a series from its discrete power series summability under certain conditions. As special cases of our main results, we get discrete analogues of some well-known Tauberian theorems in the literature.

Key words: Tauberian conditions, discrete summability, power series methods

1. Introduction and preliminaries

Let $\sum_{k=0}^{\infty} a_k$ be a series of real or complex numbers and (s_n) be its corresponding sequence of partial sums. We suppose throughout this paper that (p_n) is nonnegative for all n with $p_0 > 0$ and satisfies

$$P_n = \sum_{k=0}^{n} p_k \to \infty, \quad n \to \infty. \tag{1.1}$$

The weighted mean of (s_n) is given by the sequence

$$\sigma_n = \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k.$$

Assume that the radius of convergence of the power series $p(x) = \sum_{k=0}^{\infty} p_k x^k$ is 1. Let

$$p_s(x) = \frac{1}{p(x)} \sum_{k=0}^{\infty} p_k s_k x^k.$$

Now, we give definitions of some well-known summability methods which will be used in the sequel.

(i) The weighted mean method.

The series $\sum_{k=0}^{\infty} a_k$ is called $(\bar{N}, p)$ summable by the weighted mean method determined by the sequence p; in short, $(\bar{N}, p)$ summable to L, if

$$\lim_{n\to\infty} \sigma_n = L.$$

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(ii) The power series method (P).

The series $\sum_{k=0}^{\infty} a_k$ is called summable (P) to L, if $p_s(x)$ exists for each $x \in (0,1)$ and if

$$\lim_{x \to 1^-} p_s(x) = L.$$

(iii) The discrete power series method (P_λ).

Let (λ_n) be a strictly increasing sequence such that $\lambda_0 \geq 1$ and $\lambda_n \to \infty$ as $n \to \infty$ and define the sequence (x_n) by $x_n = 1 - \lambda_n^{-1}$ for all n. It is obvious that $0 \leq x_0 < x_1 < ... < x_n \to 1$. The series $\sum_{k=0}^{\infty} a_k$ is called summable (P_λ) to L, if $p_s(x_n)$ exists for each n and if

$$\lim_{n \to \infty} p_s(x_n) = L.$$

As opposed to the method (P) in which $x \to 1^-$ continuously we refer to (P_λ) as a discrete method, since $x_n \to 1^-$ through a sequence of numbers. The method (P_λ) reduces to the method (A_λ) (the discrete Abel method) when $p_k = 1$; to the method (L_λ) (the discrete logarithmic method) when $p_k = \frac{1}{k+1}$.

A summability method is said to be regular, if it sums every convergent series to its ordinary sum. The condition (1.1) assures the regularity of all three methods $(\bar{N}, p)$, (P) and (P_λ) (see Watson [17]). Given two summability methods A, and B, we write $A \subseteq B$ and say B includes A if a series summable A is summable B to the same value. If there is a series summable B but not summable A, we say B strictly includes A and write $A \subset B$. If $A \subseteq B$ and $B \subseteq A$, two methods are called equivalent and we write $A \simeq B$.

It is known that $(\bar{N}, p) \subseteq (P)$ (see Ishiguro [6]). Watson [17] obtained the following inclusion results for the (P_λ) method.

Theorem 1.1 *Let $E_\lambda = \{\lambda_n : n = 0, 1, 2, ...\}$ and $E_\mu = \{\mu_n : n = 0, 1, 2, ...\}$. Then*

(i) $(P_\lambda) \subseteq (P_\mu)$ *if and only if $E_\mu \backslash E_\lambda$ is a finite set,*

(ii) $(P_\lambda) \simeq (P_\mu)$ *if and only if the symmetric difference $E_\mu \Delta E_\lambda$ is a finite set,*

(iii) *For any λ, $(P) \subset (P_\lambda)$.*

Recently, conditions under which (P) and (P_λ) methods are equivalent for bounded sequences have been determined in our previous work [12].

The interest in summability methods is that they allow for a better understanding of divergent series. In general, any theorem asserting the regularity of a method of summation is said to be an Abelian theorem. The direct converse of an Abelian theorem is not always true, since if a regularity theorem for a method of summability is reversible, then the method would be trivial because it applies just to convergent series. It is therefore significant to get recovered forms of conditional converses to Abelian theorems, by imposing additional restrictions. Such restrictions are called Tauberian conditions, and the conditionally converse results, Tauberian theorems, honoring Alfred Tauber who first obtained a result of this type.

Tauberian theorems for continuous-type power series methods have a long history; see for example the books in [5, 8] and related recent papers in [2, 4, 13, 16]. The studies for the discrete-type summability methods began with the work of Armitage and Maddox [1] on discrete Abel means (A_λ). Maddox [9, 10] proved several Tauberian theorems for the summability (A_λ). Later, Watson [17] introduced the discrete power series summability (P_λ) and generalized the (A_λ) method. Further, Patterson et al. [11] extended the (P_λ) method of Watson [17] by using Bürmann series and obtained a Tauberian theorem. Çanak and Totur [3] gave a Tauberian theorem for the discrete M_φ summability method and as a special case they obtained a Tauberian theorem for the discrete logarithmic summability method. Furthermore, converse theorems for the discrete Bürmann power series summability are established in our previous work [14]. In this study we present new Tauberian conditions for the summability (P_λ). Our main results improve some well-known Tauberian theorems in the literature.

2. Main results

In this section, we use capital letter C to denote a positive number independent from the variable under consideration which is not necessarily the same at each occurence. We also adopt the following familiar conventions:

(i) $f_n = o(g_n)$ means $f_n/g_n \to 0$ as $n \to \infty$,

(ii) $f_n = O(g_n)$ means $|f_n| \leq Cg_n$ for large enough n.

In [18], Watson proved the following Tauberian theorem for the (P_λ) method as a generalization of Ishiguro's [6] result for the summability (P).

Theorem 2.1 *Assume that*

$$\frac{P_n}{p(x_n)} = O(1), \quad n \to \infty, \tag{2.1}$$

$$0 < p_n \leq C \textit{ for all } n \geq 0, \tag{2.2}$$

and

$$\lambda_n = O(P_n). \tag{2.3}$$

If $\sum_{n=0}^{\infty} a_n$ *is summable* (P_λ) *to* L *and*

$$a_n = o\left(\frac{p_n}{P_{n-1}}\right), \quad n \to \infty, \tag{2.4}$$

then $\sum_{n=0}^{\infty} a_n$ *converges to* L.

Since the weighted mean method $(\overline{N}, p)$ is regular, (2.4) implies

$$\sum_{k=0}^{n} P_{k-1} a_k = o(P_n), \tag{2.5}$$

where $P_{-1} = 0$. As $\sum_{k=0}^{n} k a_k = o(n)$ is a well-known Tauberian condition for the Abel summability (Tauber's second theorem, [15]), one may think that (2.5) is a Tauberian condition for the (P_λ) summability. We prove the following theorem as an extension of Tauber's second theorem. It also generalizes Theorem 2.1.

Theorem 2.2 *Let the conditions (2.1), (2.2), and (2.3) be satisfied. If* $\sum_{k=0}^{\infty} a_k$ *is summable* (P_λ) *to* L *and*

$$\sum_{k=0}^{n} P_{k-1} a_k = o(P_n), \quad n \to \infty, \tag{2.6}$$

then $\sum_{k=0}^{\infty} a_k$ *converges to* L.

Proof Let

$$v_n = \frac{1}{P_n} \sum_{k=0}^{n} P_{k-1} a_k, \ P_{-1} = 0. \tag{2.7}$$

We then obtain

$$\begin{aligned}
\frac{1}{P_n} \sum_{k=0}^{n} p_k s_k &= \frac{1}{P_n} \{p_0 s_0 + p_1 s_1 + ... + p_n s_n\} \\
&= \frac{1}{P_n} \{p_0 a_0 + p_1 (a_0 + a_1) + ... + p_n (a_0 + a_1 + ... + a_n)\} \\
&= \frac{1}{P_n} \{(p_0 + p_1 + ... + p_n) a_0 + (p_1 + p_2 + ... + p_n) a_1 + ... + p_n a_n\} \\
&= \frac{1}{P_n} \sum_{k=0}^{n} (P_n - P_{k-1}) a_k \\
&= s_n - \frac{1}{P_n} \sum_{k=0}^{n} P_{k-1} a_k.
\end{aligned}$$

Consequently, from the last identity above and (2.7) we get

$$\sigma_n = s_n - v_n. \tag{2.8}$$

By (2.6) and the regularity of (P_λ) method, (v_n) is summable (P_λ) to 0. Since $\sum_{k=0}^{\infty} a_k$ is summable (P_λ) to L, it follows from (2.8) that (σ_n) is also summable (P_λ) to L. Besides, considering (2.8) we have

$$\begin{aligned}
\frac{P_{n-1}}{p_n} \Delta \sigma_n &= \frac{P_{n-1}}{p_n} \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k - \frac{P_{n-1}}{p_n} \frac{1}{P_{n-1}} \sum_{k=0}^{n-1} p_k s_k \\
&= \frac{P_{n-1}}{p_n} \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k - \frac{1}{p_n} \sum_{k=0}^{n} p_k s_k + s_n \\
&= \frac{P_{n-1}}{p_n} \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k - \frac{P_n}{p_n} \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k + s_n \\
&= s_n - \frac{1}{P_n} \sum_{k=0}^{n} p_k s_k \\
&= v_n \\
&= \frac{1}{P_n} \sum_{k=0}^{n} P_{k-1} a_k
\end{aligned}$$

Therefore, (2.6) implies that $\Delta\sigma_n = o\left(\frac{p_n}{P_{n-1}}\right)$. Now, applying Theorem 2.1 to the sequence (σ_n) we get $\lim_{n\to\infty}\sigma_n = L$. Taking (2.6) and (2.8) into account, we conclude $\lim_{n\to\infty} s_n = \sum_{k=0}^{\infty} a_k = L$. □

Corollary 2.3 *Let $p_k > 0$ for all $k = 0, 1, 2, \ldots$ and the conditions (2.1), (2.2), and (2.3) be satisfied. If $\sum_{k=0}^{\infty} a_k$ is summable (P_λ) to L and*

$$\sum_{k=0}^{n} |a_k|^p \frac{P_{k-1}^p}{p_k^{p-1}} = o(P_n), \quad n\to\infty,\ p>1, \tag{2.9}$$

then $\sum_{k=0}^{\infty} a_k$ converges to L.

Proof We complete the proof by showing that (2.9) implies (2.6). Indeed

$$\begin{aligned}\left|\frac{1}{P_n}\sum_{k=0}^{n} P_{k-1}a_k\right| &\le \frac{1}{P_n}\sum_{k=0}^{n} p_k\,|a_k|\,\frac{P_{k-1}}{p_k}\\ &\le \frac{1}{P_n}\left(\sum_{k=0}^{n} p_k\,|a_k|^p\left(\frac{P_{k-1}}{p_k}\right)^p\right)^{\frac{1}{p}}\left(\sum_{k=0}^{n} p_k\right)^{\frac{1}{q}}\\ &= \left(\frac{1}{P_n}\sum_{k=0}^{n} |a_k|^p\,\frac{P_{k-1}^p}{p_k^{p-1}}\right)^{\frac{1}{p}},\end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Hence, condition (2.6) holds from (2.9) and Theorem 2.2. □

In the special case of the discrete Abel method (A_λ), we have the following theorem.

Corollary 2.4 *Assume that there exist two positive numbers γ_1 and γ_2 such that $\gamma_1 \le \frac{\lambda_k}{k} \le \gamma_2$. If $\sum_{k=0}^{\infty} a_k$ is summable (A_λ) to L and*

$$\sum_{k=0}^{n} k a_k = o(n), \quad n\to\infty,$$

then $\sum_{k=0}^{\infty} a_k$ converges to L.

Next, we prove two Tauberian theorems for the summability (P_λ). We inspired by the results in [7].

Theorem 2.5 *Assume that*

$$\frac{P_k}{p(x_k)} = O(1), \quad k\to\infty, \tag{2.10}$$

and

$$\lambda_k = O\left(\frac{1}{p_k}\right). \tag{2.11}$$

If $\sum_{k=0}^{\infty} a_k$ *is summable* (P_λ) *to* L *and*

$$a_k = o\left(\frac{p_k}{P_{k-1}}\right), \quad k \to \infty, \tag{2.12}$$

then $\sum_{k=0}^{\infty} a_k$ *converges to* L.

Proof We demonstrate this theorem by showing that the difference below tends to 0 as $n \to \infty$.

$$\begin{aligned} s_n - p_s(x_n) &= \frac{1}{p(x_n)} \sum_{k=0}^{\infty} (s_n - s_k) p_k x_n^k \\ &= \frac{1}{p(x_n)} \sum_{k=0}^{n} (s_n - s_k) p_k x_n^k + \frac{1}{p(x_n)} \sum_{k=n+1}^{\infty} (s_n - s_k) p_k x_n^k \\ &= I + J. \end{aligned}$$

If x_n is chosen to be equal to $1 - \dfrac{1}{\lambda_n}$, we have

$$I = o(1) \quad \text{as} \quad n \to \infty,$$

from (2.12) and the fact that $(\overline{N}, p)$ is regular (see Watson [18]).

Fix $\epsilon > 0$ and consider J. Since $a_k = o\left(\dfrac{p_k}{P_{k-1}}\right)$, there exists a $N \in \mathbb{N}$ such that $|a_k| \le \epsilon \dfrac{p_k}{P_{k-1}}$ for $k > N$. Suppose that $k > n > N$. Then, from (2.11) and (2.12) we find

$$\begin{aligned} |s_n - s_k| &= |a_{n+1} + a_{n+2} + ... + a_k| \\ &\le \epsilon \left(\frac{p_{n+1}}{P_n} + \frac{p_{n+2}}{P_{n+1}} + ... + \frac{p_k}{P_{k-1}} \right) \\ &\le \frac{\epsilon}{P_n} \left(p_{n+1} + p_{n+2} + ... + p_k \right) \\ &\le \frac{\epsilon C}{P_n} \left(\frac{1}{\lambda_{n+1}} + \frac{1}{\lambda_{n+2}} + ... + \frac{1}{\lambda_k} \right) \\ &\le \frac{\epsilon C}{P_n} \frac{k}{\lambda_n}. \end{aligned}$$

Considering (2.11), it yields

$$\begin{aligned} |J| &\le \frac{1}{p(x_n)} \frac{\epsilon C}{P_n \lambda_n} \sum_{k=n+1}^{\infty} k p_k x_n^k \\ &\le \frac{1}{p(x_n)} \frac{\epsilon C}{P_n \lambda_n^2} \sum_{k=n+1}^{\infty} k x_n^k \\ &\le \frac{P_n}{p(x_n)} \frac{\epsilon C}{P_n^2 \lambda_n^2} \sum_{k=0}^{\infty} k x_n^k \\ &= \epsilon \frac{P_n}{p(x_n)} \frac{C}{P_n^2} \left(1 - \frac{1}{\lambda_n} \right). \end{aligned}$$

Thus, by using (2.10) we obtain $J = o(1)$ as $n \to \infty$, which proves the theorem. □

Theorem 2.6 *Assume that*

$$(p_k) \text{ decreases monotonically,} \tag{2.13}$$

and

$$\gamma_1 \leq \frac{\lambda_k}{k} \leq \gamma_2 \text{ for some positive numbers } \gamma_1 \text{ and } \gamma_2. \tag{2.14}$$

If $\sum_{k=0}^{\infty} a_k$ *is summable* (P_λ) *to* L *and*

$$a_k = o\left(\frac{p_k}{P_{k-1}}\right), \quad k \to \infty, \tag{2.15}$$

then $\sum_{k=0}^{\infty} a_k$ *converges to* L.

Proof As in the proof of Theorem 2.5, let

$$s_n - p_s(x_n) = I + J.$$

We get from (2.15) and the regularity of $(\overline{N}, p)$ method that

$$I = o(1) \quad \text{as} \quad n \to \infty,$$

when $x_n = 1 - \dfrac{1}{\lambda_n}$.

We shall estimate J. Firstly, for sufficiently large n we have

$$P_n\left(1 - \frac{1}{\lambda_n}\right)^n < \sum_{k=0}^{n} p_k \left(1 - \frac{1}{\lambda_n}\right)^k < \sum_{k=0}^{\infty} p_k \left(1 - \frac{1}{\lambda_n}\right)^k$$

and

$$\left(1 - \frac{1}{\lambda_n}\right)^{-n} > \frac{P_n}{\sum\limits_{k=0}^{n} p_k \left(1 - \frac{1}{\lambda_n}\right)^k} > \frac{P_n}{\sum\limits_{k=0}^{\infty} p_k \left(1 - \frac{1}{\lambda_n}\right)^k}. \tag{2.16}$$

Then, by (2.14) and (2.16), we get $\dfrac{P_n}{p(x_n)} = O(1)$ as $n \to \infty$. Now, considering (2.15) and (2.13), respectively, we obtain

$$\begin{aligned}
|s_n - s_k| &= |a_{n+1} + a_{n+2} + ... a_k| \\
&\leq \epsilon \left(\frac{p_{n+1}}{P_{n+1}} + \frac{p_{n+2}}{P_{n+2}} + ... + \frac{p_k}{P_k}\right) \\
&\leq \frac{\epsilon}{P_n}\left(p_{n+1} + p_{n+2} + ... + p_k\right) \\
&\leq \epsilon \frac{P_k}{P_n},
\end{aligned}$$

and so

$$
\begin{aligned}
|J| &\leq \frac{1}{p(x_n)} \frac{\epsilon}{P_n} \sum_{k=n+1}^{\infty} P_k p_k x_n^k \\
&\leq \frac{P_n}{p(x_n)} \frac{\epsilon p_n}{P_n^2} \sum_{k=n+1}^{\infty} P_k x_n^k \\
&\leq \epsilon C \frac{p_n}{P_n^2} \sum_{k=n+1}^{\infty} P_k x_n^k.
\end{aligned}
$$

Defining

$$
\begin{aligned}
Q_k &= \sum_{j=k}^{\infty} x_n^j \\
&= \sum_{j=0}^{\infty} \left(1 - \frac{1}{\lambda_n}\right)^j - \sum_{j=0}^{k-1} \left(1 - \frac{1}{\lambda_n}\right)^j \\
&= \lambda_n - \frac{1 - \left(1 - \frac{1}{\lambda_n}\right)^k}{1 - \left(1 - \frac{1}{\lambda_n}\right)} = \lambda_n \left(1 - \frac{1}{\lambda_n}\right)^k,
\end{aligned}
$$

we find that

$$
\begin{aligned}
\sum_{k=n+1}^{\infty} P_k x_n^k &= \sum_{k=n+1}^{\infty} P_k \left(Q_k - Q_{k+1}\right) \\
&= P_{n+1} Q_{n+1} + \sum_{k=n+2}^{\infty} p_k Q_k \\
&= P_{n+1} \lambda_n \left(1 - \frac{1}{\lambda_n}\right)^{n+1} + \sum_{k=n+2}^{\infty} p_k \lambda_n \left(1 - \frac{1}{\lambda_n}\right)^k.
\end{aligned}
$$

Thus, we observe

$$
\begin{aligned}
|J| &\leq \epsilon C \frac{p_n}{P_n^2} P_{n+1} \lambda_n \left(1 - \frac{1}{\lambda_n}\right)^{n+1} + \epsilon C \frac{p_n}{P_n^2} \sum_{k=n+2}^{\infty} p_k \lambda_n \left(1 - \frac{1}{\lambda_n}\right)^k \\
&= S_1 + S_2.
\end{aligned}
$$

It suffices to show that $S_1, S_2 \to 0$ as $n \to \infty$. Then, by (2.13) and (2.14) we see that

$$
\begin{aligned}
0 \leq S_1 &\leq \epsilon C \frac{\lambda_n}{n} \frac{P_{n+1}}{P_n} \left(1 - \frac{1}{\lambda_n}\right)^{n+1} \\
&\leq \epsilon C \frac{\lambda_n}{n} \left(1 + \frac{1}{n}\right) \left(1 - \frac{1}{\lambda_n}\right)^{n+1} \\
&\leq \epsilon C,
\end{aligned}
$$

and further

$$\begin{aligned} 0 \leq S_2 &\leq \epsilon C \frac{p_n^2}{P_n^2} \lambda_n \sum_{k=0}^{\infty} \left(1 - \frac{1}{\lambda_n}\right)^k \\ &\leq \epsilon C \left(\frac{p_n \lambda_n}{P_n}\right)^2 \\ &\leq \epsilon C \left(\frac{\lambda_n}{n}\right)^2 \\ &\leq \epsilon C. \end{aligned}$$

Hence, we conclude $|J| \leq \epsilon C$ for large enough n. Therefore, letting $n \to \infty$, it follows $\lim_{n\to\infty} s_n = \lim_{n\to\infty} p_s(x_n) = L$, which completes the proof. □

Remark 2.7 *In previous theorems, conditions* (2.12) *and* (2.15) *may be replaced by condition* (2.6).

As a special case, we can give the following Tauberian theorem for the discrete logarithmic method (L_λ).

Corollary 2.8 *Let there exist two positive numbers* γ_1 *and* γ_2 *such that* $\gamma_1 \leq \dfrac{\lambda_k}{k} \leq \gamma_2$. *If* $\sum_{k=0}^{\infty} a_k$ *is summable* (L_λ) *to* L *and*

$$a_k = o\left(\frac{1}{k \log k}\right), \quad k \to \infty,$$

then $\sum_{k=0}^{\infty} a_k$ *converges to* L.

3. Conclusion

Discrete power series methods of summability were presented by Watson [17] in 1998. In the same study, their regularity and Abelian properties were developed and it was shown that each strictly includes their corresponding power series method. Recently, conditions for the equivalence of (P) and (P_λ) methods were examined in our previous work [12]. In the present work, we have proved Tauberian theorems for the (P_λ) method of summability inspired by the results in [7].

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