Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 2921 – 2930
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doi:10.3906/mat-1904-36

Zeros of the extended Selberg class zeta-functions and of their derivatives

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Received: 05.04.2019 • **Accepted/Published Online:** 27.10.2019 • **Final Version:** 22.11.2019

Abstract: Levinson and Montgomery proved that the Riemann zeta-function $\zeta(s)$ and its derivative have approximately the same number of nonreal zeros left of the critical line. Spira showed that $\zeta'(1/2+it)=0$ implies that $\zeta(1/2+it)=0$. Here we obtain that in small areas located to the left of the critical line and near it the functions $\zeta(s)$ and $\zeta'(s)$ have the same number of zeros. We prove our result for more general zeta-functions from the extended Selberg class S. We also consider zero trajectories of a certain family of zeta-functions from S.

Key words: Riemann zeta-function, extended Selberg class, nontrivial zeros, Speiser's equivalent for the Riemann hypothesis

1. Introduction

Let $s=\sigma+it$. In this paper, T always tends to plus infinity.

Speiser [17] showed that the Riemann hypothesis (RH) is equivalent to the absence of nonreal zeros of the derivative of the Riemann zeta-function $\zeta(s)$ left of the critical line $\sigma=1/2$. Later on, Levinson and Montgomery [11] obtained the quantitative version of the Speiser's result:

Theorem 1.1 *(Levinson-Montgomery) Let $N^-(T)$ be the number of zeros of $\zeta(s)$ in $R: 0<t<T, 0<\sigma<1/2$. Let $N_1^-(T)$ be the number of zeros of $\zeta'(s)$ in R. Then $N_1^-(T)=N(T)+O(\log T)$.*

Unless $N^-(T)>T/2$ for all large T there exists a sequence $\{T_j\}$, $T_j\to\infty$ as $j\to\infty$ such that $N_1^-(T_j)=N^-(T_j)$.

Here we prove the following theorem.

Theorem 1.2 *There is an absolute constant $T_0>0$ such that, for any $T>T_0$ and any $A>0.17$, there is a radius r,*

$$\exp(-T^A)\le r\le \exp(-T^{A-0.17}),$$

such that in the region

$$\{s:|s-(1/2+iT)|\le r \text{ and } \sigma<1/2\}$$

the functions $\zeta(s)$ and $\zeta'(s)$ have the same number of zeros.

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2010 *AMS Mathematics Subject Classification:* 11M26, 11M41

Nonreal zeros of $\zeta(s)$ lie symmetrically with respect to the critical line. In this sense, the result of Spira [18, Corollary 3] that $\zeta(1/2+it)=0$ if $\zeta'(1/2+it)=0$ can be regarded as a border case of the above theorem when $r=0$.

Note that for both $\zeta(s)$ and $\zeta'(s)$ the average gap between zeros is $2\pi/\log T$ around height T (Titchmarsh [21, Section 9.4] and Berndt [2]). This is much larger than the radius r in Theorem 1.2.

In Theorem 1.2, the constant 0.17 is related to the number of zeros of $\zeta(s)$ in the strip $|t-T|\leq 1/T$. For details see Section 3 which contains the proof of Theorem 1.2. Moreover, in Section 2 we consider a more general version of Theorem 1.2 devoted to the extended Selberg class S. The extended Selberg class contains most of the classical L-functions (Kaczorowski [7]). This class also includes zeta-functions for which RH is not true, a well-known example being the Davenport-Heilbronn zeta-function, which is defined as a suitable linear combination of two Dirichlet L-functions (Titchmarsh [21, Section 10.25], see also Kaczorowski and Kulas [8]). In the next section we also investigate zero trajectories of the following family of zeta-functions from S:

$$f(s,\tau):=(1-\tau)(1+\sqrt{5}/5^s)\zeta(s)+\tau L(s,\psi), \tag{1.1}$$

where $\tau\in[0,1]$ and $L(s,\psi)$ is the Dirichlet L-function with the Dirichlet character $\psi \bmod 5$, $\psi(2)=-1$.

2. Extended Selberg class

We consider Theorem 1.2 in the broader context of the extended Selberg class. Note that Levinson and Montgomery's [11, Theorem 1] approach, which is used here, usually works for zeta-functions having nontrivial zeros distributed symmetrically with respect to the critical line. See Yıldırım [23] for Dirichlet L-functions; Šleževičienė [20] for the Selberg class; Luo [12], Garunkštis [3], Minamide [13–15], Jorgenson and Smailović [6] for Selberg zeta-functions and related functions; Garunkštis and Šimėnas [5] for the extended Selberg class. In Garunkštis and Tamošiūnas [4] the Levinson and Montgomery result was generalized to the Lerch zeta-function with equal parameters. Such function has an almost symmetrical distribution of nontrivial zeros with respect to the line $\sigma=1/2$. Insights which helped to overcome difficulties raised by "almost symmetricity" in [4] led to Theorem 1.2 of this paper, although $\zeta(s)$ has a strictly symmetrical zero-distribution.

We recall the definition of the extended Selberg class (see [7, 9, 19]). A not identically vanishing Dirichlet series

$$F(s)=\sum_{n=1}^{\infty}\frac{a_n}{n^s},$$

which converges absolutely for $\sigma>1$, belongs to the *extended Selberg class* S if

(i) (Meromorphic continuation) There exists $k\in\mathbb{N}$ such that $(s-1)^kF(s)$ is an entire function of finite order.

(ii) (Functional equation) $F(s)$ satisfies the functional equation:

$$\Phi(s)=\omega\overline{\Phi(1-\overline{s})}, \tag{2.1}$$

where $\Phi(s):=F(s)Q^s\prod_{j=1}^{r}\Gamma(\lambda_js+\mu_j)$, with $Q>0$, $\lambda_j>0$, $\Re(\mu_j)\geq 0$ and $|\omega|=1$.

The data Q, λ_j, μ_j, and ω of the functional equation are not uniquely determined by F, but the value $d_F=2\sum_{j=1}^{r}\lambda_j$ is an invariant. It is called the *degree* of F.

If the element of S also satisfies the Ramanujan hypothesis ($a_n \ll_\varepsilon n^\varepsilon$ for any $\varepsilon > 0$) and has a certain Euler product, then it belongs to the Selberg class introduced by Selberg [16].

We collect several properties of $F(s) \in S$. The functional equation (2.1) gives, for $F(1/2+it) \neq 0$,

$$\Re \frac{F'}{F}(1/2+it) = -\Re \sum_{j=1}^{r} \lambda_j \frac{\Gamma'}{\Gamma}(\lambda_j(1/2+it)+\mu_j) - \log Q.$$

Then by the formula

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O\left(|s|^{-1}\right) \quad (\Re(s) \geq 0,\ |s| \to \infty)$$

we get, for $F(1/2+it) \neq 0$ and $d_F > 0$,

$$\Re \frac{F'}{F}(1/2+it) = -\frac{d_F}{2}\log t - \log Q + O\left(\frac{1}{t}\right) \qquad (t \to \infty), \tag{2.2}$$

where the implied constant may depend only on λ_j, μ_j, $j = 1, \ldots, r$.

Every $F \in S$ has a zero-free half-plane, say $\sigma > \sigma_F$. By the functional equation, $F(s)$ has no zeros for $\sigma < -\sigma_F$, apart from possible trivial zeros coming from the poles of the Γ-factors. Let $\rho = \beta + i\gamma$ denote a generic zero of $F(s)$ and

$$N_F(T) = \#\left\{\rho : F(\rho) = 0, |\beta| \leq \sigma_F, |\gamma| < T\right\}.$$

Then (Kaczorowski and Perelli [9, Section 2])

$$N_F(T) = \frac{d_F}{\pi} T \log T + c_F T + O(\log T) \tag{2.3}$$

with a certain constant c_F, for any fixed $F \in S$ with $d_F > 0$.

From the Dirichlet series expression for F we see that there are constants $\sigma_1 = \sigma_1(F) > 1$ and $c = c(\sigma_1, F) > 0$ such that

$$|F(\sigma_1 + it)| \geq c, \qquad t \in \mathbb{R}. \tag{2.4}$$

It is known (Garunkštis and Šimėnas [5, formula (12)]) that there is $B = B(F, \sigma_1) > 0$ such that

$$|F(\sigma + iT)| < T^B, \quad (T > 10), \tag{2.5}$$

for $\sigma \geq -4\sigma_1$. The specific constant $-4\sigma_1$ will be useful in the proof of Theorem 2.1 below.

In view of above, for given positive constants σ_1, c, B, ε, δ, $\bar{T}$, λ_j, and complex constants μ_j ($\Re(\mu_j) > 0$, $j = 1, \ldots, r$), we define a subclass $\bar{S} \subset S$ as follows: it consists of functions satisfying (2.4), (2.5), (2.1), with any $|\omega| = 1$; we require that any function from $\bar{S}$ has no more than

$$\frac{\varepsilon}{\log(2+\delta)} \log T - 2 \tag{2.6}$$

zeros in the area $|t - T| \leq 1/T$, $T > \bar{T}$. For each function from S the Riemann-von Mangoldt type formula (2.3) yields the existence of ε, δ, and $\bar{T}$ such that the zero number bound (2.6) is fulfilled.

Theorem 1.2 will be derived from the following more general statement.

Theorem 2.1 *Let $F(s)$ be an element of $\bar{S}$ with $d_F > 0$. Then there is a constant $T_0 = T_0(\bar{S}) > 0$ for which the following statement is true.*

If A and $s_0 = \sigma_0 + iT$ satisfy the inequalities

$$A > \varepsilon, \quad T > T_0, \quad 1/2 - \exp(-T^A) < \sigma_0 \leq 1/2, \tag{2.7}$$

then there is a radius $r = r(F)$,

$$\exp(-T^A) \leq r \leq \exp(-T^{A-\varepsilon}),$$

such that in the area

$$\{s : |s - s_0| \leq r \text{ and } \sigma < 1/2\} \tag{2.8}$$

functions $F(s)$ and $F'(s)$ have the same number of zeros.

Note that in Theorem 2.1 the constant T_0 is independent of A and σ_0. This will be important in the proof of Theorem 2.2 below.

In [5] zeta-functions $f(s, \tau)$ defined by (1.1) were considered. By Kaczorowski and Kulas [8, Theorem 2] we have that for any $0<\tau<1$ and any interval $(a, b) \subset (1/2, 1)$ the function $f(s, \tau)$ has infinitely many zeros in the half-strip $a < \sigma < b$, $t > 0$. Let $\theta > 0$ and let

$$\rho : (\tau_0 - \theta, \tau_0 + \theta) \to \mathbb{C}$$

be a continuous function such that $f(\rho(\tau), \tau) = 0$ for $\tau \in (\tau_0 - \theta, \tau_0 + \theta)$. We say that $\rho(\tau)$ is a zero trajectory of the function $f(s, \tau)$. Analogously we define a zero trajectory $\tilde{\rho}(\tau)$ of the derivative $f'_s(s, \tau)$. See also the discussion below the formula (6) in [5]. In [5] several zero trajectories $\rho(\tau)$ of $f(s, \tau)$ and $\tilde{\rho}(\tau)$ of $f'_s(s, \tau)$ were computed. The behavior of these zero trajectories correspond well to Theorem 2.1. Computations in [5] should be considered as heuristic because the accuracy was not controlled explicitly. Next we present a rigorous statement concerning zero trajectories of $f(s, \tau)$ and $f'_s(s, \tau)$.

Theorem 2.2 *Let $\tau_0 \in [0, 1]$. Let $s = \rho_0$ be a second-order zero of $f(s) = f(s, \tau_0)$ with $\Re(\rho_0) = 1/2$ and sufficiently large $\Im(\rho_0)$. Then the following two statements are equivalent.*

1) There is a zero trajectory $\rho(\tau)$, $\tau \in (\tau_0 - \theta, \tau_0 + \theta)$, $\theta > 0$, of $f(s, \tau)$ such that

- *(i)* $\rho(\tau_0) = \rho_0$*;*
- *(ii)* $\Re(\rho(\tau)) = 1/2$ *if* $\tau < \tau_0$*;*
- *(iii)* $\Re(\rho(\tau)) < 1/2$*, if* $\tau > \tau_0$*.*

2) There is a zero trajectory $\tilde{\rho}(\tau)$, $\tau \in (\tau_0 - \eta, \tau_0 + \eta)$, $\eta > 0$, of $f'_s(s, \tau)$ such that

- *(i)* $\tilde{\rho}(\tau_0) = \rho_0$*;*
- *(ii)* $\Re(\tilde{\rho}(\tau)) > 1/2$ *if* $\tau < \tau_0$*;*
- *(iii)* $\Re(\tilde{\rho}(\tau)) < 1/2$*, if* $\tau > \tau_0$*.*

From the proof we see that Theorem 2.2 remains true if all inequalities $\tau < \tau_0$ and $\tau > \tau_0$ are simultaneously replaced by opposite inequalities.

According to computations of [5] there are 1452 zero trajectories $\rho(\tau)$ of $f(s,\tau)$ with $0<\Im\rho(0)\le 1500$, 1166 of these trajectories stay on the critical line, while the remaining 286 leave it. The points at which mentioned trajectories leave the critical line are double zeros of $f(s)=f(s,\tau)$ (see also a discussion at the end of Section 3 in Balanzario and Sánchez-Ortiz [1]). In view of this we expect that the family $f(s,\tau)$, $\tau\in[0,1]$ has infinitely many double zeros lying on the line $\sigma=1/2$. Moreover, we think that the similar statement to Theorem 2.2 can also be proved in the case where $s=\rho$ is a higher order zero of $f(s)=f(s,\tau)$ with $\Re\rho=1/2$; however, there is no evidence that such zeros exist.

The next section is devoted to the proofs of Theorems 1.2, 2.1, and 2.2.

3. Proofs

Proof of Theorem 2.1 is based on the next lemma. Recall that the subclass $\bar{S}$ depends on constants σ_1, c, B, ε, δ, $\bar{T}$, λ_j, μ_j, $(j=1,\dots,r)$.

Lemma 3.1 *Let $F(s)$ be an element of $\bar{S}$ with $d_F>0$. Suppose that $s_0=\sigma_0+iT$ satisfies the inequality $1/2-\exp(-T^A)<\sigma_0\le 1/2$, where $T>\bar{T}$ and $A>\varepsilon$. Then there is a radius $r=r(F)$,*

$$\exp(-T^A)\le r\le \exp(-T^{A-\varepsilon}), \tag{3.1}$$

such that, for $|s-s_0|=r$, $\sigma\le 1/2$,

$$\Re\frac{F'}{F}(s)\le -\frac{d_F}{2}\log T-\log Q+O\left(\frac{1}{T}\right), \tag{3.2}$$

uniformly for $F(s)\in\bar{S}$.

Proof We repeat the steps of the proof of Proposition 4 in [4]. Contrary to Proposition 4, here we do not need the upper bound for ε (see (2.6)). This is because the "symmetric" functional equation (2.1) leads to the convenient formula (2.2), while the "almost symmetric" functional equation of the Lerch zeta-function with equal parameters in [4] leads to a more restricted version of (2.2) (see [4, Lemma 3]).

Let $T>\bar{T}$ and $r_k=\exp\left(-(2+\delta)^{-k}T^A\right)$, $k=1,\dots,[\frac{\varepsilon}{\log(2+\delta)}\log T]$. By (2.6) and Dirichlet's box principle there is $j=j(F)\in\{2,\dots,[\frac{\varepsilon}{\log(2+\delta)}\log T]\}$ such that the region

$$r_{j-1}<|s-s_0|\le r_j \tag{3.3}$$

has no zeros of $F(s)$. Then the auxiliary function

$$g(s):=\frac{F'}{F}(s)-\sum_{\rho:\,|\rho-s_0|\le r_{j-1}}\frac{1}{s-\rho} \tag{3.4}$$

is analytic in the disc $|s-s_0|\le r_j$ and in this disc we have

$$g(s)=\sum_{n=0}^{\infty}a_n(s-s_0)^n \quad\text{and}\quad a_n=\frac{1}{2\pi i}\int\limits_{|s-s_0|=r_j}\frac{g(s)ds}{(s-s_0)^{n+1}}. \tag{3.5}$$

In view of bounds (2.4) and (2.5), Lemma α from Titchmarsh [21, Section 3.9] gives that, for $|s-s_0| \leq r_j$,

$$\frac{F'}{F}(s) = \sum_{\rho\,:\,|\rho-(\sigma_1+iT)|\leq 2\sigma_1} \frac{1}{s-\rho} + O(\log T).$$

Recall that σ_1 was defined before (2.4). Here and elsewhere in this proof the constants in big-O and $\ll$ notations may only depend on the subclass $\bar{S}$. By the last equality, the zero-free region (3.3), and (3.4) we get

$$g(s) = \sum_{\substack{\rho\,:\,|\rho-(\sigma_1+iT)|\leq 2\sigma_1 \text{ and} \\ |\rho-s_0|>r_j}} \frac{1}{s-\rho} + O(\log T).$$

Using this expression in the integral for a_n we obtain that

$$a_n \ll r_j^{-n} \log T \quad (n \geq 1). \tag{3.6}$$

Let us choose

$$r = r_j^{1+\delta/3}.$$

Clearly, the bounds (3.1) are satisfied. By (3.5) and (3.6), for $|s-s_0| = r$, we have

$$g(s) = a_0 + O\left(r_j^{\delta/3} \log T\right).$$

Hence, for $|s-s_0| = r$, the expression (3.4) gives

$$\Re\frac{F'}{F}(s) = \Re a_0 + \sum_{\rho\,:\,|\rho-s_0|\leq r_{j-1}} \frac{\sigma-\beta}{|s-\rho|^2} + O\left(r_j^{\delta/3}\log T\right). \tag{3.7}$$

For $|\rho-s_0| \leq r_{j-1}$, $|s-s_0| = r$, $1/2-(\Re s_0 - 1/2 + r_{j-1}) \leq \sigma \leq 1/2$, and large T, we have that $|\sigma-\beta| \leq 4r_{j-1}$ and $|s-\rho|^2 > r_j^{2+2\delta/3}/2$. Then by (2.6) we get

$$\sum_{\rho\,:\,|\rho-s_0|\leq r_{j-1}} \frac{\sigma-\beta}{|s-\rho|^2} \ll r_j^{\delta/3}\log T.$$

Consequently, by (3.7),

$$\Re\frac{F'}{F}(s) = \Re a_0 + O\left(r_j^{\delta/3}\log T\right). \tag{3.8}$$

The region (3.3) is zero-free. Thus, $F(s)$ does not vanish on the circle $|s-s_0| = r$. By instantiating (2.2) and (3.8) to a single s on the intersection of $|s-s_0| = r$ and $\sigma = 1/2$ we obtain that

$$\Re a_0 = -\frac{d_F}{2}\log T - \log Q + O\left(\frac{1}{T}\right) + O\left(r_j^{\delta/3}\log T\right). \tag{3.9}$$

Hence, for $|s-s_0| = r$ and $1/2 - (\Re s_0 - 1/2 + r_{j-1}) \leq \sigma \leq 1/2$,

$$\Re\frac{F'}{F}(s) = -\frac{d_F}{2}\log T - \log Q + O\left(\frac{1}{T}\right). \tag{3.10}$$

If $|s-s_0|=r$ and $\sigma<1/2-(\Re s_0-1/2+r_{j-1})$, then

$$\sum_{\rho:|\rho-s_0|\le r_{j-1}}\frac{\sigma-\beta}{|s-\rho|^2}\le 0$$

and, in view of formulas (3.7), (3.9),

$$\Re\frac{F'}{F}(s)\le-\frac{d_F}{2}\log T-\log Q+O\left(\frac{1}{T}\right). \tag{3.11}$$

The expressions (3.10) and (3.11), together with the zero-free region (3.3), prove Lemma 3.1. □

Proof of Theorem 2.1 Let

$$R=\{s:|s-s_0|\le r \text{ and } \sigma<1/2\},$$

where r is from Lemma 3.1. To prove the theorem, it is enough to consider the difference in the number of zeros of $F(s)$ and $F'(s)$ in the region R.

We consider the change of $\arg F'/F(s)$ along the appropriately indented boundary R' of the region R. More precisely, the left side of R' coincides with the circle segment $\{s:|s-s_0|=r,\sigma\le 1/2\}$. To obtain the right-hand side of the contour of R', we take the right-hand side boundary of R and deform it to bypass the zeros of $F(1/2+it)$ by left semicircles with an arbitrarily small radius. In [5, proof of Theorem 1.2] it is showed that on the right-hand side of R' the inequality

$$\Re\frac{F'}{F}(s)<0 \tag{3.12}$$

is true. Then, in view of Lemma 3.1, we have that the inequality (3.12) is valid on the whole contour R'. Therefore, the change of $\arg F'/F(s)$ along the contour R' is less than π. This proves Theorem 2.1. □

Proof of Theorem 1.2 The Riemann zeta-function is an element of degree 1 of the extended Selberg class (Kaczorowski [7]). By Trudgian [22, Corollary 1] we see that, for large T, the Riemann zeta-function has less than $0.225\log T$ zeros in the strip $|t-T|\le 1/T$. Thus, in the formula (2.6) we choose $\varepsilon=0.17$ and $\delta=0.1$. Then Theorem 1.2 follows from Theorem 2.1. □

Proof of Theorem 2.2 We will use Theorem 2.1. Next we show that there is a subclass $\bar{S}$ such that $f(s,\tau)\in\bar{S}$ for all $\tau\in[0,1]$. In view of the definition (1.1) of $f(s,\tau)$ we see that there are constants c, B, and σ_1 independent of τ for which the bounds (2.4) and (2.5) are valid. By this and Jensen's theorem, similarly as in Titchmarsh [21, Theorem 9.2], we get that there are constants ε, δ, and $\bar{T}$ independent of τ for which the zero number bound (2.6) is true. The function $f(s)=f(s,\tau)$ satisfies the functional equation ([5, formula (3)])

$$f(s)=5^{-s+1/2}2(2\pi)^{s-1}\Gamma(1-s)\sin\left(\frac{\pi s}{2}\right)f(1-s) \tag{3.13}$$

which is independent of τ; thus, the constants λ_j, μ_j are also independent of τ. This proves the existence of required $\bar{S}$. Therefore, in Theorem 2.1 with $F(s)=f(s,\tau)$ it is possible to choose T_0, which is independent of τ. Further in this proof we assume that $\Im(\rho_0)>T_0+10$.

We consider a zero trajectory $\rho(\tau)$ of $f(s,\tau)$ which satisfies $\rho(\tau_0)=\rho_0$. The two variable function $f(s,z)$ is holomorphic in a neighborhood of any

$$(s,z)\in\mathbb{C}^2\setminus\{(1,z):z\in\mathbb{C}\}.$$

By conditions of the theorem we have that $\rho_0\neq 1$, $f(\rho_0,\tau_0)=0$,

$$\frac{\partial f(\rho_0,\tau_0)}{\partial s}=0, \quad \text{and} \quad \frac{\partial^2 f(\rho_0,\tau_0)}{\partial s^2}\neq 0. \tag{3.14}$$

By (3.14) and by the Weierstrass preparation theorem (Krantz and Parks [10, Theorem 5.1.3]) there exists a polynomial

$$p(s,\tau)=s^2+a_1(\tau)s+a_0(\tau),$$

where each $a_j(\tau)$ is a holomorphic function in a neighborhood of $\tau=\tau_0$ that vanishes at $\tau=\tau_0$, and there is a function $u(s,\tau)$ holomorphic and nonvanishing in some neighborhood N of (ρ_0,τ_0) such that

$$f(s,\tau)=u(s,\tau)p(s,\tau) \tag{3.15}$$

holds in N. Solving $s^2+a_1(\tau)s+a_0(\tau)=0$ we get

$$s_{1,2}=s_{1,2}(\tau)=\frac{-a_1(\tau)\pm\sqrt{a_1(\tau)^2-4a_0(\tau)}}{2}, \tag{3.16}$$

where for the square-root we choose the branch defined by $\sqrt{1}=1$. Note that in the neighborhood N the function $f(s,\tau)$ has no other zeros except those described by (3.16).

Assume that the statement 1) of Theorem 2.2 is true. Then in some neighborhood U of $\tau=\tau_0$ the first part of trajectory $\rho(\tau)$ consists either of $\{s_1(\tau):\tau<\tau_0,\tau\in U\}$ or of $\{s_2(\tau):\tau<\tau_0,\tau\in U\}$. Similarly, the remaining part of trajectory $\rho(\tau)$ consists either of $\{s_1(\tau):\tau>\tau_0,\tau\in U\}$ or of $\{s_2(\tau):\tau>\tau_0,\tau\in U\}$.

If $\Re s_1(\tau)\neq 1/2$ or $\Re s_2(\tau)\neq 1/2$ for some τ, then by the functional equation (3.13) we see that $s_2(\tau)=1-\overline{s_1(\tau)}$. This and the condition *(iii)* give that

$$s_1(\tau)\neq s_2(\tau), \quad \text{if} \quad \tau>0,\ \tau\in U. \tag{3.17}$$

Thus, $a_1(\tau)^2-4a_0(\tau)\neq 0$ if $\tau>0$, $\tau\in U$. By the condition *(i)* we see that $\rho(\tau_0)=s_1(\tau_0)=s_2(\tau_0)$ is a double zero of $P(s)=P(s,\tau)$; therefore, $a_1(\tau_0)^2-4a_0(\tau_0)=0$. Hence, $a_1(\tau)^2-4a_0(\tau)$ is a nonconstant holomorphic function. Then there is a neighborhood of $\tau=\tau_0$, where

$$s_1(\tau)\neq s_2(\tau), \quad \text{if} \quad \tau<0. \tag{3.18}$$

In view of formulas (3.14), the implicit function theorem ([10, Theorem 2.4.1])) yields the existence of $\eta>0$ and of a continuous function

$$\tilde\rho:(\tau_0-\eta,\tau_0+\eta)\to\mathbb{C},$$

such that $\tilde\rho(\tau_0)=\rho(\tau_0)=0$ and $f'_s(\tilde\rho(\tau),\tau)=0$. By this we get condition *(a)* of the second statement.

We assume that $\eta > 0$ is such that the set

$$\{(\tilde{\rho}(\tau), \tau) : \tau \in (\tau_0 - \eta, \tau_0]\}$$

is a subset of the neighborhood N (defined by (3.15)). We have ([5, Proposition 1.4]) that $f_s'(1/2+it,\tau)=0$ implies $f(1/2+it,\tau)=0$. Then in view of (3.18) we obtain that $\Re\tilde{\rho}(\tau) \neq 1/2$ if $\tau \in (\tau_0-\eta,\tau_0)$. By condition *(ii)* and by above there is a neighborhood of (ρ_0,τ_0), where $f(s,\tau)\neq 0$ if $\tau<\tau_0$. Then condition *(b)* follows from Theorem 2.1.

Theorem 2.1 and condition *(iii)* lead to $\Re(\tilde{\rho}(\tau)) < 1/2$ if $\tau \in (\tau_0, \tau_0+\eta)$ and $\eta > 0$ is sufficiently small. We get condition *(c)*. By this we proved that the statement 1) implies the statement 2).

Assume the second statement of Theorem 2.2. Then by applying Theorem 2.1 and reasoning similarly as above, we see that from the trajectories defined by (3.16) we can construct a trajectory $\rho(\tau)$ which satisfies conditions of the first statement. □

Acknowledgment

This research is funded by the European Social Fund according to the activity 'Improvement of researchers' qualification by implementing world-class R&D projects' of Measure No. 09.3.3-LMT-K-712-01-0037.

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