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A variational study on a natural Hamiltonian for curves

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Abstract: A variational study of finding critical points of the total squared torsion functional for curves in Euclidean 3−spaces is presented. Critical points of this functional also known as one of the natural Hamiltonians of curves are characterized by two Euler−Lagrange equations in terms of curvature and torsion of a curve. To solve these balance equations, the curvature of the critical curve is expressed by its torsion so that equations are completely solved by quadratures. Then two Killing fields along the critical curve are found for integrating the structural equations of the critical curve and this curve is expressed by quadratures in a system of cylindrical coordinate. Finally, the problem is generalized to finding extremals of total squared torsion functional for nonnull curves in Minkowski 3−space.

Key words: Calculus of variation, Euler−Lagrange equations, torsion, Killing fields

1. Introduction

The mechanics of thin rods have a long history, and are used to tackle a number of problems from different fields today. Finding the equilibrium position of these structures is a mechanical problem and the balance equations can be reached using classical differential geometry techniques. These models often give rise to variational problems on curves in a form that is invariant under Euclidean motions. The corresponding equilibrium equations are usually expressed in abstract geometrical form and do not seem to be widely known in the physics and mechanics literature [15].

Consider that γ is a unit speed curve in Euclidean 3−space $\mathbb{R}^3$. Let $\{T, N, B\}$ denote the Frenet frame of γ at the point $\gamma(s)$. Then, the Frenet formulas for the curve γ are given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{1.1}$$

where

$$\kappa = \|\gamma''\| \text{ and } \tau =< N', B > \tag{1.2}$$

are the curvature and the torsion of γ, respectively [11]. If the curvature $\kappa > 0$ and the torsion τ of a curve are known, then the curve can be completely characterized. Moreover, the Hamiltonian for curves of the form $H = \int f(\kappa, \tau, \kappa', \tau', ...)ds$ is defined by curvature and torsion of a curve. Such Hamiltonians play a role both in static and kinematic description of curves. A simple model for these Hamiltonians is in the form $\int f(\kappa)ds$

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which depends on the curvature (see [2]). Especially, a natural Hamiltonian $\int \kappa^2 ds$ generated by $< T', T' >$ is known as a bending energy functional and critical points of this functional under suitable conditions are called as elastic curves proposed by Daniel Bernoulli to Leonhard Euler in 1744 (see, [1, 2, 13]). Elastic curves and their generalization which are the critical points of the functional of a form $\int p(\kappa)ds$, where $p(\kappa)$ is a polynomial of κ with degree ≥ 2 under given first order boundary data have been worked and developed by many authors up to now and are being continued to be developed see for example [1, 3, 5–9, 16]. Although the natural Hamiltonian in this form has been extensively studied, there is not much work on natural Hamiltonians produced by $< N', N' >$ and $< B', B' >$ which are $\int \kappa^2 + \tau^2 ds$ and $\int \tau^2 ds$. In [2], Capovilla et al. examine local reparametrization invariant Hamiltonians for curves of the form $H = \int f(\kappa, \tau)ds$ and show that a pure torsion Hamiltonian also leads to integrable equilibrium conditions as well as a pure bending Hamiltonian. Moreover, they derive Euler−Lagrange equations for the form bending and torsion and obtain equilibrium equations by using the theory of deformations of a curve tailored to the Frenet-Serret frame. This method used by Capovilla et al. to characterize the natural Hamiltonians differs from that of the Langer and Singer (see [6–9, 13], which is often used to minimize pure curvature functionals. The idea is that Langer and Singer's approach can also be useful to minimize pure torsion functional. The critical points of pure torsion functional in Euclidean and Minkowski 3−spaces are investigated using that approach in this paper. This problem now offers a pure mathematical flavor, but considering the role of variational calculus in geometric control theory, the results of this paper are believed to be a good reference to new research studies in this concept.

To sum up, the natural Hamiltonian variational problem which is produced by inner multiplication of a binormal derivative of a Frenet curve is considered with a different approach (the Langer and Singer's approach) in this paper. Equilibrium configurations for the natural Hamiltonian with appropriate boundary conditions both in Euclidean and in Minkowski 3−spaces are investigated. The geometrical state of the balance equations characterized by its curvature and torsion is obtained and solved by quadratures in both spaces. Two Killing fields, introduced by Langer and Singer [8], are constructed along the critical curves. By using these Killing fields, a cylindrical coordinate system is constructed. Finally, critical curves are expressed in $\mathbb{R}^3$ and $\mathbb{R}^3_1$ by quadratures in these cylindrical coordinates, respectively.

2. Critical points of the torsion energy action in Euclidean 3-space

In this section, information about the structure of a curve in Euclidean 3−space is given. Some variational conditions are presented. Then, the main problem which is finding critical points of the total squared torsion functional under some boundary conditions in Euclidean 3−space are constructed and solutions are studied.

2.1. Variational formulas

In this subsection, basic facts for the Euclidean curves and a geometrical construction that will be needed in Subsection 2.2 are given.

Let $\gamma = \gamma(s) : I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve in Euclidean 3−space $\mathbb{R}^3$. $T = T(s) = \gamma'$ denotes the unit tangent vector field of γ, $N = \frac{\gamma''}{\|\gamma''\|}$ the principle unit normal vector field and $B = T \times N$ the binormal vector field. Then $\{T, N, B\}$ is the Frenet frame along the curve γ, and Frenet equations are given by the equations (1.1) with the curvature and the torsion (1.2).

To find the extremals of the total squared torsion functional, its first variation must be computed. Consider a map $\gamma\colon (-\varepsilon, \varepsilon) \times I \to \mathbb{R}^3$ so that $(w, t) \to \gamma(w, t) = \gamma_w(t)$ and the curve $\gamma_w(t)$ goes throughout

γ provided that $\gamma(0,t)=\gamma(t)$. Then $\gamma_w(t)$ is known as a variation of γ. Two vector fields $V(w,t)=\frac{\partial\gamma(w,t)}{\partial t}$ and $W(w,t)=\frac{\partial\gamma(w,t)}{\partial w}$ are defined such that $V(0,t)=\gamma'(t)$. Furthermore, $W(t)=W(0,t)$ is a variational vector field along $\gamma(t)$ so that $\left.\frac{\partial\gamma(w,t)}{\partial w}\right|_{w=0}=W(t)$ [1, 13]. If s denotes the arc length parameter, then $\gamma(s)$, $\kappa^2(w,s)$, $V(s)$, etc. can be written for the corresponding reparametrizations, where $s\in[0,\ell]$ and ℓ is arc length of γ.

The following lemma is needed for some variational calculations in Section 2.2.

Lemma 2.1 *(see [6, 7]). Let $\gamma(w,t)$ be a variation of a curve $\gamma\in\mathbb{R}^3$. Then the following formulas are satisfied;*

i) $[W,V]=0$,

ii) $W(v)=<W',T>v$,

iii) $W(\kappa)=<W'',N>-2\kappa<W',T>$

iv) $W(\tau^2)=2\tau\left(\frac{1}{\kappa}<W'',B>\right)'-2\tau<W',(\tau T-\kappa B)>$.

2.2. Finding critical Euclidean curves for the natural Hamiltonian

Suppose that $\mathcal{L}$ is the space of smooth curves $\gamma:[0,\ell]\subset\mathbb{R}\to\mathbb{R}^3$ satisfying

$$\gamma(i\ell)=p_i,\quad \gamma'(i\ell)=v_i$$

for $p_i\in\mathbb{R}^3$ and $v_i\in T_{p_i}\mathbb{R}^3$, $i=0,1$. Then, a natural Hamiltonian generated by the inner product of derivative of binormal of a curve, i.e. $<B',B'>$ is defined as

$$\begin{array}{rccl}\mathcal{F}: & \mathcal{L} & \to & [0,\infty)\\ & \gamma & \to & \mathcal{F}(\gamma)=\mathcal{F}_\gamma=\int_0^\ell\tau^2ds=\int_0^1\tau^2vdt,\end{array}\tag{2.1}$$

where τ is the torsion of the curve and $v=\|\gamma'(t)\|=\frac{ds}{dt}\neq0$ is the speed function. In this section, the critical points of the functional (2.1) are investigated.

Let γ be a critical point of the functional $\mathcal{F}$. Then for a variation γ_w associated with a variation vector field W along γ, the following equations are obtained from Lemma 2.1.

$$\begin{aligned}\delta\mathcal{F}_\gamma(W)&=\int_0^1(W(\tau^2)v+\tau^2W(v))dt,\\ &=\int_0^\ell(2\tfrac{\tau}{\kappa}<W''',B>+2\tau<W'',\left(\tfrac{1}{\kappa}B\right)'>-2\tau<W',(\tau T-\kappa B)>+\tau^2<W',T>)ds.\end{aligned}$$

By the integrating by parts, we get

$$\delta\mathcal{F}_\gamma(W)=\int_0^\ell<E[\gamma],W>ds+\left.(S[\gamma,W])\right|_0^\ell$$

where

$$E[\gamma]=\left(3\tau^2\kappa+2\frac{(\tau')^2}{\kappa}+4\tau\left(\frac{\tau'}{\kappa}\right)'\right)N+2\left(\tau'\frac{\tau^2}{\kappa}-(\tau\kappa)'-\left(\frac{\tau'}{\kappa}\right)''\right)B$$

and

$$(S[\gamma,W])\big|_0^\ell =< 2\frac{\tau}{\kappa}B, W'' > - < 2\frac{\tau'}{\kappa}B, W' > + < -\tau^2 T - 2\tau'\frac{\tau}{\kappa}N + 2\left(\tau\kappa + \left(\frac{\tau'}{\kappa}\right)'\right)B, W >\Big|_0^\ell$$

If γ is a critical point of the functional $\mathcal{F}$, then $E[\gamma]$ vanishes identically [4]. Then, the first variation reduces to

$$\begin{aligned} \delta\mathcal{F}_\gamma(W) &= (S[\gamma,W])\big|_0^\ell \\ &= < 2\frac{\tau}{\kappa}B, W'' > - < 2\frac{\tau'}{\kappa}B, W' > + < J, W >\big|_0^\ell, \end{aligned} \tag{2.2}$$

where

$$J = -\tau^2 T - 2\tau'\frac{\tau}{\kappa}N + 2\left(\tau\kappa + \left(\frac{\tau'}{\kappa}\right)'\right)B.$$

According to the Noether theorem, the first variation of $\mathcal{F}_\gamma$ is zero for the constant vector field W (see [4, 13]), this means that Eq. (2.2) is equal to zero. One can easily see that J is constant along a critical point of the functional (2.1) because of

$$\begin{aligned} J' &= (-2\tau\tau' + 2\tau\tau')T + \left(-\tau^2\kappa - \left(2\tau'\frac{\tau}{\kappa}\right)' - 2\tau\left(\tau\kappa + \left(\frac{\tau'}{\kappa}\right)'\right)\right)N + 2\left(\left(\tau\kappa + \left(\frac{\tau'}{\kappa}\right)'\right)' - \tau'\frac{\tau^2}{\kappa}\right)B \\ &= -\left(-3\tau^2\kappa - 2\frac{(\tau')^2}{\kappa} - 4\tau\left(\frac{\tau'}{\kappa}\right)'\right)N - 2\left(\tau'\frac{\tau^2}{\kappa} - (\tau\kappa)' - \left(\frac{\tau'}{\kappa}\right)''\right)B \\ &= -E[\gamma] = 0. \end{aligned}$$

Then we have

$$\|J\|^2 = \tau^4 + 4(\tau')^2\frac{\tau^2}{\kappa^2} + 4\left(\tau\kappa + \left(\frac{\tau'}{\kappa}\right)'\right)^2 = \frac{a^2}{4}, \tag{2.3}$$

where a is a constant. On the other hand, since $E[\gamma] = 0$, we have

$$3\tau^2\kappa + 2\frac{(\tau')^2}{\kappa} + 4\tau\left(\frac{\tau'}{\kappa}\right)' = 0 \tag{2.4}$$

and

$$\frac{1}{3\kappa}\left(\tau^3\right)' - (\tau\kappa)' - \left(\frac{\tau'}{\kappa}\right)'' = 0. \tag{2.5}$$

Theorem 2.2 *The critical points of the total squared torsion functional are characterized by the Euler−Lagrange equations* (2.4) *and* (2.5)*.*

If the curve γ is a critical point of the functional (2.1) and W is an infinitesimal symmetry, then $(S[\gamma, W])$ is a constant. On the other hand, if W is the restriction of a rotational field, then it can be written as

$$W = \gamma \times W_0 \tag{2.6}$$

for constant W_0 in [4, 13]. Substituting the first and second derivatives of Eq. (8) into $(S[\gamma, W])$ and using Frenet equations (1.1) yield

$$\begin{aligned} const. &= (S[\gamma, W]) =< 2\frac{\tau}{\kappa}B, W'' > - < 2\frac{\tau'}{\kappa}B, W' > + < J, W > \\ &= 2\frac{\tau}{\kappa} < B, \gamma'' \times W_0 > -2\frac{\tau'}{\kappa} < B, \gamma' \times W_0 > + < J, \gamma \times W_0 > \\ &= 2\tau < B, N \times W_0 > -2\frac{\tau'}{\kappa} < B, T \times W_0 > + < J \times \gamma, W_0 > \\ &= < -2\tau T - 2\frac{\tau'}{\kappa}N + J \times \gamma, W_0 > \end{aligned}$$

which gives

$$-2\tau T - 2\frac{\tau'}{\kappa}N + J \times \gamma = A,$$

where A is a constant vector field. Then the vector field

$$I = -2\tau T - 2\frac{\tau'}{\kappa}N = A + \gamma \times J$$

is the restriction of an isometry to the curve γ. These equations and Euler$-$Lagrange equations (2.4) and (2.5) show that I and J are Killing fields along the critical curve γ of the functional (2.1) (see [8]). Observe that

$$< I, J >=< A, J >= 2\tau^3 + 4\left(\tau'\right)^2 \frac{\tau}{\kappa^2} = c, \tag{2.7}$$

where c is a constant. Multiplying $\frac{\tau}{\kappa}$ of Eq. (2.4) and substituting (2.7) into the obtained equation,(2.4) reduces to

$$\left(\frac{\tau'}{\kappa}\right)' = -\frac{\kappa}{4\tau^2}\left(2\tau^3 + \frac{c}{2}\right). \tag{2.8}$$

From Eqs. (2.7) and (2.3), Eqs. (2.4) and (2.5) are solved for κ as a function of $\tau:$

$$\kappa^2 = \frac{4\tau^8 - 4c\tau^5 + a^2\tau^4}{\left(2\tau^3 - \frac{c}{2}\right)^2}. \tag{2.9}$$

Substituting (2.9) into (2.7), the following equation is obtained

$$\left(\tau'\right)^2 = \frac{\left(2\tau^3 - c\right)}{\left(2\tau^3 - \frac{c}{2}\right)^2}\left(-\tau^7 + c\tau^4 - \frac{a^2}{4}\tau^3\right).$$

Thus, the torsion $\tau(s)$ can be expressed by quadratures

$$\pm\int \frac{\left(2\tau^3 - \frac{c}{2}\right)}{\sqrt{\left(2\tau^3 - c\right)\left(-\tau^7 + c\tau^4 - \frac{a^2}{4}\tau^3\right)}}d\tau = \int ds.$$

This gives rise to the following theorem.

Theorem 2.3 *Euler− Lagrange equations* (2.4) *and* (2.5) *of the functional* (2.1) *can be completely solved by quadratures.*

Now motivated by [8], Euler-Lagrange equations of the functional (2.1) could be solved. A preferred cylindrical coordinate system (r, θ, z) can be constructed by using Killing fields I and J. Since $J' = 0$ in $\mathbb{R}^3$, the Killing field J is a translation vector field, so one coordinate field is obtained as $\frac{\partial}{\partial z} = \frac{J}{\|J\|}$. Because of $< I, J >= c$, the Killing field I defines a rotation along $z-$direction. $J_1 = J - \left(\frac{1}{c}\right)\|J\|^2 I$ is a rotation field perpendicular to J. Thus, for a normalization factor $Q = \frac{c}{\|J\|^3}$, the second coordinate field is given by $\frac{\partial}{\partial \theta} = QJ_1$. Then $\frac{\partial}{\partial r}$ is given in terms of a cross product $\frac{\partial}{\partial r} = \frac{J \times B}{\|J \times B\|}$.

In the cylindrical coordinate system, the unit tangent vector T can be written as $T = r_s \frac{\partial}{\partial r} + \theta_s \frac{\partial}{\partial \theta} + z_s \frac{\partial}{\partial z}$. Taking inner product T with $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial z}$, one can easily obtain

$$
\begin{aligned}
r_s &= < T, \frac{\partial}{\partial r} >= -\frac{2\tau'}{\sqrt{\tau^2\kappa^2 + 4\left(\tau'\right)^2}}, \\
\theta_s &= \frac{1}{\left\| \frac{\partial}{\partial \theta} \right\|^2} < T, \frac{\partial}{\partial \theta} >= \frac{a\left(\tau - \frac{2c}{a^2}\tau^2\right)}{4\left(\tau^2 + \frac{(\tau')^2}{\kappa^2} - \frac{c^2}{a^2}\right)}, \\
z_s &= < T, \frac{\partial}{\partial z} >= -\frac{2}{a}\tau^2.
\end{aligned}
\tag{2.10}
$$

Therefore, the following theorem can be given.

Theorem 2.4 *Let* (r, θ, z) *be cylindrical coordinates whose coordinate fields defined above. Consider that* $\gamma(s) = (r(s), \theta(s), z(s))$ *is a critical point of the functional* (2.1)*. Then the equalities* (12) *are satisfied.*

3. Critical Points of the Functional in Minkowski 3-space

Let $\mathbb{R}^3_1$ denote Minkowski $3-$space with symmetric, bilinear and non-degenerate metric $\langle , \rangle$ such that for vectors $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ in $\mathbb{R}^3_1$

$$\langle x, y \rangle = -x_1y_1 + x_2y_2 + x_3y_3.$$

There are three families of curves depending on their causal character in Minkowski $3-$space. A curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3_1$ is a spacelike, timelike or null (lightlike) at t in I if its velocity vector $\gamma'(t)$ is a spacelike, timelike or null, respectively [10, 12]. In this section, the basic facts for regular nonnull curves and geometrical set up are given. Then, extremals of the functional produced by a binormal derivative of a regular non-null curve are studied in Subsection 3.2.

3.1. Variational formulas for nonnull curves with nonnull normal vector field

Consider $\gamma = \gamma(s) : I \subset \mathbb{R} \to \mathbb{R}^3_1$ be a nonnull unit-speed curve in Minkowski $3-$space $\mathbb{R}^3_1$. At a point $\gamma(s)$ of γ, let $T = \gamma^{'}(s)$ denote the unit tangent vector field to γ, $N(s)$ the unit principal normal vector field.

Then $\varepsilon_2 B(s) = T(s) \times N(s)$ is the unit binormal vector field, where $\varepsilon_2 =< B, B >$. Then $\{T, N, B\}$ is an orthonormal basis known as the Frenet frame along γ for all vectors at $\gamma(s)$ on γ. The derivative equations of Frenet frame $\{T, N, B\}$ are given by

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_1\kappa & 0 \\ -\varepsilon_0\kappa & 0 & \varepsilon_2\tau \\ 0 & -\varepsilon_1\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{3.1}$$

where $\varepsilon_0 =< T, T >$ and $\varepsilon_1 =< N, N >$. Also $\kappa > 0$ and τ are the curvature and torsion of γ, respectively [10, 12].

Assume W is a variational vector field along γ, then the following lemma is needed for some variatianol calculations in Section 3.2.

Lemma 3.1 (*see* [7, 14]). *Let $\gamma(w, t)$ be a variation of a curve $\gamma \in \mathbb{R}^3_1$. Then the following formulas are satisfied;*

$i)$ $[W, V] = 0,$

$ii)$ $W(v) = \varepsilon_0 < W', T > v,$

$iii)$ $W(\kappa) =< W'', N > -2\varepsilon_0\kappa < W', T >$

$iv)$ $W(\tau^2) = 2\varepsilon_1\tau\left(\frac{1}{\kappa} < W'', B >\right)' - 2\varepsilon_0\tau < W', (\tau T - \kappa B) > .$

3.2. Finding critical curves for the natural Hamiltonian in Minkowski 3-space

In this subsection, critical points of the functional $\mathcal{F}_\gamma = \int_\gamma \varepsilon_1\tau^2 ds$ which is a natural Hamiltonian generated by the inner product of the binormal derivative of a curve, i.e. $< B', B' >= \varepsilon_1\tau^2$, among a family of curves length ℓ in $\mathbb{R}^3_1$ with fixed end points and directions are investigated. For this purpose, making similar calculations as in Euclidean $3-$space, but using the Frenet equations (3.1) and Lemma 3.1, the first variation of $\mathcal{F}_\gamma$ is found as follows

$$\delta\mathcal{F}_\gamma(W) = \int_0^\ell < E[\gamma], W > ds + \left(S[\gamma, W]\right)\Big|_0^\ell$$

where

$$E[\gamma] = \left(3\varepsilon_0\tau^2\kappa + 2\varepsilon_1\frac{(\tau')^2}{\kappa} + 4\varepsilon_1\tau\left(\frac{\tau'}{\kappa}\right)'\right)N + 2\left(\varepsilon_1\varepsilon_2\tau'\frac{\tau^2}{\kappa} - \varepsilon_0\varepsilon_1(\tau\kappa)' - \left(\frac{\tau'}{\kappa}\right)''\right)B$$

and

$$\left(S[\gamma, W]\right)\Big|_0^\ell =< 2\frac{\tau}{\kappa}B, W'' > - < 2\frac{\tau'}{\kappa}B, W' > + < -\varepsilon_0\varepsilon_1\tau^2 T - 2\varepsilon_1\tau'\frac{\tau}{\kappa}N + 2\left(\varepsilon_0\varepsilon_1\tau\kappa + \left(\frac{\tau'}{\kappa}\right)'\right)B, W > \Bigg|_0^\ell .$$

According to the Noether theorem (see [4]), the first variation of $\mathcal{F}_\gamma$ is zero for the constant vector field W, so $J = -\varepsilon_0\varepsilon_1\tau^2 T - 2\varepsilon_1\tau'\frac{\tau}{\kappa}N + 2\left(\varepsilon_0\varepsilon_1\tau\kappa + \left(\frac{\tau'}{\kappa}\right)'\right)B$ is constant along a critical point of the functional $\mathcal{F}_\gamma$

because of $J' = -E[\gamma] = 0$. One can see that

$$\|J\|^2 = \varepsilon_0\tau^4 + 4\varepsilon_1 (\tau')^2 \frac{\tau^2}{\kappa^2} + 4\varepsilon_2 \left(\varepsilon_0\varepsilon_1\tau\kappa + \left(\frac{\tau'}{\kappa}\right)'\right)^2 = \frac{a^2}{4}, \tag{3.2}$$

where a is a constant. Moreover, the following equations are obtained

$$3\varepsilon_0\tau^2\kappa + 2\varepsilon_1 \frac{(\tau')^2}{\kappa} + 4\varepsilon_1\tau \left(\frac{\tau'}{\kappa}\right)' = 0 \tag{3.3}$$

and

$$\frac{\varepsilon_1\varepsilon_2}{3\kappa} \left(\tau^3\right)' - \varepsilon_0\varepsilon_1 (\tau\kappa)' - \left(\frac{\tau'}{\kappa}\right)'' = 0. \tag{3.4}$$

Theorem 3.2 *The critical points of the total squared torsion functional $\mathcal{F}_\gamma$ in Minkowski $3-$space are characterized by the Euler$-$Lagrange equations (3.3) and (3.4).*

Observe that

$$I = -2\varepsilon_0\varepsilon_1\tau T - 2\varepsilon_1 \frac{\tau'}{\kappa} N = A + \gamma \times J$$

and J are the Killing fields along the curve γ of the functional $\mathcal{F}_\gamma$. Thus, the following equation is found

$$< I, J >=< A, J >= 2\varepsilon_0\tau^3 + 4\varepsilon_1 (\tau')^2 \frac{\tau}{\kappa^2} = c, \tag{3.5}$$

where c is a constant. From (3.5), Euler$-$Lagrange equations (3.3) and (3.4) are solved for κ as a function of τ :

$$\kappa^2 = \frac{4\varepsilon_0\varepsilon_1\tau^8 - 4\varepsilon_2 c\tau^5 + \varepsilon_2 a^2\tau^4}{\left((4 - 2\varepsilon_1\varepsilon_2)\tau^3 - \varepsilon_1\frac{c}{2}\right)^2}.$$

These equations show that the torsion $\tau(s)$ can be expressed by quadratures

$$\pm \int \frac{\left((4 - 2\varepsilon_1\varepsilon_2)\tau^3 - \varepsilon_1\frac{c}{2}\right)}{\sqrt{(2\varepsilon_0\tau^3 - c)\left(-\varepsilon_0\varepsilon_1\varepsilon_2\tau^7 + \varepsilon_1\varepsilon_2 c\tau^4 - \varepsilon_1\varepsilon_2\frac{\alpha^2}{4}\tau^3\right)}} d\tau = \int ds.$$

This gives rise to the following theorem.

Theorem 3.3 *Euler$-$Lagrange equations* (15) *and* (16) *of the functional $\mathcal{F}_\gamma$ in Minkowski $3-$space can be completely solved by quadratures.*

These Euler$-$Lagrange equations could be solved similarly as Euclidean $3-$space. Killing fields I and J can be used to construct a system of cylindrical coordinates. Then the coordinates are obtained as $\frac{\partial}{\partial z} = \frac{J}{\|J\|}$, $\frac{\partial}{\partial\theta} = \frac{c}{\|J\|^3}(J - \left(\frac{1}{c}\right)\|J\|^2 I)$ and $\frac{\partial}{\partial r} = \frac{J\times B}{\|J\times B\|}$ (see[7, 8]).

The unit tangent vector T can be written as $T = r_s \frac{\partial}{\partial r} + \theta_s \frac{\partial}{\partial \theta} + z_s \frac{\partial}{\partial z}$. Taking inner product T with $\frac{\partial}{\partial r}$, $\frac{\partial}{\partial \theta}$ and $\frac{\partial}{\partial z}$, one can easily obtain

$$\begin{aligned} r_s &= -\frac{2\varepsilon_1 \tau'}{\sqrt{\left|\varepsilon_1 \tau^2 \kappa^2 + 4\varepsilon_0 \left(\tau'\right)^2\right|}}, \\ \theta_s &= \frac{\varepsilon_1 a \left(\tau - \frac{2c}{a^2}\tau^2\right)}{4\left(\varepsilon_0 \tau^2 + \varepsilon_1 \frac{(\tau')^2}{\kappa^2} - \frac{c^2}{a^2}\right)}, \\ z_s &= -\frac{2}{a}\varepsilon_1 \tau^2. \end{aligned} \tag{3.6}$$

Therefore, the following theorem can be given.

Theorem 3.4 *Let* (r, θ, z) *be cylindrical coordinates whose coordinate fields defined above in Minkowski* $3-$*space. Consider that* $\gamma(s) = (r(s), \theta(s), z(s))$ *is a critical point of the functional* $\mathcal{F}_\gamma$. *Then the equalities* (3.6) *are satisfied.*

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