Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2019) 43: 2941 – 2953
© TÜBİTAK
doi:10.3906/mat-1907-3

Approximately rings in proximal relator spaces

Ebubekir İNAN*
Department of Mathematics, Faculty of Arts and Sciences, Adıyaman University, Adıyaman, Turkey

Received: 01.07.2019 • **Accepted/Published Online:** 04.11.2019 • **Final Version:** 22.11.2019

Abstract: This article introduces approximately rings, approximately ideals, and approximately rings of all approximately cosets by considering new operations on the set of all approximately cosets. Afterwards, some properties of approximately rings and ideals were given.

Key words: Proximity space, relator space, descriptive approximation, approximately ring.

1. Introduction

Proximal relator space is a pair $(X, \mathcal{R}_\delta)$ that consists of a nonempty set X and set of proximity relations $\mathcal{R}_\delta$ defined on X. There are different types of proximity relations such as Efremovič proximity, Wallman proximity, descriptive proximity, and Lodato proximity [1, 7, 11]. In proximal relator space, the sets consist of nonabstract points which have location and features.

The aim of this concept is to obtain algebraic structures in proximal relator spaces using descriptively upper approximations of the subsets of nonabstract points. In 2017 and 2018, approximately semigroups and approximately ideals, approximately groups, and approximately subgroups were introduced by İnan [2–4]. Approximately Γ-semigroups were also introduced [5]. In these articles some examples of these approximately algebraic structures in digital images endowed with proximity relations were given. Approximately algebraic structures satisfy a framework for further applied areas such as image analysis or classification problems. The other theories of proximity spaces were introduced by Kula [8]. In the present article an explicit characterization of the separation properties and Tychonoff objects are given in the topological category of proximity space.

Essentially, the focus of this article is to obtain approximately rings, approximately ideals, and approximately rings of all descriptive approximately cosets by considering new operations on the set of all descriptive approximately cosets. Furthermore, some properties of approximately rings and approximately ideals will be introduced.

2. Preliminaries

Let X be a nonempty set and $\mathcal{R}$ be a family of relations on X. Let $\mathcal{R}$ be a family of proximity relations on X, then $(X, \mathcal{R}_\delta)$ is called proximal relator space. $\mathcal{R}_\delta$ contains proximity relations, for example Efremovič proximity δ_E [1], Lodato proximity $\delta_{\mathcal{L}}$ [7], Wallman proximity δ_ω, or descriptive proximity δ_Φ [10, 11, 13].

*Correspondence: einan@adiyaman.edu.tr
2010 *AMS Mathematics Subject Classification:* 08A05, 68Q32, 54E05

Throughout this article, the Efremovič proximity [1] and the descriptive proximity relations are considered.

An Efremovič proximity δ is a relation on $P(X)$ that satisfies the following conditions:

- $A\ \delta\ B \Rightarrow B\ \delta\ A$.
- $A\ \delta\ B \Rightarrow A \neq \emptyset$ and $B \neq \emptyset$.
- $A \cap B \neq \emptyset \Rightarrow A\ \delta\ B$.
- $A\ \delta\ (B \cup C) \Leftrightarrow A\ \delta\ B$ or $A\ \delta\ C$.
- $\{x\}\ \delta\ \{y\} \Leftrightarrow x = y$.
- $A\ \underline{\delta}\ B \Rightarrow \exists E \subseteq X$ such that $A\ \underline{\delta}\ E$ and $E^c\ \underline{\delta}\ B$ (*Efremovič Axiom*).

Lodato proximity [7] swaps the *Efremovič Axiom* with:

$$A\ \delta\ B \text{ and } \forall b \in B, \{b\}\ \delta\ C \Rightarrow A\ \delta\ C\ (\textit{Lodato Axiom}).$$

Let X be a set of nonabstract points which has a location and features [6, §3] in $(X, \mathcal{R}_{\delta_\Phi})$. Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a set of probe functions that represents features of $x \in X$.

A *probe functions* $\varphi_i : X \to \mathbb{R}$ represents features of a sample nonabstract point. Let $\Phi(x) = (\varphi_1(x), \ldots, \varphi_n(x)), (n \in \mathbb{N})$ be an object description denoting a feature vector of x, which provides a description of each $x \in X$. After choosing a set of probe functions, one obtains a descriptive proximity relation δ_Φ.

[9] Let X be a set of nonabstract points and $A, B \subseteq X$. The *set description* of $A \subseteq X$ is defined with

$$\mathcal{Q}(A) = \{\Phi(a) \mid a \in A\}.$$

The descriptive intersection of A and B is defined with

$$A \underset{\Phi}{\cap} B = \{x \in A \cup B \mid \Phi(x) \in \mathcal{Q}(A) \ \textit{and}\ \Phi(x) \in \mathcal{Q}(B)\}.$$

[10] If $\mathcal{Q}(A) \cap \mathcal{Q}(B) \neq \emptyset$, then A is called *descriptively near* or *descriptively proximal* to B. And it is denoted by $A\delta_\Phi B$.

[12] Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and $A \subseteq X$. Let $(A, \circ)$ and $(\mathcal{Q}(A), \cdot)$ be groupoids. Let us consider the object description Φ by means of a function

$$\Phi : A \subseteq X \longrightarrow \mathcal{Q}(A) \subset \mathbb{R}^n,\ x \mapsto \Phi(x),\ x \in A.$$

The object description Φ of A into $\mathcal{Q}(A)$ is an *object descriptive homomorphism* if $\Phi(x \circ y) = \Phi(x) \cdot \Phi(y)$ for all $x, y \in A$.

Definition 2.1 *[3] Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and $A \subseteq X$. A descriptively upper approximation of A is defined with*

$$\Phi^* A = \{x \in X \mid x\delta_\Phi A\}.$$

It is clear that $A \subseteq \Phi^* A$ for all $A \subseteq X$.

Lemma 2.2 *[3] Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be a descriptive proximal relator space and* A, B *be subsets of* X. *Then*

(i) $\mathcal{Q}(A \cap B) = \mathcal{Q}(A) \cap \mathcal{Q}(B)$,

(ii) $\mathcal{Q}(A \cup B) = \mathcal{Q}(A) \cup \mathcal{Q}(B)$.

Definition 2.3 *[3] Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be a descriptive proximal relator space and let "* $\cdot$ *" be a binary operation on* X. $G \subseteq X$ *is called an approximately groupoid if* $x \cdot y \in \Phi^* G$ *for all* $x, y \in G$.

Definition 2.4 *[2] Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be a descriptive proximal relator space and let "* $\cdot$ *" be a binary operation on* X. $G \subseteq X$ *is called an approximately group if the following conditions are true:*

$(\mathcal{A}G_1)$ $x \cdot y \in \Phi^* G$ *for all* $x, y \in G$,

$(\mathcal{A}G_2)$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ *property holds in* $\Phi^* G$ *for all* $x, y, z \in G$,

$(\mathcal{A}G_3)$ *There exists* $e \in \Phi^* G$ *such that* $x \cdot e = e \cdot x = x$ *for all* $x \in G$ *(*e *is called the approximately identity element of* G*),*

$(\mathcal{A}G_4)$ *There exists* $y \in G$ *such that* $x \cdot y = y \cdot x = e$ *for all* $x \in G$ *(*y *is called the inverse of* x *in* G *and denoted as* x^{-1}*).*

$S \subseteq X$ is called an *approximately semigroup* if

$(\mathcal{A}S_1)$ $x \cdot y \in \Phi^* S$ for all $x, y \in S$,

$(\mathcal{A}S_2)$ $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ property holds in $\Phi^* S$ for all $x, y, z \in S$

properties are satisfied.

If approximately semigroups have an approximately identity element $e \in \Phi^* S$ such that $x \cdot e = e \cdot x = x$ for all $x \in S$, then S is called an *approximately monoid*.

If $x \cdot y = y \cdot x$ for all $x, y \in S$ holds in $\Phi^* G$, then G is called *commutative approximately groupoid (semigroup, monoid, or group)*.

Theorem 2.5 *Let* $(X, \mathcal{R}_{\delta_\Phi})$ *be a descriptive proximal relator space and* $G \subseteq X$ *be an approximately group. Then the following are true:*

(i) There is one and only one approximately identity element in G.

(ii) There is one and only one inverse of elements in G.

(iii) If either $x \cdot z = y \cdot z$ *or* $z \cdot x = z \cdot y$, *then* $x = y$ *for all* $x, y, z \in G$.

Theorem 2.6 *[2] Let* G *be an approximately group,* H *be a nonempty subset of* G *and* $\Phi^* H$ *be a groupoid.* H *is an approximately subgroup of* G *if and only if* $x^{-1} \in H$ *for all* $x \in H$.

Let G be an approximately groupoid in $(X, \mathcal{R}_{\delta_\Phi})$, $x \in G$ and $A, B \subseteq G$. Then the subsets $x \cdot A, A \cdot x, A \cdot B \subseteq \Phi^* G \subseteq X$ are defined as:

$$x \cdot A = xA = \{xa \mid a \in A\},$$
$$A \cdot x = Ax = \{ax \mid a \in A\},$$
$$A \cdot B = AB = \{ab \mid a \in A, b \in B\}.$$

Lemma 2.7 *[2] Let (X, δ_Φ) be a descriptive proximity space, $A, B \subseteq X$ and $A, B, \mathcal{Q}(A), \mathcal{Q}(B)$ be groupoids. If $\Phi : X \longrightarrow \mathbb{R}^n$ is an object descriptive homomorphism, then*

$$\mathcal{Q}(A)\mathcal{Q}(B) = \mathcal{Q}(AB).$$

Theorem 2.8 *Let G be an approximately group, H be an approximately subgroup of G and $G/_{\rho_l}$ be a set of all descriptive approximately left cosets of G by H. If $(\Phi^* G)/_{\rho_l} \subseteq \Phi^*(G/_{\rho_l})$, then $G/_{\rho_l}$ is an approximately group under the operation given by $xH \odot yH = (x \cdot y) H$ for all $x, y \in G$.*

3. Approximately rings

Definition 3.1 *Let $(X, \mathcal{R}_{\delta_\Phi})$ be a descriptive proximal relator space and "$+$", "$\cdot$" be binary operations on X. $R \subseteq X$ is called an approximately ring if the following conditions are satisfied:*

$(\mathcal{A}R_1)$ *R is an abelian approximately group with "$+$",*

$(\mathcal{A}R_2)$ *R is an approximately semigroup with "$\cdot$",*

$(\mathcal{A}R_3)$ *For all $x, y, z \in R$, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ properties hold in $\Phi^* R$.*

In addition,

$(\mathcal{A}R_4)$ *If $x \cdot y = y \cdot x$ for all $x, y \in R$, then R is a commutative approximately ring.*

$(\mathcal{A}R_5)$ *If $\Phi^* R$ contains an element 1_R such that $1_R \cdot x = x \cdot 1_R = x$ for all $x \in R$,*

then R is called an approximately ring with identity.

These conditions have to hold in $\Phi^* R$. Addition or multiplying of finite number of elements in R may not always belong to $\Phi^* R$. Therefore, we cannot always say that $nx \in \Phi^* R$ or $x^n \in \Phi^* R$ for all $x \in R$ and some $n \in \mathbb{Z}^+$. If $(\Phi^* R, +)$ and $(\Phi^* R, \cdot)$ are groupoids, then $nx \in \Phi^* R$ for all integer n or $x^n \in \Phi^* R$ for all positive integer n, for all $x \in R$.

An element x in approximately ring R with identity is called *left (resp. right) invertible* if there exists $y \in R$ (resp. $z \in R$) such that $y \cdot x = 1_R$ (resp. $x \cdot z = 1_R$). The element y (resp. z) is called *a left (resp. right) inverse* of x. If $x \in R$ is both left and right invertible, then x is called *approximately invertible* or *approximately unit*. The set of approximately units in an approximately ring R with identity forms is an approximately group with multiplication.

An approximately ring R is an approximately division ring iff $(R \setminus \{0\}, \cdot)$ is an approximately group, that is, each nonzero element in R is an approximately unit. Moreover, an approximately ring R is an approximately field iff $(R \setminus \{0\}, \cdot)$ is a commutative approximately group.

Example 3.2 *Let I be a digital image endowed with δ_Φ. I is composed of 16 pixels (image elements) as shown in the Figure 1.*

An image element x_{ij} is a pixel in the location (i, j). Let φ be a probe function that represents (Red, Green, Blue) RGB codes of pixels that are given in Table .

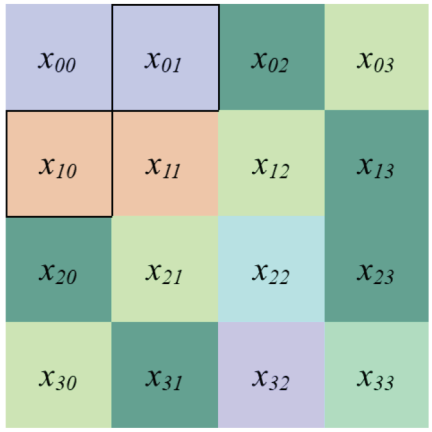

Figure 1. Digital image I and subimage R.

Table 1. RGB codes of pixels.

	Red	Green	Blue		Red	Green	Blue
x_{00}	193	202	253	x_{20}	100	160	145
x_{01}	193	202	253	x_{21}	204	245	185
x_{02}	100	160	145	x_{22}	181	232	231
x_{03}	204	245	185	x_{23}	100	160	145
x_{10}	237	198	169	x_{30}	204	245	185
x_{11}	237	198	169	x_{31}	100	160	145
x_{12}	204	245	185	x_{32}	200	200	250
x_{13}	100	160	145	x_{33}	170	240	200

Let

$$+ \quad : \quad \begin{array}{ll} I \times I & \longrightarrow I \\ (x_{ij}, x_{kl}) & \longmapsto x_{ij} + x_{kl} \end{array} \quad ,$$

$$x_{ij} + x_{kl} = x_{mn} \quad , i + k \equiv m \ (mod\ 2) \ and \ j + l \equiv n \ (mod\ 2)$$

be a binary operation on I *such that* $0 \leq i, j, k, l \leq 3$. *Let* $R = \{x_{01}, x_{10}\} \subseteq I$.

From Definition 2.1, descriptively upper approximation of R *is* $\Phi^* R = \{x_{ij} \in X \mid x_{ij} \delta_{\varphi} R\}$. *Hence,* $\varphi(x_{ij}) \cap Q(R) \neq \emptyset$ *such that* $x_{ij} \in I$, $Q(R) = \{\varphi(x_{ij}) \mid x_{ij} \in R\}$. *From Table,*

$$\begin{aligned} \mathcal{Q}(R) &= \{\varphi(x_{01}), \varphi(x_{10})\} \\ &= \{(193, 202, 253), (237, 198, 169)\}. \end{aligned}$$

Hence, we get $\Phi^* R = \{x_{00}, x_{01}, x_{10}, x_{11}\}$ *as in Figure 2.*

Thus, R *is an abelian approximately group with "+" in* $(I, \mathcal{R}_{\delta_\Phi})$ *from Definition 2.4. Furthermore, let*

$$\begin{array}{ll} \cdot : I \times I & \longrightarrow I \\ (x_{ij}, x_{kl}) & \longmapsto x_{ij} \cdot x_{kl} = x_{pr},\ p = \min\{i, k\} \ and \ r = \min\{j, l\} \end{array}$$

be a binary operation on I. *Then it is obivious that* R *is an approximately semigroup with "·" in* $(X, \mathcal{R}_{\delta_\Phi})$. *Also for all* $x_{ij}, x_{kl}, x_{mn} \in R$,

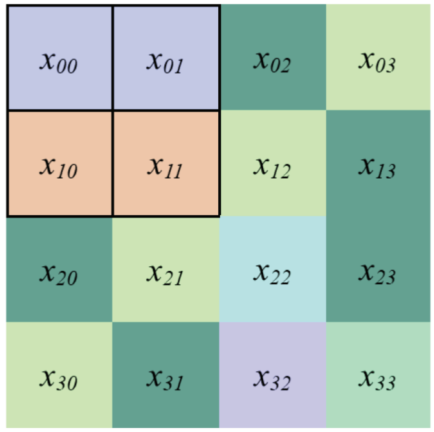

Figure 2. Descriptively upper approximation of R.

$x_{ij} \cdot (x_{kl} + x_{mn}) = x_{ij} \cdot x_{kl} + x_{ij} \cdot x_{mn}$ and $(x_{ij} + x_{kl}) \cdot x_{mn} = x_{ij} \cdot x_{mn} + x_{kl} \cdot x_{mn}$ properties hold in $\Phi^* R$. Consequently, R is an approximately ring.

In addition, since for all $x_{ij}, x_{kl} \in R$, $x_{ij} \cdot x_{kl} = x_{kl} \cdot x_{ij}$, R is a commutative approximately ring in digital image I.

Lemma 3.3 *All ordinary rings in proximal relator spaces are approximately rings.*

Proposition 3.4 *Let (X, δ_Φ) be a descriptive proximity space, $R \subseteq X$ be an approximately ring and $0 \in R$. If $0 \cdot x, x \cdot 0 \in R$, then for all $x, y \in R$*

- $x \cdot 0 = 0 \cdot x = 0$,
- $x \cdot (-y) = (-x) \cdot y = -(x \cdot y)$,
- $(-x) \cdot (-y) = x \cdot y$.

Definition 3.5 *Let (X, δ_Φ) be a descriptive proximity space, $R \subseteq X$ be an approximately ring and S be a nonempty subset of R. S is called approximately subring of R, if S is an approximately ring with binary operations "$+$" and "$\cdot$" on approximately ring R.*

Definition 3.6 *Let R be an approximately field and $S \subseteq R$. S is called approximately subfield of R, if S is an approximately field.*

Theorem 3.7 *Let (X, δ_Φ) be a descriptive proximity space, $R \subseteq X$ be an approximately ring and $(\Phi^* S, +)$, $(\Phi^* S, \cdot)$ be groupoids. A nonempty subset S of R is an approximately subring of R iff $-x \in S$ for all $x \in S$.*

Proof Let S be an approximately subring of R. Hence, S is an approximately ring and $-x \in S$ for all $x \in S$. On the other hand, let $-x \in S$ for all $x \in S$. Then since $(\Phi^* S, +)$ is a groupoid, $(S, +)$ is a commutative approximately group in $(X, \mathcal{R}_{\delta_\Phi})$ by Theorem 2.6. From the hypothesis, since $(\Phi^* S, \cdot)$ is a groupoid and $S \subseteq R$, then associative condition is satisfied in $\Phi^* S$. Hence, $(S, \cdot)$ is an approximately semigroup. For all $x, y, z \in S \subseteq R$, $y + z \in \Phi^* S$ and $x \cdot (y + z) \in \Phi^* S$. Moreover, $x \cdot y + x \cdot z \in \Phi^* S$. Since R is an approximately ring, $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ property holds in $\Phi^* S$. Moreover, it is clear that $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ property satisfies in $\Phi^* S$. Therefore, S is an approximately subring of R. □

Theorem 3.8 *Let (X, δ_Φ) be a descriptive proximity space, $R \subseteq X$ be an approximately ring, S_1 and S_2 two approximately subrings of R and $\Phi^* S_1$, $\Phi^* S_2$ be groupoids with the binary operations "$+$" and "$\cdot$". If*

$$(\Phi^* S_1) \cap (\Phi^* S_2) = \Phi^* (S_1 \cap S_2),$$

then $S_1 \cap S_2$ is an approximately subring of R.

Corollary 3.9 *Let (X, δ_Φ) be a descriptive proximity space, $R \subseteq X$ be an approximately ring, $\{S_i \mid i \in \Delta\}$ be a nonempty family of approximately subrings of R, and $\Phi^* S_i$ be groupoids. If*

$$\bigcap_{i \in \Delta} (\Phi^* S_i) = \Phi^* \left(\bigcap_{i \in \Delta} S_i \right),$$

then $\bigcap_{i \in \Delta} S_i$ is an approximately subring of R.

Definition 3.10 *Let (X, δ_Φ) be a descriptive proximity space, $R \subseteq X$ be an approximately ring and $I \subseteq R$. I is a left (right) approximately ideal of R provided for all $x, y \in I$ and for all $r \in R$, $x + y \in \Phi^* I$, $-x \in I$ and $r \cdot x \in \Phi^* I$ ($x \cdot r \in \Phi^* I$).*

A nonempty set I of an approximately ring R is called an *approximately ideal* of R if I is both a left and a right approximately ideal of R.

There is only one trivial approximately ideal of approximately ring R, that is, R itself. Moreover, $\{0\}$ is a trivial approximately ideal of approximately ring R iff $0 \in R$.

Lemma 3.11 *Every approximately ideal is an approximately subring of approximately ring R.*

Theorem 3.12 *Let (X, δ_Φ) be a descriptive proximity space, $R \subseteq X$ be an approximately ring, I_1 and I_2 two approximately ideals of R and $\Phi^* I_1$, $\Phi^* I_2$ be groupoids with the binary operations "$+$" and "$\cdot$". If*

$$(\Phi^* I_1) \cap (\Phi^* I_2) = \Phi^* (I_1 \cap I_2),$$

then $I_1 \cap I_2$ is an approximately ideal of R.

Proof Let I_1 and I_2 be two approximately ideals of R. It is obvious that $I_1 \cap I_2 \subseteq R$. Consider $x, y \in I_1 \cap I_2$. Since I_1 and I_2 are approximately ideals, $I_1 \subseteq \Phi^* I_1$ and $I_2 \subseteq \Phi^* I_2$, we have $x + y, -x, r \cdot x \in \Phi^* I_1$ and $x + y, -x, r \cdot x \in \Phi^* I_2$, that is, $x + y, -x, r \cdot x \in (\Phi^* I_1) \cap (\Phi^* I_2)$ for all $x, y \in I_1 \cap I_2$ and $r \in R$. Assuming $(\Phi^* I_1) \cap (\Phi^* I_2) = \Phi^* (I_1 \cap I_2)$, we have $x + y, -x, r \cdot x \in \Phi^* (I_1 \cap I_2)$. From Definition 3.10, $I_1 \cap I_2$ is an approximately ideal of R. □

Corollary 3.13 *Let (X, δ_Φ) be a descriptive proximity space, $R \subseteq X$ be an approximately ring, $\{I_i \mid i \in \Delta\}$ be a nonempty family of approximately ideals of R and $\Phi^* I_i$ be groupoids with the binary operations "$+$" and "$\cdot$". If*

$$\bigcap_{i \in \Delta} (\Phi^* I_i) = \Phi^* \left(\bigcap_{i \in \Delta} I_i \right),$$

then $\bigcap_{i \in \Delta} I_i$ is an approximately ideal of R.

Let R be an approximately ring and S be an approximately subring of R. The left weak equivalence relation "ρ_ℓ" defined as

$$x\rho_\ell y :\Leftrightarrow -x+y \in S \cup \{e\}.$$

A weak class defined by relation "ρ_ℓ" is called an *approximately left coset*. The approximately left coset that contains the element $x \in R$ is denoted by $\tilde{x}_\ell$, that is,

$$\tilde{x}_\ell = \{x+s \mid s \in S,\ x \in R,\ x+s \in R\} \cup \{x\}.$$

Similarly, we can define the approximately right coset that contains the element $x \in R$ denoted by $\tilde{x}_r$, that is,

$$\tilde{x}_r = \{s+x \mid s \in S,\ x \in R,\ s+x \in R\} \cup \{x\}.$$

We can easily show that $\tilde{x}_\ell = x+S$ and $\tilde{x}_r = S+x$. Since $(R,+)$ is an abelian approximately group, $\tilde{x}_\ell = \tilde{x}_r$ and so we use only notation $\tilde{x}$. Then

$$R/_\rho = \{x+S \mid x \in R\}$$

is a set of all approximately cosets of R by S. In this case, if we consider $\Phi^* R$ instead of approximately ring R

$$(\Phi^* R)/_\rho = \{x+S \mid x \in \Phi^* R\}.$$

Definition 3.14 *Let (X, δ_Φ) be a descriptive proximity space, $R \subseteq X$ be an approximately ring, and S be an approximately subring of R. For $x, y \in R$, let $x+S$ and $y+S$ be two approximately cosets that determined the elements x and y, respectively. Then the addition of two approximately cosets determined by $x+y \in \Phi^* R$ can be defined as*

$$(x+y)+S = \{(x+y)+s \mid s \in S,\ x+y \in \Phi^* R,\ \ (x+y)+s \in R\} \cup \{x+y\}$$

and denoted by

$$(x+S) \oplus (y+S) = (x+y)+S.$$

Definition 3.15 *Let (X, δ_Φ) be a descriptive proximity space, $R \subseteq X$ be an approximately ring, and S be an approximately subring of R. For $x, y \in R$, let $x+S$ and $y+S$ be two approximately cosets that determine the elements x and y, respectively. Then the multiplying of two approximately cosets that are determined by $x \cdot y \in \Phi^* R$ can be defined as*

$$(x \cdot y)+S = \{(x \cdot y)+s \mid s \in S,\ x \cdot y \in \Phi^* R,\ \ (x \cdot y)+s \in R\} \cup \{x \cdot y\}$$

and denoted by

$$(x+S) \odot (y+S) = (x \cdot y)+S.$$

Definition 3.16 *Let $R/_\rho$ be a set of all approximately cosets of R by S, $\xi_\Phi(A)$ be a descriptive approximately collections and $A \in P(X)$. Then*

$$\Phi^*(R/_\rho) = \bigcup_{\xi_\Phi(A) \underset{\Phi}{\cap} R/_\rho \neq \emptyset} \xi_\Phi(A)$$

is called upper approximation of $R/_\rho$.

Theorem 3.17 *Let $R \subseteq X$ be an approximately ring, S be an approximately subring of R and $R/_{\rho}$ be a set of all approximately cosets of R by S. If $(\Phi^{*}R)/_{\rho} \subseteq \Phi^{*}(R/_{\rho})$, then $R/_{\rho}$ is an approximately ring under the operations given by $(x+S) \oplus (y+S) = (x+y)+S$ and $(x+S) \odot (y+S) = (x \cdot y)+S$ for all $x, y \in R$.*

Proof $(\mathcal{A}R_1)$ Let $(\Phi^{*}R)/_{\rho} \subseteq \Phi^{*}(R/_{\rho})$. Since R is an approximately ring, $(R/_{\rho}, \oplus)$ is an abelian approximately group of all approximately cosets of R by S from Theorem 2.8.

$(\mathcal{A}R_2)$ $(\mathcal{A}S_1)$ Since $(R, \cdot)$ is an approximately semigroup, $x \cdot y \in \Phi^{*}R$ for all $x, y \in R$ and $(x+S) \odot (y+S) = (x \cdot y)+S \in (\Phi^{*}R)/_{\rho}$ for all $(x+S), (y+S) \in R/_{\rho}$. From the hypothesis, $(x+S) \odot (y+S) = (x \cdot y)+S \in N_r(B)^{*}(R/_{\rho})$ for all $(x+S), (y+S) \in R/_{\rho}$.

$(\mathcal{A}S_2)$ Since $(R, \cdot)$ is an approximately semigroup, associative property holds in $\Phi^{*}R$. Hence,

$$\begin{aligned} & ((x+S) \odot (y+S)) \odot (z+S) \\ = & ((x \cdot y)+S) \odot (z+S) = ((x \cdot y) \cdot z)+S \\ = & (x \cdot (y \cdot z))+S = (x+S) \odot ((y \cdot z)+S) \\ = & (x+S) \odot ((y+S) \odot (z+S)) \end{aligned}$$

holds in $(\Phi^{*}R)/_{\rho}$ for all $(x+S), (y+S), (z+S) \in R/_{\rho}$. By the hypothesis, associative property holds in $\Phi^{*}(R/_{\rho})$. Therefore, $(R/_{\rho}, \odot)$ is an approximately semigroup of all approximately left cosets of R by S.

$(\mathcal{A}R_3)$ Since R is an approximately ring, left distributive law holds in $\Phi^{*}R$. For all $(x+S), (y+S), (z+S) \in R/_{\rho}$,

$$\begin{aligned} & ((x+S) \odot (y+S) \oplus (z+S)) \\ = & ((x+S) \odot ((y+z)+S) \\ = & (x \cdot (y+z))+S = ((x \cdot y)+(x \cdot z))+S \\ = & ((x \cdot y)+S) \oplus ((x \cdot z)+S) \\ = & ((x+S) \odot (y+S)) \oplus ((x+S) \odot (z+S)). \end{aligned}$$

Hence, left distributive law holds in $(\Phi^{*}R)/_{\rho}$. Moreover, it is clear that right distributive law is satisfied in $(\Phi^{*}R)/_{\rho}$,

$$((x+S) \oplus (y+S)) \odot (z+S) = ((x+S) \odot (z+S)) \oplus ((x+S) \odot (z+S))$$

for all $(x+S), (y+S), (z+S) \in R/_{\rho}$.

By the hypothesis, distributive laws are satisfied in $\Phi^{*}(R/_{\rho})$. Consequently, $R/_{\rho}$ is an approximately ring. □

Definition 3.18 *Let $R \subseteq X$ be an approximately ring, S be an approximately subring of R. The approximately ring $R/_{\rho}$ is called an approximately ring of all approximately cosets of R by S and denoted by $R/_{\rho}S$.*

Definition 3.19 *Let (X, δ_{Φ}) be a descriptive proximity space, $R_1, R_2 \subseteq X$ be two approximately rings and ψ be a mapping from $\Phi^{*}R_1$ into $\Phi^{*}R_2$. If $\psi(x+y) = \psi(x)+\psi(y)$ and $\psi(x \cdot y) = \psi(x) \cdot \psi(y)$ for all $x, y \in R_1$, then ψ is called an approximately ring homomorphism and also, R_1 is called approximately homomorphic to R_2, denoted by $R_1 \simeq_n R_2$. An approximately ring homomorphism ψ of $\Phi^{*}R_1$ into $\Phi^{*}R_2$ is called*
(i) an approximately monomorphism if ψ is one-to-one,
(ii) an approximately epimorphism if ψ is onto,
(iii) an approximately isomorphism if ψ is one-to-one and onto.

Theorem 3.20 *Let $R_1, R_2 \subseteq X$ be two approximately rings and ψ be an approximately homomorphism of $\Phi^{*}R_1$ into $\Phi^{*}R_2$. Then the following properties are true:*

(i) $\psi(0_{R_1}) = 0_{R_2}$, *where* $0_{R_2} \in \Phi^* R_2$ *is the approximately zero of* R_2.

(ii) $\psi(-x) = -\psi(x)$ *for all* $x \in R_1$.

Proof (i) Since ψ is an approximately homomorphism, $\psi(0_{R_1}) \cdot \psi(0_{R_1}) = \psi(0_{R_1} \cdot 0_{R_1}) = \psi(0_{R_1}) = \psi(0_{R_1}) \cdot 0_{R_2}$. Thus, we have $\psi(0_{R_1}) = 0_{R_2}$ by the Theorem 2.5 (iii).

(ii) Let $x \in R_1$. Then $\psi(x) + \psi(-x) = \psi(x + (-x)) = \psi(0_{R_1}) = 0_{R_2}$. Moreover, it is clear that $\psi(-x) + \psi(x) = 0_{R_2}$ for all $x \in R_1$. From Theorem 2.5 (ii), since $\psi(x)$ has a unique inverse, $\psi(-x) = -\psi(x)$ for all $x \in R_1$. $\square$

Theorem 3.21 *Let* (X, δ_Φ) *be a descriptive proximity space,* $R_1, R_2 \subseteq X$ *be two approximately rings and* ψ *be an approximately homomorphism of* $\Phi^* R_1$ *into* $\Phi^* R_2$ *and* $\Phi^* S$ *be a groupoid. Then the following properties hold.*

(i) If S *is an approximately subring of* R_1 *and* $\psi(\Phi^* S) = \Phi^* \psi(S)$, *then* $\psi(S) = \{\psi(x) \mid x \in S\}$ *is an approximately subring of* R_2.

(ii) If S *is a commutative approximately subring* R_1 *and* $\psi(\Phi^* S) = \Phi^* \psi(S)$, *then* $\psi(S)$ *is a commutative approximately subring of* R_2.

Proof (i) Let S be an approximately subring of R_1. Then $0_S \in \Phi^* S$ and by Theorem 3.20 (i), $\psi(0_S) = 0_{R_2}$, where $0_{R_2} \in \Phi^* R_2$. Thus, $0_{R_2} = \psi(0_S) \in \psi(\Phi^* S) = \Phi^* \psi(S)$. Hence, $\psi(S) \neq \emptyset$. Let $\psi(x) \in \psi(S)$, $x \in S$. Since S is an approximately subring of R_1, $-x \in S \subseteq \Phi^* S$ for all $x \in S$. Thus, $-\psi(x) = \psi(-x) \in \psi(\Phi^* S) = \Phi^* \psi(S)$ for all $\psi(x) \in \psi(S)$. Hence, by Theorem 3.7, $\psi(S)$ is an approximately subring of R_2.

(ii) Let S be a commutative approximately subring and $\psi(x), \psi(y) \in \psi(S)$. We have $\psi(S)$ which is an approximately subring of R_2 by (i), that is, $\psi(S)$ is an approximately ring. Then $\psi(x) \cdot \psi(y) = \psi(x \cdot y) = \psi(y \cdot x) = \psi(y) \cdot \psi(x)$ for all $\psi(x), \psi(y) \in \psi(R_1)$. Hence, $\psi(S)$ is a commutative approximately subring of R_2. $\square$

Definition 3.22 *Let* $R_1, R_2 \subseteq X$ *be two approximately rings in* (X, δ_Φ) *and* ψ *be an approximately homomorphism of* $\Phi^* R_1$ *into* $\Phi^* R_2$. *The kernel of* ψ, *denoted by* $Ker\psi$, *is defined as*

$$Ker\psi = \{x \in R_1 \mid \psi(x) = 0_{R_2}\}.$$

Theorem 3.23 *Let* $R_1, R_2 \subseteq X$ *be two approximately rings in* (X, δ_Φ) *and* ψ *be an approximately homomorphism of* $\Phi^* R_1$ *into* $\Phi^* R_2$, $\Phi^* Ker\psi$ *be a groupoid with "$+$" and "$\cdot$". Then* $\emptyset \neq Ker\psi$ *is an approximately ideal of* R_1.

Proof Let $x, y \in Ker\psi$. Then $\psi(x + (-y)) = \psi(x) + \psi(-y) = \psi(x) - \psi(y) = 0_{R_2} - 0_{R_2} = 0_{R_2} \in \Phi^* R_2$ and so $x + (-y) \in \Phi^*(Ker\psi)$. Let $r \in R_1$. Then $\psi(r \cdot x) = \psi(r) \cdot \psi(x) = \psi(r) \cdot 0_{R_2} = 0_{R_2} \in \Phi^* R_2$ and so $r \cdot x \in \Phi^*(Ker\psi)$. Similarly, $x \cdot r \in \Phi^*(Ker\psi)$. Hence, from Definition 3.10, $Ker\psi$ is an approximately ideal of R_1. $\square$

Theorem 3.24 *Let* R *be an approximately ring in* (X, δ_Φ) *and* S *be an approximately subring of* R. *Then the mapping* $\Pi : \Phi^* R \to \Phi^*(R/_\rho S)$ *defined by* $\Pi(x) = x + S$ *for all* $x \in \Phi^* R$ *is an approximately homomorphism.*

Proof By definition of Π and Definition 3.15,

$$\Pi(x+y) = (x+y)+S = (x+S)\oplus(y+S) = \Pi(x)\oplus\Pi(y),$$
$$\Pi(x\cdot y) = (x\cdot y)+S = (x+S)\odot(y+S) = \Pi(x)\odot\Pi(y)$$

for all $x, y \in R$. As a result, from Definition 3.19 Π is an approximately homomorphism. □

Definition 3.25 *The approximately homomorphism* Π *is called an approximately natural homomorphism from* $\Phi^* R$ *into* $\Phi^*(R/_\rho S)$.

Definition 3.26 *Let* R_1, R_2 *be two approximately rings in* (X, δ_Φ) *and* S *be a nonempty subset of* R_1. *Let*

$$\chi : \Phi^* R_1 \longrightarrow \Phi^* R_2$$

be a mapping and

$$\chi_S = \chi\big|_S : S \longrightarrow \Phi^* R_2$$

be a restricted mapping. If $\chi(x+y) = \chi_S(x+y) = \chi_S(x)+\chi_S(y) = \chi(x)+\chi(y)$ *and* $\chi(x\cdot y) = \chi_S(x\cdot y) = \chi_S(x)\cdot\chi_S(y) = \chi(x)\cdot\chi(y)$ *for all* $x, y \in S$, *then* χ *is called a restricted approximately homomorphism and also,* R_1 *is called restricted approximately homomorphic to* R_2, *denoted by* $R_1 \simeq_r R_2$.

Theorem 3.27 *Let* $R_1, R_2 \subseteq X$ *be two approximately rings in* (X, δ_Φ) *and* χ *be an approximately homomorphism from* $\Phi^* R_1$ *into* $\Phi^* R_2$. *Let* $(\Phi^* Ker\chi, +)$ *and* $(\Phi^* Ker\chi, \cdot)$ *be groupoids and* $(\Phi^* R_1)/_\rho$ *be a set of all approximately cosets of* $\Phi^* R_1$ *by* $Ker\chi$. *If* $(\Phi^* R_1)/_\rho \subseteq \Phi^*(R_1/_\rho Ker\chi)$ *and* $\Phi^*\chi(R_1) = \chi(\Phi^* R_1)$, *then*

$$R_1/_\rho Ker\chi \simeq_r \chi(R_1).$$

Proof Since $(\Phi^* Ker\chi, +)$ and $(\Phi^* Ker\chi, \cdot)$ are groupoids, from Theorem 3.23 $Ker\chi$ is an approximately subring of R_1. Since $Ker\chi$ is an approximately subring of R_1 and $(\Phi^* R_1)/_\rho \subseteq \Phi^*(R_1/_\rho Ker\chi)$, then $R_1/_\rho Ker\chi$ is an approximately ring of all approximately cosets of R_1 by $Ker\chi$ by Theorem 3.17. Since $\Phi^*\chi(R_1) = \chi(\Phi^* R_1)$, $\chi(R_1)$ is an approximately subring of R_2. Define

$$\begin{array}{rcl} \psi : \Phi^*(R_1/_\rho Ker\chi) & \longrightarrow & \Phi^*\chi(R_1) \\ A & \longmapsto & \psi(A) = \begin{cases} \psi_{R_1/_\rho Ker\chi}(A) & , A \in (\Phi^* R_1)/_\rho \\ e_{\chi(R_1)} & , A \notin (\Phi^* R_1)/_\rho \end{cases} \end{array}$$

where

$$\begin{array}{rcl} \psi_{R_1/_\rho Ker\chi} = \psi\big|_{R_1/_\rho Ker\chi} : R_1/_\rho Ker\chi & \longrightarrow & \Phi^*\chi(R_1) \\ x + Ker\chi & \longmapsto & \psi_{R_1/_\rho Ker\chi}(x+Ker\chi) = \chi(x) \end{array}$$

for all $x + Ker\chi \in R_1/_\rho Ker\chi$.

Since

$$x + Ker\chi = \{x+k \mid k \in Ker\chi, x+k \in R_1\} \cup \{x\},$$
$$y + Ker\chi = \{y+k' \mid k' \in Ker\chi, y+k' \in R_1\} \cup \{y\}$$

and the mapping χ is an approximately homomorphism,

$$\begin{array}{ll} & x + Ker\chi = y + Ker\chi \\ \Rightarrow & x \in y + Ker\chi \\ \Rightarrow & x \in \{y + k' \mid k' \in Ker\chi, y + k' \in R_1\} \text{ or } x \in \{y\} \\ \Rightarrow & x = y + k',\ y + k' \in R_1 \text{ or } x = y \\ \Rightarrow & -y + x = (-y + y) + k', \text{ or } \chi(x) = \chi(y) \\ \Rightarrow & -y + x = k' \\ \Rightarrow & -y + x \in Ker\chi \\ \Rightarrow & \chi(-y + x) = e_{\chi(R_1)} \\ \Rightarrow & \chi(-y) + \chi(x) = e_{\chi(R_1)} \\ \Rightarrow & -\chi(y) + \chi(x) = e_{\chi(R_1)} \\ \Rightarrow & \chi(x) = \chi(y) \\ \Rightarrow & \psi_{R_1/_\rho Ker\chi}(x + Ker\chi) = \psi_{R_1/_\rho Ker\chi}(y + Ker\chi) \end{array}$$

Therefore, $\psi_{R_1/_\rho Ker\chi}$ is well defined.

For $A, B \in \Phi^*(R_1/_\rho Ker\chi)$, let $A = B$. Since the mapping $\psi_{R_1/_\rho Ker\chi}$ is well defined,

$$\begin{aligned} \psi(A) &= \begin{cases} \psi_{R_1/_\rho Ker\chi}(A) & , A \in (\Phi^* R_1)/_\rho \\ e_\chi(R_1) & , A \notin (\Phi^* R_1)/\rho \end{cases} \\ &= \begin{cases} \psi_{R_1/_\rho Ker\chi}(B) & , B \in (\Phi^* R_1)/_\rho \\ e_\chi(R_1) & , B \notin (\Phi^* R_1)/\rho \end{cases} \\ &= \psi(B) \end{aligned}$$

Therefore, ψ is well defined.

For all $x + Ker\chi, y + Ker\chi \in R_1/_\rho Ker\chi \subset \Phi^*(R_1/_\rho Ker\chi)$,

$$\begin{array}{ll} & \psi((x + Ker\chi) \oplus (y + Ker\chi)) \\ = & \psi_{R_1/_\rho Ker\chi}((x + Ker\chi) \oplus (y + Ker\chi)) \\ = & \psi_{R_1/_\rho Ker\chi}((x + y) + Ker\chi) \\ = & \chi(x + y) \\ = & \chi(x) + \chi(y) \\ = & \psi_{R_1/_\rho Ker\chi}(x + Ker\chi) + \psi_{R_1/_\rho Ker\chi}(y + Ker\chi) \\ = & \psi(x + Ker\chi) + \psi(y + Ker\chi) \end{array}$$

and

$$\begin{array}{ll} & \psi((x + Ker\chi) \odot (y + Ker\chi)) \\ = & \psi_{R_1/_\rho Ker\chi}((x + Ker\chi) \odot (y + Ker\chi)) \\ = & \chi_{R_1/_\rho Ker\chi}((x \cdot y) + Ker\chi) \\ = & \chi(x \cdot y) \\ = & \chi(x) \cdot \chi(y) \\ = & \psi_{R_1/_\rho Ker\chi}(x + Ker\chi) \cdot \psi_{R_1/_\rho Ker\chi}(y + Ker\chi) \\ = & \psi(x + Ker\chi) \cdot \psi(y + Ker\chi). \end{array}$$

Therefore, ψ is a restricted approximately homomorphism by Definition 3.26. Hence, $R_1/_\rho Ker\chi \simeq_r \chi(R_1)$.

$\square$

4. Conclusion

To extend this work, one could study the properties of other approximately algebraic structures arising from proximal relator spaces. Hopefully this concept provides a fundamental framework for some theoretical and applied sciences.

References

[1] Efremovič VA. The geometry of proximity I. Matematicheski Sbornik (N.S.) 1952; 31 (73) (1): 189-200.

[2] İnan E. Approximately groups in proximal relator spaces. Communication Faculty Science University Ankara Ser. A1 Mathematics Statistics 2019; 68 (1): 572-582.

[3] İnan E. Approximately semigroups and ideals: an algebraic view of digital images. Afyon Kocatepe University Journal of Science and Engineering. 2017; 17: 479-487.

[4] İnan E. Approximately subgroups in proximal relator spaces. Adıyaman University Journal of Science 2018; 8 (1): 24-41.

[5] İnan E, Uçkun M. Approximately Γ-semigroups in proximal relator spaces. Applicable Algebra in Engineering Communication and Computing 2019; 30 (4): 299-311.

[6] Kovăr M. A new causal topology and why the universe is co-compact, arXiv:1112.0817 [math-ph] 2011; 1-15.

[7] Lodato M. On topologically induced generalized proximity relations. PhD, Rutgers University, 1962.

[8] Muammer K. ST_2, ΔT_2, ST_3, ΔT_3, Tychonoff, compact and ∂-connected objects in the category of proximity spaces. Hacettepe Journal of Mathematics and Statistics 2019; 48: 490-500.

[9] Naimpally SA, Peters JF. Topology with Applications: Topological Spaces via Near and Far. Singapore: World Scientific, 2013.

[10] Peters JF. Near Sets: An Introduction. Mathematics Computer Science 2013; 7 (1): 3-9.

[11] Peters JF. Proximal relator spaces. Filomat 2019; 30 (2): 469-472.

[12] Peters JF, Öztürk MA, Uçkun M. Exactness of Proximal Groupoid Homomorphisms. Adıyaman University Journal of Science 2015; 5 (1): 1-13.

[13] Peters JF, İnan E, Öztürk MA. Spatial and descriptive isometries in proximity spaces. General Mathematics Notes 2014; 21 (2): 1-10.