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Principal parts of a vector bundle on projective line and the fractional derivative

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Abstract: This work is an exposition on computational aspects of principal parts of a vector bundle on projective line over the field of characteristic zero. Principal parts help determine the possibility of algebraically formalizing infinitesimal-neighborhoods of subschemes inside some ambient scheme. The purpose of this study is to look for the possibility of formalizing the algebraic geometric interpretation of fractional derivative. For the latter, this study follows the approach proposed by Vasily Tarasov. The difference is that Tarasov proposed a geometric interpretation using finite-order jet bundles from differential geometry. Present study proposes finite-order principal parts of the structure-sheaf of real projective line as its formal algebraic geometric parallel.

Key words: Principal parts of a vector bundle, fractional derivative, formal completion

1. Introduction

1.1. Principal parts and formal completion

Let X be a smooth scheme over some field K of characteristic zero.* Then from $X \overset{p_X}{\leftarrow} X \times X \overset{q_X}{\rightarrow} X$ and the diagonal embedding of X in $X \times X$ (i.e. $\triangle : X \to X \times X$), we obtain the exact sequence on $X \times X$,

$$0 \longrightarrow \mathcal{I}^n_{\triangle_X}/\mathcal{I}^{n+1}_{\triangle_X} \longrightarrow \mathcal{O}_{X\times X}/\mathcal{I}^{n+1}_{\triangle_X} \longrightarrow \mathcal{O}_{X\times X}/\mathcal{I}^{n}_{\triangle_X} \longrightarrow 0.$$

We define an nth order principal part of a vector bundle (locally free sheaf) $\mathcal{E}$ on X to be

$$J^n(\mathcal{E}) = p_{X*}(\mathcal{O}_{X\times X}/\mathcal{I}^{n+1}_{\triangle_X} \otimes q_X^*(\mathcal{E})).$$

$J^n(\mathcal{E})$ are vector bundles on $X, \forall n \in \mathbb{N} \cup \{0\}$ [11]. On the other hand, if Y is any closed subscheme of X, then there exists an abstract formal account of what counts an infinitesimal neighborhood of Y in X, when X is any noetherian scheme. If $\mathcal{I}_Y$ is the sheaf of ideals of Y in $\mathcal{O}_X$, then for each $n \in \mathbb{N}$, we can obtain a sequence of sheaves,

$$\ldots \to \mathcal{O}_X/\mathcal{I}^n_Y \to \mathcal{O}_X/\mathcal{I}^{n-1}_Y \to \cdots \to \mathcal{O}_X/\mathcal{I}^2_Y \to \mathcal{O}_X/\mathcal{I}^1_Y \to 0$$

which form an inverse system $(Y_n, \mathcal{O}_X/\mathcal{I}^n_Y)_{n\in\mathbb{N}}$ such that the topological structure of each Y_n is the same as Y.† These Y_n define the nth-order infinitesimal neighborhoods of Y in X. Since category of sheaves is

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*Unless stated otherwise, all schemes in this exposition are understood to be over a characteristic zero field K, i.e. they come equipped with a scheme morphism $X \to Spec(K)$.

†Cf. [7] II.9 for the detail of this construction.

closed under projective limits, we can define the infinite-order infinitesimal neighborhood (or simply infinitesimal neighborhood) as the formal completion of X along Y which is the ringed space $(\hat{X}, \mathcal{O}_{\hat{X}}) = (Y, \varprojlim(\mathcal{O}_X/\mathcal{I}_Y^n))$. $\hat{X}$ which corresponds to this structure is called the formal scheme. Its topological structure is also the same as Y. The case which is most important for us is when $Y = p$ is a closed point. In this case $\hat{X} = (\{p\}, \hat{\mathcal{O}}_p)$, which is just a one point scheme with structure sheaf $\hat{\mathcal{O}}_p = \varprojlim(R_p/I_p^n)$ such that $R = \Gamma(U, \mathcal{O}_X)$, $U =$Spec(R), $I_p^n = p^n R_{(p)}$ the maximal ideal in the local ring $R_{(p)}$ and U is an open set from local trivialization of X which contains p.

1.2. Noninteger fractional derivative

The derivatives of noninteger order have been proposed in many different formulations: for instance, Riemann–Liouville, Caputo, Nishimoto. [14, 15]. These different formulations sometimes lead to different versions of performing fractional calculus which has many significant applications in electrodynamics, fractal distribution, hydrodynamics, rigid-body dynamics, quantum systems of different types, fields, and media with long-range interactions [19, 21]. Much of these proposals are based upon ad hoc assumptions attested by the fact that many have hard time in making sense of these proposals both analytically and geometrically. For instance, generalization of the classical vector calculus in case of Riemann–Liouville fractional integral coupled with Caputo fractional derivative has been achieved with much progress. However, similar generalization is still an open problem in cases, for instance, Riesz, Grünwal-Latnikov, Weyl, Nishimito [21]. To account geometric sense of fractional derivative, several interpretations have been proposed depending upon the formulation [2, 3, 20], but none of these senses capitalize to become a proper framework for geometry. For instance, Adda's geometric interpretation ([2, 3]) which makes use of Nishimoto's fractional derivative presents the notion of fractional differential as a map $(d^\alpha f)|_{x_0} : \mathbb{R} \to \mathbb{R} := (d^\alpha f)|_{x_0}(h) = N^\alpha f(x_0)\frac{1}{\Gamma(\alpha+1)}h^\alpha, \alpha \in (0,1] \subset \mathbb{R}$ which becomes complex-valued as soon as $\alpha \neq \frac{1}{2n+1}, \forall n \in \mathbb{N}$. What is required is a comprehensive approach to ground any attempt of geometrically interpreting fractional differential calculus according to any particular formulation. To serve this end, Tarasov (cf. [22]) proposed an outline of differential geometric perspective that makes use of the notion of jet bundles in cases of fractional derivative of Riemann–Liouville, Caputo, and Hadamard type. We intend to propose an algebraic geometric perspective on it using a modification of projective limit of finite-order principal parts of vector bundles on real projective line $\mathbb{P}^1$ in section 3. For this, we need to make the following observations about the cases of Riemann–Liouville and Caputo fractional derivative.

1.2.1. The case of Riemann–Liouville

From [19] (15.3), every noninteger Riemann–Liouville (or RL) fractional derivative admits a series expansion in terms of integer-order derivatives as follows:

$$^{RL}\mathbb{D}^\alpha_{a+} f(x) = \sum_{n=0}^{\infty} \binom{\alpha}{n} \frac{(x-a)^{(n-\alpha)}}{\Gamma(n-\alpha+1)} f^{(n)}(x),$$

where f is any real analytic function on open interval (a,b), $f^{(n)}(x)$ is the standard n^{th}-derivative and $\alpha \in (0,\infty) \subset \mathbb{R}$. This gives rise to a general form

$$^{RL}\mathbb{D}^\alpha_{a+} f(x) = \sum_{n=0}^{\infty} A_n(\alpha)(x-a)^n f^{(n)}(x), \tag{1.1}$$

where $A_n(\alpha) \in \mathbb{R}\ \forall n \in \mathbb{N}$ at a particular point $x_0 \in (a,b)$.

1.2.2. The case of Caputo

From [10] (2.4), Caputo fractional derivative can be written in terms of RL-derivative as

$$^{C}\mathbb{D}^{\alpha}_{a+}f(x) = \left(^{RL}\mathbb{D}^{\alpha}_{a+}\Big[f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{(k)}}{\Gamma(k+1)} f^{(k)}(a)\Big]\right)(x)$$

, where $n = [\alpha]+1$ if $\alpha \notin \mathbb{N}$, $n = \alpha$ if $\alpha \in \mathbb{N}$, for every $\alpha > 0$, $\alpha = [\alpha] + \{\alpha\}$ such that $[\alpha] =$greatest integer less than or equal to $\alpha > 0$, $\{\alpha\} =$ fractional part of α, $x \in (a,b)$, $f \in C^{\infty}(I), (a,b) \subset I \subseteq \mathbb{R}$ without f necessarily being analytic over I. However, if f is analytic over I, then one obtains:

$$^{C}\mathbb{D}^{\alpha}_{a+}f(x) = \left(^{RL}\mathbb{D}^{\alpha}_{a+}\Big[\sum_{k=[\alpha]+1}^{\infty} \frac{(t-a)^{(k)}}{\Gamma(k+1)} f^{(k)}(a)\Big]\right)(x). \tag{1.2}$$

Thus, in each case, we see a recurring pattern of fractional derivatives being written in terms of convergent power series such that the coefficients of these power series a_n can be written in the form $B_n(\alpha)f^n(x)$, where f^n is the standard n^{th}-derivative of f and $B_n(\alpha) \in \mathbb{R}, \forall n \in \mathbb{N} \cup \{0\}$. Thus, if there is a geometric structure that contains all the information about all possible power series expansions with x a formal symbol, then one should expect to find the geometric interpretation of fractional derivative in this structure. Our claim is that this structure is a weighted-modified formal completion of n^{th}-order principal parts of the line bundle $\mathcal{O}_{\mathbb{P}^1}$. In case of positive integer order derivatives or standard derivative, the definition depends upon the point in an open interval. In physical applications, this leads to forgetting the history of the phenomenon being modeled [9] (cf. pp. 87-106). However, once we fractionalize this classical derivative through, for instance Riemann–Liouville sense, Caputo sense or Gr$\ddot{u}$nwal-Latnikov, we arrive at the possibility of modeling physical phenomena with memory. This is described and interpreted theoretically as the nonlocality of fractional derivative [9, 25]. Although there have been several attempts to devise local versions of fractional derivative, for instance, conformal local derivative [18], local fractional derivative of KG-type [4] (cf. pp. 250ff), nonconformable local-type fractional derivative [6]. However, after Tarasov's argument for the principle of nonlocality, attempts to found a fractional calculus on local operator now seriously require a theoretical rigor to avoid becoming objectionable [23, 24]. In particular, Abdelhakim [1] has shown that conformable fractional derivative is in fact integer-order derivative in fractional disguise, thereby advising the fractional calculus researchers against its use.

This paper is a formal algebraic geometric interpretation of fractional derivative aimed at opening future topics of research to relate algebraic geometry with fractional analysis just like it happened in case of classical analysis, for instance, in cases of differential forms, classical integration and derivations (just to give few examples). This historically correlated with the development of homological algebra of sheaves and schemes [5]. We first introduce some computational tools involving principal parts which are seen as algebraic finite-order neighborhoods. The results proved and calculations presented will help develop a view to interpret noninteger fractional derivative as weighted-modified formal completion of finite-order neighborhoods. In particular, this presentation is independent of the locality of classical derivative. Section 2 presents the algebraic geometric side necessary to make this required connection. Section 3 will present this connection.

2. On local description of principal parts

This section presents calculations involving principal parts of a vector bundle on smooth schemes defined over the field of characteristic zero. This algebraic side is motivated from the works of David Perkinson [17], Dan Laksov, Anders Thorup [11], and Maakestad [12, 13]. Primary source of basic computational methods is [7].

Proposition 2.1 *J^n is a functor $\forall n \in \mathbb{N}$. If $f : X \to Y$ is any morphism between smooth schemes, then the pullback functor f^* may not commute with J^n, i.e. $f^* \circ J^n_Y \neq J^n_X \circ f^*$; however, there exists a natural transformation $f^* \circ J^n_Y \to J^n_X \circ f^*$.*[‡]

Proof We first show that J^n is a functor. Let $\mathcal{F}$ be any coherent sheaf on X, then $J^n(\mathcal{F})$ is coherent, since pullbacks, pushforwards and tensor products of coherent sheaves are coherent [7]. To show that for any $g : \mathcal{F}_1 \to \mathcal{F}_2, h : \mathcal{F}_2 \to \mathcal{F}_3, \mathcal{F}_i$ all coherent,

$$J^n : J^n(\mathcal{F}_1) \overset{J^n(h\circ g)\simeq J^n(h)\circ J^n(g)}{\longrightarrow} J^n(\mathcal{F}_3)$$

and that $J^n(1_{\mathcal{F}}) = 1_{J^n(\mathcal{F})}$ are well-defined, one just has to verify that J^n is a composition of three functors, i.e. $J^n = p_* \circ F \circ q^*$, where $F = (\mathcal{O}_{\mathbb{P}\times\mathbb{P}}/\mathcal{I}^{n+1}_{\triangle} \otimes -)$. To show $f^* \circ J^n_Y \to J^n_X \circ f^*$, we proceed as follows. Consider the diagram in the category of smooth schemes,

$$\begin{tikzcd}
X \arrow[d, "f"] & X \times X \arrow[l, "p_X"'] \arrow[r, "q_X"] \arrow[d, "f\times f"] & X \arrow[d, "f"] \\
Y & Y \times Y \arrow[l, "p_Y"'] \arrow[r, "q_Y"] & Y.
\end{tikzcd}$$

Let $\mathcal{E}$ be a locally free sheaf (i.e. a vector bundle) on Y, then

$$\begin{aligned}
(f^* \circ J^n_Y)(\mathcal{E}) &= f^*(p_{Y*}(\mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}_{\triangle_Y} \otimes q^*_Y(\mathcal{E})) \\
&= (f^* \circ p_{Y*})(\mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}_{\triangle_Y} \otimes q^*_Y(\mathcal{E}))
\end{aligned}$$

applying [7] (III.9.3) on the left square to the above diagram, we obtain from the natural transformation $f^* \circ p_{Y*} \to p_{X*} \circ (f \times f)^*$, a map

$$(f^* \circ p_{Y*})(\mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}_{\triangle_Y} \otimes q^*_Y(\mathcal{E})) \xrightarrow{\alpha_1} (p_{X*} \circ (f \times f)^*)(\mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}_{\triangle_Y} \otimes q^*_Y(\mathcal{E})),$$

then,

$$\begin{aligned}
(p_{X*} \circ (f \times f)^*)(\mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}_{\triangle_Y} \otimes q^*_Y(\mathcal{E})) &= p_{X*}((f \times f)^*(\mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}_{\triangle_Y} \otimes q^*_Y(\mathcal{E}))) \\
&\simeq p_{X*}((f \times f)^*(\mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}_{\triangle_Y}) \otimes (f \times f)^* q^*_Y(\mathcal{E})) \\
&\simeq p_{X*}((f \times f)^*(\mathcal{O}_{Y\times Y}/\mathcal{I}^{n+1}_{\triangle_Y}) \otimes (q_Y(f \times f))^*(\mathcal{E})).
\end{aligned}$$

[‡]David Perkinson [17] proves a proposition from which the morphism $f^* \circ J^n_Y \to J^n_X \circ f^*$ can be derived as a special case. But the present proof is different. Maakestad [12, 13] also assumed the existence of morphism $f^* \circ J^n_Y \to J^n_X \circ f^*$ from the way Perkinson proved it.

We similarly obtain from right square (cf. [7](III.9.3)), a natural transformation $(q_Y \circ (f \times f))^* \to (fq_X)^*$, which gives a morphism of coherent sheaves on X as follows

$$p_{X*}((f \times f)^*(\mathcal{O}_{Y \times Y}/\mathcal{I}^{n+1}_{\triangle_Y}) \otimes (q_Y(f \times f))^*(\mathcal{E})) \xrightarrow{\alpha_2} p_{X*}((f \times f)^*(\mathcal{O}_{Y \times Y}/\mathcal{I}^{n+1}_{\triangle_Y}) \otimes (fq_X)^*(\mathcal{E})).$$

On the other hand, from $(f \times f) : X \times X \to Y \times Y$, one obtains the structure sheaf morphism $\beta_1 : \mathcal{O}_{Y \times Y} \to (f \times f)_*(\mathcal{O}_{X \times X})$, which induces the morphism $\beta_2 : \mathcal{O}_{Y \times Y}/\mathcal{I}^{n+1}_{\triangle_Y} \to (f \times f)_*(\mathcal{O}_{X \times X}/\mathcal{I}^{n+1}_{\triangle_X})$. Since $(f \times f)^*$ is left adjoint to $(f \times f)_*$ [8], this implies that

$$Hom(\mathcal{O}_{Y \times Y}/\mathcal{I}^{n+1}_{\triangle_Y}, (f \times f)_*(\mathcal{O}_{X \times X}/\mathcal{I}^{n+1}_{\triangle_X})) \simeq Hom((f \times f)^*(\mathcal{O}_{Y \times Y}/\mathcal{I}^{n+1}_{\triangle_Y}), \mathcal{O}_{X \times X}/\mathcal{I}^{n+1}_{\triangle_X}),$$

we thus obtain another morphism $\beta_4 : (f \times f)^*(\mathcal{O}_{Y \times Y}/\mathcal{I}^{n+1}_{\triangle_Y}) \to \mathcal{O}_{X \times X}/\mathcal{I}^{n+1}_{\triangle_X}$. Also since, $G = p_{X*}(- \otimes (fq_X)^*(\mathcal{E}))$ is a functor, letting $\alpha_3 = G(\beta_4)$, we obtain

$$p_{X*}((f \times f)^*(\mathcal{O}_{Y \times Y}/\mathcal{I}^{n+1}_{\triangle_Y}) \otimes (fq_X)^*(\mathcal{E}) \xrightarrow{\alpha_3} p_{X*}(\mathcal{O}_{X \times X}/\mathcal{I}^{n+1}_{\triangle_X} \otimes q_X^*(f^*(\mathcal{E}))).$$

The desired morphism $f^* \circ J^n_Y \to J^n_X \circ f^*$ is the composition $\alpha_3 \circ \alpha_2 \circ \alpha_1$. □

Corollary 2.2 *Let U be any open set in X. If $j : U \to X$ is the usual injection, then J^n commutes with j^*, i.e. $j^* \circ J^n \simeq J^n \circ j^*$.*

Proof Replacing f by j, X and Y by U and X respectively in Proposition 2.1 above, then since j is an open immersion and thus separated ([7] II.4), we obtain natural isomorphisms $j^* \circ p_{X*} \simeq p_{U*} \circ (j \times j)^*, (q_X \circ (j \times j))^* \simeq (j \circ q_U)^*$ and from the fact that $j^*(\mathcal{F}) \simeq \mathcal{F}|_U, \forall \mathcal{F}$ coherent sheaves on X, one obtains the isomorphism $(j \times j)^*(\mathcal{O}_{X \times X}/\mathcal{I}^{n+1}_{\triangle_X}) \simeq \mathcal{O}_{U \times U}/\mathcal{I}^{n+1}_{\triangle_U}$ (cf. [7] II.5). These isomorphism turns all $\alpha_i, 1 \leq i \leq 3$ in Proposition 2.1 into isomorphisms under the special case when $f = j$. We thus have the result.

□

Now consider $\mathbb{P}^1_K$ (or simply $\mathbb{P}^1$) to be the projective scheme Proj($K[x_0, x_1]$). Let $\triangle_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal embedding, where we may identify $\mathbb{P}^1 \times \mathbb{P}^1 \simeq$Proj($K[x_0, x_1]$)$\times_K$ Proj($K[y_0, y_1]$).[§] Topologically identifying $\triangle_{\mathbb{P}^1}(\mathbb{P}^1) \simeq \mathbb{P}^1$, we obtain the exact sequence

$$0 \longrightarrow \mathcal{I}_{\triangle_{\mathbb{P}^1}} \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow \triangle_{\mathbb{P}^1 *}(\mathcal{O}_{\mathbb{P}^1}) \longrightarrow 0$$

such that $\mathcal{I}_{\triangle_{\mathbb{P}^1}}$ is the sheaf of ideals of $\mathbb{P}^1$ in $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}$ and $\triangle_{\mathbb{P}^1 *}(\mathcal{O}_{\mathbb{P}^1})$ is the structure sheaf of $\triangle_{\mathbb{P}^1}(\mathbb{P}^1) \simeq \mathbb{P}^1$ pushed forward to $\mathbb{P}^1 \times \mathbb{P}^1$ which in this case reduces to extension of $\mathcal{O}_{\mathbb{P}^1}$ by zero outside $\triangle_{\mathbb{P}^1}(\mathbb{P}^1)$. From this we can obtain,

$$0 \longrightarrow \mathcal{I}^{n+1}_{\triangle_{\mathbb{P}^1}} \longrightarrow \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \longrightarrow (\triangle_{\mathbb{P}^1 *}(\mathcal{O}_{\mathbb{P}^1}))_n \longrightarrow 0, \tag{2.1}$$

where $(\triangle_{\mathbb{P}^1})_n$ is the n$^{\text{th}}$-infinitesimal neighborhood of $\triangle_{\mathbb{P}^1}$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Let $\mathcal{E}$ be any locally free sheaf of rank r on $\mathbb{P}$, then we can define n$^{\text{th}}$-order principal part of $\mathcal{E}$ to be $J^n(\mathcal{E}) = p_*(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}/\mathcal{I}^{n+1}_{\triangle} \otimes q^*(\mathcal{E}))$. However, since every vector bundle on $\mathbb{P}^1$ splits as direct sum of twisted line bundles $\mathcal{O}_{\mathbb{P}^1}(d), d \in \mathbb{Z}$, thus, working with finite-order principal parts of rank-r vector bundle $\mathcal{E}$ reduces to working with $\oplus_{i=1}^r(\mathcal{O}_{\mathbb{P}^1}(d_i)), d_i \in \mathbb{Z}, 1 \leq i \leq r$. We now restrict our attention only to the cases when $\mathcal{E} = \mathcal{O}_{\mathbb{P}^1}(d)$. Then, we have the following.

[§]Proj($K[x_0, x_1]$)$\times_K$ Proj($K[y_0, y_1]$) is the fiber product of $\mathbb{P}^1$ with itself over Spec(K).

Proposition 2.3 *Let U_0 be the open set in $\mathbb{P}^1$ determined by Spec$(K[\frac{x_1}{x_0}])$, where both x_i are homogeneous coordinates on $\mathbb{P}^1$, for $0 \leq i \leq 1$. Then the n^{th}-order principal part of $\mathcal{O}_{\mathbb{P}^1}|_{U_0}$ is globally generated by $\{1, (s-t), (s-t)^2, \ldots, (s-t)^n\}$ as a free $K[t]$-module, where s and t are local affine coordinates respectively on $\mathbb{P}^1 \times \mathbb{P}^1 \simeq Proj(K[x_0, x_1]) \times_{Spec(K)} Proj(K[x_0, x_1])$.*

Proof From the short exact sequence (s.e.s.) 2.1 above, we obtain

$$0 \longrightarrow \mathcal{I}^n_{\triangle_{\mathbb{P}^1}}/\mathcal{I}^{n+1}_{\triangle_{\mathbb{P}^1}} \longrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}/\mathcal{I}^{n+1}_{\triangle_{\mathbb{P}^1}} \longrightarrow \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}/\mathcal{I}^{n+1}_{\triangle_{\mathbb{P}^1}} \longrightarrow 0$$

from which we straightforwardly get the following exact sequences one after the other ([11]);

$$0 \longrightarrow \mathcal{I}^n_{\triangle_{\mathbb{P}^1}}/\mathcal{I}^{n+1}_{\triangle_{\mathbb{P}^1}} \longrightarrow J^n(\mathcal{E}) \longrightarrow J^{(n-1)}(\mathcal{E}) \longrightarrow 0$$

and

$$0 \longrightarrow \mathrm{Sym}^n_{\mathcal{O}_{\mathbb{P}^1}}(\Omega_{\mathbb{P}^1}) \otimes \mathcal{E} \longrightarrow J^n(\mathcal{E}) \longrightarrow J^{(n-1)}(\mathcal{E}) \longrightarrow 0. \tag{2.2}$$

Here $\Omega_{\mathbb{P}^1}$ denotes the sheaf of differentials on $\mathbb{P}^1$, $\mathrm{Sym}^n_{\mathcal{O}_{\mathbb{P}^1}}(\ )$ is the usual n-fold symmetric product of coherent sheaves considered as $\mathcal{O}_{\mathbb{P}^1}$-module. Since $J^0(\mathcal{O}_{\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1} \Rightarrow J^0(\mathcal{O}_{\mathbb{P}^1}|_{U_0}) = K[t]$, the statement is trivially true for $n = 0$. Let $n = 1$. Since $\Omega_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(-2)$, s.e.s. 2.2 gives

$$0 \longrightarrow \Omega_{\mathbb{P}^1} \longrightarrow J^1(\mathcal{O}_{\mathbb{P}^1}) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0.$$

Since the right most term is locally free, this exact sequence must split.[¶] Therefore,

$$J^1(\mathcal{O}_{\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$$
$$J^1(\mathcal{O}_{\mathbb{P}^1}|_{U_0}) = K[t] \oplus K[t]\{(s-t)\}$$

The last isomorphism follows from

$$\mathcal{O}_{\mathbb{P}^1}(-2) = \triangle^*_{\mathbb{P}^1}(\mathcal{I}_{\triangle_{\mathbb{P}^1}}/\mathcal{I}^2_{\triangle_{\mathbb{P}^1}})$$
$$(\mathcal{I}_{\triangle_{\mathbb{P}^1}}/\mathcal{I}^2_{\triangle_{\mathbb{P}^1}})|_{U_0\times U_0} \simeq ((1\otimes t - s\otimes 1)(K[s]\otimes_K K[t])/(1\otimes t - s\otimes 1)^2)^{Ash}$$
$$(\triangle^*_{\mathbb{P}^1}(\mathcal{I}_{\triangle_{\mathbb{P}^1}}/\mathcal{I}^2_{\triangle_{\mathbb{P}^1}}))|_{U_0} \simeq ((s-t)K[s,t]/(s-t)^2)^{Ash},$$

where $(\ )^{Ash}$ correspond to the fully faithful exact affine sheafification functor from the category of R-module (R any commutative ring with $1 \neq 0$) to the category of $\mathcal{O}_X$-module (cf. detail of its construction in [7] II.5.2) such that pushing forward along p_{U_0} gives $((s-t)K[s,t]/(s-t)^2)^{Ash}$ as an $\mathcal{O}_{Spec(K[s])}-$module.

For $n = 2$, we obtain

$$0 \longrightarrow \mathrm{Sym}^2_{\mathcal{O}_{\mathbb{P}^1}}(\Omega_{\mathbb{P}^1}) \longrightarrow J^2(\mathcal{O}_{\mathbb{P}^1}) \longrightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0,$$

which gives $J^2(\mathcal{O}_{\mathbb{P}^1}) \simeq \mathrm{Sym}^2_{\mathcal{O}_{\mathbb{P}^1}}(\Omega_{\mathbb{P}^1}) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}$. Since, $\mathrm{Sym}^2_{\mathcal{O}_{\mathbb{P}^1}}(\Omega_{\mathbb{P}^1}) \simeq p_{U_0 *}(\mathcal{I}^2_{\triangle_{\mathbb{P}^1}}/\mathcal{I}^3_{\triangle_{\mathbb{P}^1}})$ ([11]), restricting to U_0, we get

$$\mathrm{Sym}^2_{\mathcal{O}_{\mathbb{P}^1}}(\Omega_{\mathbb{P}^1})|_{U_0} \simeq p_{U_0 *}(\mathcal{I}^2_{\triangle_{\mathbb{A}^1}}/\mathcal{I}^3_{\triangle_{\mathbb{A}^1}})$$
$$\simeq ((s-t)^2 K[s,t]/(s-t)^3)^{Ash}$$

[¶]There is another way for this. Note that this s.e.s. is isomorphic to the fundamental Euler sequence of $\mathbb{P}^1$. In this case, $J^1(\mathcal{O}_{\mathbb{P}^1}) = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$

which implies

$$J^2(\mathcal{O}_{\mathbb{P}^1}|_{U_0}) = K[t] \oplus K[t]\{(s-t)\} \oplus K[t]\{(s-t)^2\}.$$

Using induction on n for,

$$\begin{aligned} J^n(\mathcal{O}_{\mathbb{P}^1}) &= \mathrm{Sym}^n_{\mathcal{O}_{\mathbb{P}^1}}(\Omega_{\mathbb{P}^1}) \oplus J^{(n-1)}(\mathcal{O}_{\mathbb{P}^1}) \\ &\simeq \mathrm{Sym}^n_{\mathcal{O}_{\mathbb{P}^1}}(\Omega_{\mathbb{P}^1}) \oplus \mathrm{Sym}^{(n-1)}_{\mathcal{O}_{\mathbb{P}^1}}(\Omega_{\mathbb{P}^1}) \oplus \cdots \oplus \Omega_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \end{aligned}$$

we obtain the result. □

Lemma 2.4 *If $\mathcal{L}$ is a line bundle on $\mathbb{P}^1$ then the n^{th}-order principal part of $\mathcal{L}$ on U_0 is globally generated by the sections $\{1 \otimes x_0^d, (s-t) \otimes x_0^d, \dots, (s-t)^n \otimes x_0^d\}$ as a free $K[t]$-module, for some $d \in \mathbb{Z}$.*[‖]

Proof Since $\mathcal{L} \in$Pic$(\mathbb{P}^1)$ implies that $\mathcal{L} \simeq \mathcal{O}_{\mathbb{P}^1}(d)$ for some $d \in \mathbb{Z}$, hence, using above proposition, we just have to show that $J^n(\mathcal{O}_{\mathbb{P}^1}(d)|_{U_0}) \simeq ((\mathbb{R}[t])[1 \otimes x_0^d, (s-t) \otimes x_0^d, \dots, (s-t)^n \otimes x_0^d])^{Ash}$. Since

$$\begin{aligned} J^n(\mathcal{O}_{\mathbb{P}^1}(d)) &= J^n(\mathcal{O}_{\mathbb{P}^1}) \otimes \mathcal{O}_{\mathbb{P}^1}(d) \\ &\simeq (\mathrm{Sym}^n_{\mathcal{O}_{\mathbb{P}^1}}(\Omega_{\mathbb{P}^1}) \oplus \mathrm{Sym}^{(n-1)}_{\mathcal{O}_{\mathbb{P}^1}}(\Omega_{\mathbb{P}^1}) \oplus \cdots \oplus \Omega_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}) \otimes \mathcal{O}_{\mathbb{P}^1}(d) \end{aligned}$$

then from Proposition 2.3 above, we are only left to locally compute the tensoring with $\mathcal{O}_{\mathbb{P}^1}(d)$ to verify the statement.

Consider $\mathbb{P}^1$ =Proj(S), $S = \oplus_{k \geq 0} S_k = \oplus_k (K[x_0, x_1])_k$, where $(K[x_0, x_1])_k$ corresponds to the k$^{\text{th}}$-homogeneous part of $K[x_0, x_1]$. If $(\ )^{Psh}$ denotes the projective sheafification of the graded ring S (cf. [7] II.5 for the detail of its construction), then $\mathcal{O}_{\mathbb{P}^1}(d) \simeq (S(d))^{Psh} \Rightarrow \mathcal{O}_{\mathbb{P}^1}(d)|_{U_0} \simeq \mathcal{O}_{Spec(S(d)_{(x_0)})} = (S(d))_{(x_0)})^{Ash}$ where $S(d) = \oplus_{k \geq 0} S_{k+d}, S(d)_{(x_0)} \subseteq S_{x_0}$. We prove the statement by showing that $S(d)_{(x_0)}$ is a free $S_{(x_0)}$-module of rank 1. Consider $f \in S(d)_{(x_0)}$, then locally, $f = \frac{P(x_0,x_1)}{x_0^k}$, with $P(x_0, x_1)$ being a homogeneous polynomial of degree $(k+d)$. Then f can be written as $f = x_0^d P(t)$. Define a map $S(d)_{(x_0)} \xrightarrow{\varphi} K[t][x^d], \varphi(f) = x_0^d P(t)$, then it is fairly straightforward to check that φ is an isomorphism. Hence, we obtain:

$$\mathcal{O}_{\mathbb{P}^1}(d)|_{U_0} \simeq (K[t]\{x_0^d\})^{Ash}$$

which is a free $K[t]$-module of rank 1. Thus, locally on affine neighborhood U_0, we obtain,

$$J^n(\mathcal{O}_{\mathbb{P}^1}(d))|_{U_0} = ((K[t])^{Ash} \oplus (K[t]\{(s-t)\})^{Ash} \oplus \cdots \oplus (K[t]\{(s-t)^n\})^{Ash}) \otimes (K[t]\{x_0^d\})^{Ash}.$$

The result follows by commuting tensor product with direct sum. □

3. The algebraic geometric view of noninteger fractional derivative

Let J^n denote the sheaf $J^n(\mathcal{O}_{\mathbb{P}^1})$ (unless stated otherwie) $\forall n \in \mathbb{N} \cup \{0\}$ and fix $K = \mathbb{R}$. Then from [11], we obtain $i^n_{\mathbb{P}^1} : \mathcal{O}_{\mathbb{P}^1} \to J^n$ as the structure map determining J^n as an $\mathcal{O}_{\mathbb{P}^1}$-algebra such that $i^n_{\mathbb{P}^1}$

[‖]Helge Maakestadt in [12, 13] refers to this statement without proving it. Since there is no proof as such presented anywhere else, authors have presented their own proof here.

is the same as the canonical structure-sheaf morphism which comes from the scheme morphism $(\triangle_{\mathbb{P}^1})_n = (\triangle_{\mathbb{P}^1}(\mathbb{P}^1), \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}/\mathcal{I}^n_{\triangle_{\mathbb{P}^1}}) \overset{p|_{\triangle_{\mathbb{P}^1}}}{\rightarrow} \mathbb{P}^1$, where $p|_{\triangle_{\mathbb{P}^1}}$ is the restriction of usual projection $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{p} \mathbb{P}^1$. In other words, $i^n_{\mathbb{P}^1} : \mathcal{O}_{\mathbb{P}^1} \to p|_{\triangle_{\mathbb{P}^1}*}(\mathcal{O}_{\triangle_{\mathbb{P}^1}})$.

We also get $d^n_{J^n} : \mathcal{O}_{\mathbb{P}^1} \to J^n$ such that d^n are all $\mathcal{O}_{Spec(\mathbb{R})}$-algebra morphisms (which are locally nothing but homomorphisms of $\mathbb{R}$-algebras determined by $J^n|_{U_0}$). Here, local amounts to the transition from $\mathbb{P}^1$ to $\mathbb{R}$ (or respectively from $\mathbb{P}^1\times\mathbb{P}^1$ to $\mathbb{R}\times\mathbb{R}$) in terms of affine coordinate s (or respectively both s,t).** For $n=0$, $d^0_{J^0}$ is the identity. For $n=1$ we get the formal local description of $i^1_{\mathbb{P}^1}$ and $d^1_{J^1}$ as follows: f is any section of $U_0 = \mathbb{A}^1 = \mathbb{R}$ in $\mathcal{O}_{\mathbb{P}^1}$, then $i^1_{U_0}(f) = f(t) \in \mathbb{R}[t]$, $(d^1_{J^1})_{U_0}(f) = f(t) + D(f)(t)dt$, such that $D(f)$ is the second formal coordinate of $(d^1_{J^1})_{U_0}(f)$ in $\Gamma(U_0, J^1)$. It is fairly straightforward that $i^1_{\mathbb{P}^1}$ provides the splitting of the exact sequence

$$0 \longrightarrow \Omega_{\mathbb{P}^1} \longrightarrow J^1(\mathcal{O}_{\mathbb{P}^1}) \longrightarrow \mathcal{O}_{\mathbb{P}^1} \longrightarrow 0$$

such that $d =_{\text{df}} d^1 - i^1_{\mathbb{P}^1} : \mathcal{O}_{\mathbb{P}^1} \to \Omega_{\mathbb{P}^1}$ is the universal derivation. Here, the symbols 'd' (as universal derivation), 'D' (as second coordinate), etc. are formal algebraic symbols. However, they do admit usual univariate calculus definitions in the form of derivative and differential once we are able to interpret this in the definition of $J^1 = p_*((\mathcal{O}_{\triangle_{\mathbb{P}^1}}/\mathcal{I}^2_{\triangle_{\mathbb{P}^1}}) \otimes q^*(\mathcal{O}_{\mathbb{P}^1})$. This happens as follows: locally, pullback q^* is just taking $f(t)$ to $(t, f(t))$ which is only formally differentiated when tensored with $(\mathcal{O}_{\triangle_{\mathbb{P}^1}}/\mathcal{I}^2_{\triangle_{\mathbb{P}^1}})$ (using Proposition 2.3 above) such that push-forward p_* then gives the information as $f(t) + D(f)(t)dt$, whereas $(s-t)$ is identified as dt. In general, $d^n_{J^n}(f)|_{t_0} = f(t_0) + D(f)(t_0)(s-t_0) + \frac{1}{2}D^2(f)(t_0)(t)(s-t_0)^2 \cdots \frac{1}{n!}D^n(f)(t_0)(t)(s-t_0)^n$ ([11]). This gives $(J^n, d^n_{J^n})$ such that the fibers or stalks $J^n_{t_0}$ determine the structure of n$^{\text{th}}$-order Taylor polynomial associated with the section f at the point t_0. Assuming that there will not be any ambiguity, we set $J^n = d^n_{J^n}$, then $J^n|_{t_0}$ satisfies all the linearity and leibnizian properties that we should expect. Since $(U_0, J^n(\mathcal{O}_{\mathbb{P}^1})|_{U_0})_{n\geq 0}$ forms an inverse system, taking projective limit, we get, $(U_0, \varprojlim(J^n((\mathcal{O}_{\mathbb{P}^1})|_{U_0})_{n\geq 0})) =_{\text{df}} J^\infty$ which corresponds to the formal completion discussed in Subsection 1.1 above.†† From Proposition 2.3, we find that it is the ring of formal power series $\mathbb{R}[[(s-t_0)]]$ with s and t both being the local affine coordinates on $\mathbb{P}^1\times\mathbb{P}^1$ respectively, such that the coordinates of every section in the fiber $J^\infty(f)|_{t_0} = \varinjlim(\Gamma(V, \mathcal{O}_{\mathbb{P}^1}))$ (limit is taken over all open sets V containing t_0) is an infinite column vector $(f(t_0), D(f)(t)(t_0)), \frac{1}{2}D^2(f)(t)(t_0), \cdots, \frac{1}{n!}D^n(f)(t)(t_0), \cdots)$. This is the algebraic infinitesimal-neighborhood of the point $t_0 \in \mathbb{R}$.

Define a weighted finite-order principal part of $\mathcal{O}_{\mathbb{P}^1}$ as J^n_α such that its action on a section $(f) \in \Gamma(V, \mathcal{O}_{\mathbb{P}^1}))$ is determined as $J^n_\alpha|_{t_0}(f) = f(t_0) + A'_0(\alpha)D(f)(t_0)(s-t_0) + A'_1(\alpha)D^2(f)(t_0)(t)(s-t_0)^2 \cdots A'_n(\alpha)D^n(f)(t_0)(t)(s-t_0)^n$, such that $A'_i(\alpha) = (i!)A_i(\alpha)$, $A_i(\alpha)$ as in equation 1.1 (cf. Subsection 1.2). This gives its corresponding modified projective limiting version as the RL-fractional-derivative

$$^{RL}\mathbb{D}^\alpha_{t_0+}(f) = J^\infty_{\alpha,t_0}(f). \tag{3.1}$$

Let $\tilde{J}^\infty_{[\alpha]+1}|_{t_0}$ be defined by its action on the section f:

**cf. section 2 above.

††This J^∞ that we have have constructed is different from Tarasov's Infinite-order jet [22].

$$\tilde{J}^{\infty}_{[\alpha]+1}|_{t_0}(f) = \sum_{i=0}^{\infty} \frac{1}{\Gamma([\alpha]+1+(i+1))} D^{([\alpha]+1+i)}(f)(t_0)(s-t_0)^{([\alpha]+1+i)}.$$

This similarly defines a modified projective limiting version as the C-fractional-derivative:

$$^{C}\mathbb{D}^{\alpha}_{t_0+} = J^{\infty}_{\alpha,t_0} \circ \tilde{J}^{\infty}_{([\alpha]+1),t_0}. \tag{3.2}$$

From 3.1 and 3.2, we observe that noninteger fractional derivative of both Riemann–Liouville type and Caputo type admit an interpretation as modifications of projective limits of finite-order principal parts of the structure-sheaf $\mathcal{O}_{\mathbb{P}^1}$. However, there is a limitation of our approach in this paper. This follows from the difference of topology between Zariski topology of schemes and the manifold topology on $\mathbb{R}$. The projective limit of principal parts involve local neighborhoods which are too coarse topologically, since each of these neighborhood is dense in the ambient scheme (here $\mathbb{P}^1$). On the contrary, any neighborhood in $\mathbb{R}$ considered as manifold (which Tarasov in [22] has assumed) consists of (can be covered by) finite open intervals which are topologically too fine.[‡‡] Secondly, the weights α and $[\alpha]+1$ for defining both modified finite-order neighborhoods and infinitesimal-neighborhoods suggest that there is a deeper connection between the diagonal embedding of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and the parameterizing of this embedding by a topological space that is not Zariski, i.e. not an algebraic variety, especially if the parameterizing depends upon the weight α. This remains an open problem in this paper towards which a further rigor can be developed.

4. Conclusion

In Section 2, we have presented explicit calculations involving local description of principal parts of vector bundles on $\mathbb{P}^1$. This helped develop, in Section 3, an interpretation of fractional derivative of both Riemann–Liouville and Caputo type. According to this, noninteger fractional derivatives are formally the weighted-modified projective limits of local restriction of principal parts of the structure sheaf of $\mathbb{P}^1$. In particular, as far as the formal interpretation is concerned, this local restriction is independent of the locality of integer-order derivative. However, we have also stated some limitations on our interpretation. In particular, weighted-modification of stated projective limits suggests that there is a deeper connection between the diagonal embedding of $\mathbb{P}^1$ in $\mathbb{P}^1 \times \mathbb{P}^1$ and the parameterizing of this embedding by a topological space which remains an open problem in this paper.

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[‡‡]Here, 'topologically too fine' means that the topology of $\mathbb{R}$ as a subspace of $\mathbb{P}^1$ can be considered as a subcollection or special case of the topology that [22] is using for $\mathbb{R}$; cf. fine and coarse spaces in [16], p. 77.

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