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Study on the q-analogue of a certain family of linear operators

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Abstract: In this paper, we introduce the q-analogue of a certain family of linear operators in geometric function theory. Our main purpose is to define some subclasses of analytic functions by means of the q-analogue of linear operators and investigate various inclusion relationships with integral preserving properties.

Key words: Analytic functions, q-difference operator, q-Hurwitz–Lerch zeta function, q-Srivastava–Attiya operator, q-multiplier transformation

1. Introduction

Let $\mathbf{A}$ denote the class of analytic functions $f(z)$ in the open unit disk $\mathbf{O} = \{z : |z| < 1\}$ such that

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Subordination of two functions f and g is denoted by $f \prec g$ and defined as $f(z) = g(w(z))$, where $w(z)$ is the Schwartz function in $\mathbf{O}$ (see [18]). Let S, S^*, and C denote the subclasses of $\mathbf{A}$ of univalent functions, starlike functions, and convex functions, respectively. We recall here some basic definitions and details of q-calculus that are used in this paper.

The q-difference operator, which was introduced by Jackson [12], is defined by

$$D_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}; \quad q \neq 1, \; z \neq 0 \tag{1.2}$$

for $q \in (0,1)$. It is clear that $\lim_{q \to 1^-} D_q f(z) = f'(z)$, where $f'(z)$ is the ordinary derivative of the function. For more properties of D_q see [5–7, 16, 27].

It can easily be seen that for $n \in \mathbb{N} = \{1, 2, 3, ..\}$ and $z \in \mathbf{O}$

$$D_q \left\{ \sum_{n=1}^{\infty} a_n z^n \right\} = \sum_{n=1}^{\infty} [n]_q \, z^{n-1}, \tag{1.3}$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + ... + q^{n-1}. \tag{1.4}$$

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We have the following rules of D_q:

$$D_q\left(af\left(z\right)\pm bg\left(z\right)\right)=aD_qf\left(z\right)\pm bD_qg\left(z\right), \tag{1.5}$$

$$D_q\left(f\left(z\right)g\left(z\right)\right)=f\left(qz\right)D_q\left(g\left(z\right)\right)+g(z)D_q\left(f\left(qz\right)\right), \tag{1.6}$$

$$D_q\left(\frac{f(z)}{g(z)}\right)=\frac{D_q\left(f(z)\right)g(z)-f(z)D_q\left(g(z)\right)}{g(qz)g(z)},\quad g(qz)g(z)\neq 0, \tag{1.7}$$

$$D_q\left(\log f(z)\right)=\frac{D_q\left(f(z)\right)}{f(z)}. \tag{1.8}$$

In the context of geometric function theory, usage of q-difference equations was first applied by Jackson [13], Carmichael [4], Mason [17], Adams [1], and Trijitzinsky [26]. Some properties related with q-function theory were first introduced by Ismail et al. [11]. Moreover, several authors studied many applications of q-calculus associated with generalized subclasses of analytic functions; see [2, 19, 21, 28]. The study of linear operators plays an important and vital role in the field of geometric function theory. In recent days, many well-known researchers are interested in introducing and studying those linear operators in terms of q-analogues. In [15], the authors introduced the q-analogue of the Ruscheweyh derivative operator and studied some of the properties of this differential operator. Noor et al. [22] introduced the q-Bernardi integral operator. Furthermore, by using the concept of q-calculus, Arif et al. [3] defined the q-Noor integral operator and investigated a number of important properties.

Motivated by the work mentioned above, we introduce the q-Srivastava–Attiya operator and the q-multiplier transformation on $\mathbf{A}$ as follows.
First, we give the q-analogue of the Hurwitz–Lerch zeta function by the following series:

$$\varphi_q\left(s,b;z\right)=\sum_{n=0}^{\infty}\frac{z^n}{[n+b]_q^s}, \tag{1.9}$$

where $b\in\mathbb{C}\setminus\mathbb{Z}_0^-$, $s\in\mathbb{C}$ when $|z|<1$, and $Re\left(s\right)>1$ when $|z|=1$.

From (1.9) we can easily get

$$\begin{aligned}\psi_q\left(s,b;z\right) &= [1+b]_q^s\left\{\varphi_q\left(s,b;z\right)-[b]_q^s\right\}\\ &= z+\sum_{n=2}^{\infty}\left(\frac{[1+b]_q}{[n+b]_q}\right)^s z^n.\end{aligned} \tag{1.10}$$

Now by (1.10) and (1.1), we define the q-Srivastava–Attiya operator $J_{q,b}^s:\mathbf{A}\to\mathbf{A}$ by

$$\begin{aligned}J_{q,b}^s f(z) &= \psi_q\left(s,b;z\right)*f(z)\\ &= z+\sum_{n=2}^{\infty}\left(\frac{[1+b]_q}{[n+b]_q}\right)^s a_n z^n,\end{aligned} \tag{1.11}$$

where * denotes convolution (or the Hadamard product).

We note that:

(i) If $q \to 1^-$, then the function $\varphi_q(s,b;z)$ reduces to the Hurwitz–Lerch zeta function and the operator $J_{q,b}^s$ coincides with the Srivastava–Attiya operator; we refer to [24, 25].

(ii) $J_{q,0}^1 f(z) = \int_0^z \frac{f(t)}{t} d_q t$. (q-Alexander operator).

(iii) $J_{q,b}^1 f(z) = \frac{[1+b]_q}{z^b} \int_0^z t^{b-1} f(t) d_q t$. (q-Bernardi operator [22]).

(iv) $J_{q,1}^1 f(z) = \frac{[2]_q}{z} \int_0^z f(t) d_q t$. (q-Libera operator [22]).

For any real number s, the q-analogue of multiplier transformation $I_{q,s}^b : \mathbf{A} \to \mathbf{A}$ is defined as follows:

$$I_{q,s}^b f(z) = z + \sum_{n=2}^{\infty} \left(\frac{[n+b]_q}{[1+b]_q} \right)^s a_n z^n, \tag{1.12}$$

where $f \in \mathbf{A}$ and $b > -1$. It is noted that for nonnegative integers s and $b = 0$, the operator $I_{q,s}^b$ becomes the well-known Salagean q-differential operator [8], which was studied by several authors [9, 10].

From (1.11) and (1.12), we easily have the following identities:

$$zD_q\left(J_{q,b}^{s+1} f(z)\right) = \left(1 + \frac{[b]_q}{q^b}\right) J_{q,b}^s f(z) - \frac{[b]_q}{q^b} J_{q,b}^{s+1} f(z), \tag{1.13}$$

$$zD_q\left(I_{q,s}^b f(z)\right) = \left(1 + \frac{[b]_q}{q^b}\right) I_{q,s+1}^b f(z) - \frac{[b]_q}{q^b} I_{q,s}^b f(z). \tag{1.14}$$

Using the q-derivative, we now define some new classes of analytic functions below.

Let Φ be the class of analytic functions $\varphi(z)$, which are univalent convex functions in $\mathbf{O}$ with $\varphi(0) = 1$ and $Re(\varphi(z)) > 0$ in $\mathbf{O}$.

Definition 1.1 *A function $f \in \mathbf{A}$ is said to be in the class $ST_q(\varphi)$ if it satisfies the following condition:*

$$\frac{zD_q f(z)}{f(z)} \prec \varphi(z),$$

where D_q is the q-difference operator.

Analogously, a function $f \in \mathbf{A}$ is said to be in the class $CV_q(\varphi)$ if and only if

$$zD_q f(z) \in ST_q(\varphi). \tag{1.15}$$

Using the operators defined above, we define the following:

Definition 1.2 *Let $f \in \mathbf{A}$, s be real, and $b > -1$. Then*

$$f \in ST_{q,b}^s(\varphi) \text{ if and only if } J_{q,b}^s f(z) \in ST_q(\varphi),$$

and

$$f \in CV_{q,b}^s(\varphi) \text{ if and only if } J_{q,b}^s f(z) \in CV_q(\varphi). \tag{1.16}$$

It is clear that

$$f \in CV_{q,b}^s(\varphi) \text{ if and only if } z(D_q f) \in ST_{q,b}^s(\varphi). \tag{1.17}$$

Special cases:

(i) If $s = 0$ and $\varphi(z) = \frac{1+Az}{1+Bz}$ $(-1 \leq B < A \leq 1)$, then $ST^{s}_{q,b}(\varphi)$ reduces to the class $S^{*}_{q}(A,B)$ introduced by Noor et al. [22]. Moreover, if $q \to 1^{-}$, then $S^{*}_{q}(A,B)$ coincides with the class $S^{*}[A,B]$ (see [14]).

(ii) If $s = 0$ and $\varphi(z) = \frac{1}{1-qz}$, then $ST^{s}_{q,b}(\varphi)$ reduces to the class ST_q studied by Noor [20].

(iii) If $s = 0$ and $\varphi(z) = \frac{1+z}{1-qz}$, then $ST^{s}_{q,b}(\varphi)$ reduces to the class S^{*}_{q} introduced by Noor et al. [21].

Definition 1.3 *Let $f \in \mathbf{A}$, s be real, and $b > -1$. Then*

$$f \in \widetilde{ST}^{b}_{q,s}(\varphi) \text{ if and only if } I^{b}_{q,s}f(z) \in ST_q(\varphi),$$

and

$$f \in \widetilde{CV}^{b}_{q,s}(\varphi) \text{ if and only if } I^{b}_{q,s}f(z) \in CV_q(\varphi).$$

It is obvious that

$$f \in \widetilde{CV}^{b}_{q,s}(\varphi) \text{ if and only if } z(D_q f) \in \widetilde{ST}^{b}_{q,s}(\varphi). \tag{1.18}$$

Remark 1.4 *It is easy to check that for $0 < q < 1$, the function $f_1(z) = \frac{z}{1-qz}$, $z \in \mathbf{O}$, belongs to $ST_q(\varphi)$ and the function $f_2(z) = \frac{1+z}{1-qz}$, $z \in \mathbf{O}$, belongs to $CV_q(\varphi)$. Therefore, our newly defined classes are not empty; consequently, we proceed to discuss our main results.*

2. Main results

We need the following lemma to prove our results:

Lemma 2.1 *[23] Let β and γ be complex numbers with $\beta \neq 0$ and let $h(z)$ be analytic in $\mathbf{O}$ with $h(0) = 1$ and $Re\{\beta h(z) + \gamma\} > 0$. If $p(z) = 1 + p_1 z + p_2 z^2 + ...$ is analytic in $\mathbf{O}$, then*

$$p(z) + \frac{zD_q p(z)}{\beta p(z) + \gamma} \prec h(z)$$

implies that $p(z) \prec h(z)$.

2.1. Inclusion results

Theorem 2.2 *Let $\varphi(z)$ be an analytic and convex univalent function with $\varphi(0) = 1$ and $Re(\varphi(z)) > 0$ for $z \in \mathbf{O}$. Then , for $s > 0$, $b \in \mathbb{N}$,*

$$ST^{s}_{q,b}(\varphi) \subset ST^{s+1}_{q,b}(\varphi).$$

Proof Let $f \in ST^{s}_{q,b}(\varphi)$ and we set

$$\frac{zD_q\left(J^{s+1}_{q,b}f(z)\right)}{J^{s+1}_{q,b}f(z)} = p(z), \tag{2.1}$$

where $p(z)$ is analytic in $\mathbf{O}$ with $p(0) = 1$.

Using identity (1.13) and (2.1), we get

$$\frac{zD_q\left(J_{q,b}^{s+1}f(z)\right)}{J_{q,b}^{s+1}f(z)}=\left(1+\frac{[b]_q}{q^b}\right)\frac{J_{q,b}^{s}f(z)}{J_{q,b}^{s+1}f(z)}-\frac{[b]_q}{q^b},$$

or equivalently

$$\left(1+\frac{[b]_q}{q^b}\right)\frac{J_{q,b}^{s}f(z)}{J_{q,b}^{s+1}f(z)}=p(z)+\gamma_q,\quad\left(\text{for }\gamma_q=\frac{[b]_q}{q^b}\right),$$

and q-logarithmic differentiation yields

$$\frac{zD_q\left(J_{q,b}^{s}f(z)\right)}{J_{q,b}^{s}f(z)}=p(z)+\frac{zD_qp(z)}{p(z)+\gamma_q}. \tag{2.2}$$

Since $f\in ST_{q,b}^{s}(\varphi)$, from (2.2) we have

$$p(z)+\frac{zD_qp(z)}{p(z)+\gamma_q}\prec\varphi(z).$$

By applying Lemma 2.1, we conclude that $p(z)\prec\varphi(z)$. Consequently, $\frac{zD_q\left(J_{q,b}^{s+1}f(z)\right)}{J_{q,b}^{s+1}f(z)}\prec\varphi(z)$ implies $f\in ST_{q,b}^{s+1}(\varphi)$. □

Theorem 2.3 *Let s, b, and φ be given the same as in Theorem 2.2. Then*

$$CV_{q,b}^{s}(\varphi)\subset CV_{q,b}^{s+1}(\varphi).$$

Proof Let $f\in CV_{q,b}^{s}(\varphi)$. Then, by Alexander's type relation (1.17), we can write $z(D_qf)\in ST_{q,b}^{s}(\varphi)$. From Theorem 2.2, we know that $ST_{q,b}^{s}(\varphi)\subset ST_{q,b}^{s+1}(\varphi)$, so we have $z(D_qf)\in ST_{q,b}^{s+1}(\varphi)$. We get the required result by again using Alexander's type relation (1.17). □

Corollary 2.4 *Let s and b be given the same as in Theorem 2.2. Then, for $\varphi(z)=\frac{1+Az}{1+Bz}$ $(-1\leq B<A\leq 1)$,*

$$ST_{q,b}^{s}\left(\frac{1+Az}{1+Bz}\right)\subset ST_{q,b}^{s+1}\left(\frac{1+Az}{1+Bz}\right).$$

Moreover, for $A=0$ and $B=-q$ and for $A=1$ and $B=-q$,

$$ST_{q,b}^{s}\left(\frac{1}{1-qz}\right)\subset ST_{q,b}^{s+1}\left(\frac{1}{1-qz}\right)\ \text{and}\ ST_{q,b}^{s}\left(\frac{1+z}{1-qz}\right)\subset ST_{q,b}^{s+1}\left(\frac{1+z}{1-qz}\right),$$

respectively.

One can easily prove the following results by using similar arguments as used before.

Theorem 2.5 *Let $\varphi(z)$ be an analytic and convex univalent function with $\varphi(0) = 1$ and $Re(\varphi(z)) > 0$ for $z \in \mathbf{O}$. Then, for $s > 0$, $b \in \mathbb{N}$,*

$$\widetilde{ST}^{b}_{q,s+1}(\varphi) \subset \widetilde{ST}^{b}_{q,s}(\varphi)$$

and

$$\widetilde{CV}^{b}_{q,s+1}(\varphi) \subset \widetilde{CV}^{b}_{q,s}(\varphi).$$

2.2. Integral preserving property

Theorem 2.6 *Let $f \in ST^{s}_{q,b}(\varphi)$. Then $J^{1}_{q,b}f \in ST^{s}_{q,b}(\varphi)$, where*

$$J^{1}_{q,b}f(z) = \frac{[1+b]_q}{z^b}\int_0^z t^{b-1}f(t)d_qt. \tag{2.3}$$

Proof Let $f \in ST^{s}_{q,b}(\varphi)$. If we set $F(z) = J^{1}_{q,b}f(z)$,

$$\frac{zD_q\left(J^{s}_{q,b}F(z)\right)}{J^{s}_{q,b}F(z)} = Q(z), \tag{2.4}$$

where $Q(z)$ is analytic in $\mathbf{O}$ with $Q(0) = 1$.

From (2.3) we can write

$$\frac{D_q\left(z^bF(z)\right)}{[1+b]_q} = z^{b-1}f(z).$$

Using the product rule of the q-difference operator, we get

$$zD_qF(z) = \left(1 + \frac{[b]_q}{q^b}\right)f(z) - [b]_q F(z). \tag{2.5}$$

From (2.2), (2.5), and (1.11), we have

$$Q(z) = \left(1 + \frac{[b]_q}{q^b}\right)\frac{zD_q\left(J^{s}_{q,b}f(z)\right)}{J^{s}_{q,b}F(z)} - [b]_q.$$

On q-logarithmic differentiation, we get

$$\frac{zD_q\left(J^{s}_{q,b}f(z)\right)}{J^{s}_{q,b}f(z)} = Q(z) + \frac{zD_qQ(z)}{Q(z)+[b]_q}. \tag{2.6}$$

Since $f \in ST^{s}_{q,b}(\varphi)$, (2.6) implies

$$Q(z) + \frac{zD_qQ(z)}{Q(z)+[b]_q} \prec \varphi(z).$$

Now, by applying Lemma 2.1, we conclude $Q(z) \prec \varphi(z)$. Consequently, $\frac{zD_q\left(J^{s}_{q,b}F(z)\right)}{J^{s}_{q,b}F(z)} \prec \varphi(z)$. Hence, $F \in ST^{s}_{q,b}(\varphi)$. □

On similar arguments as used in Theorem 2.3, one can easily prove the following result.

Theorem 2.7 *Let $f \in CV^{s}_{q,b}(\varphi)$. Then $J^{1}_{q,b}f \in CV^{s}_{q,b}(\varphi)$, where $J^{1}_{q,b}f$ is defined by (2.3).*

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